A-LEVEL NOTES

STATISTICS
(for Jesse)

March 2021
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A-LEVEL STATISTICS
Mathematics and Further Mathematics
(March 2021)

This document is a self contained set of Probability and Statistics notes for A level Mathematics and Further Mathematics. These notes are maintained by Hugh Murrell and made available as an *open source* document from:
https://hughmurrell.github.io/.

These notes are derived from notes by **Aakash Jog** available from github under Creative Commons licence: The original versions of Aakash's notes on probability and statistics can be found here:
https://github.com/aakashjog/Introduction-to-Probability-and-Statistics

These notes have been augmented with extra content from **OpenIntro Statistics** by **David Diez**, **Mine Cetinkaya-Rundel** and **Christopher D Barr**, a set of open source notes for introductory Statistics available via github. The interested reader can view the originals here:
https://github.com/OpenIntroStat/openintro-statistics

This collection also includes worked problems from past A-level and STEP (Sixth Term Examination Paper) papers. Further worked problems will be added in due course depending on reader engagement with the collection.

The intention is that this set of Mechanics notes will provide preparation material for the following A-LEVEL papers:

Mathematics	Further Mathematics	STEP
Paper 5 (Prob & Stats)	Paper 4 (Prob & Stats)	Section C (Prob & Stats)
Paper 6 (Prob & Stats)		

Prospective A-level students and students planning to write the *Sixth Term Examination Paper* (STEP), are encouraged to make use of this text to supplement their A-level materials. Answers and hints to selected exercises can be found in the appendix of this text.

Further assistance with challenging problems can be obtained via email. To obtain help, point out errors or propose new problems or YouTube videos please feel free to email hugh.murrell@gmail.com.

Contents

Chapter 1

Basics of Probability

1.1 Terminology

Definition 1 (Experiment)**.** A situation with uncertain results is called an experiment.

Definition 2 (Sample space)**.** The set of all possible outcomes of an experiment is called the sample space. It is denoted by Ω or S.

Definition 3 (Event)**.** Any subset A of the sample space is called an event.

Definition 4 (Intersection of sets)**.** Let A and B be two events of sample space Ω. The set of all outcomes that are both in A and B is called the intersection of A and B. It is denoted by $A \cap B$.

Definition 5 (Union of sets)**.** Let A and B be two events of sample space Ω. The set of all outcomes that are in either of A and B is called the union of A and B. It is denoted by $A \cup B$.

Definition 6 (Complement of set)**.** Let A be an event of sample space Ω. The set of all outcomes that are not in A, but are in Ω is called the complement of A. It is denoted by $\overline{A}$ or A^{c}.

Definition 7 (Mutually exclusive events)**.** Two events A and B are said to be mutually exclusive, if

$$A \cap B = \emptyset$$

1.2 Basic Laws

Law 1 (Commutative Laws).

$$A \cup B = B \cup A$$
$$A \cap B = B \cap A$$

Law 2 (Associative Laws).

$$(A \cup B) \cup C = A \cup (B \cup C)$$
$$(A \cap B) \cap C = A \cap (B \cap C)$$

Law 3 (Distributive Laws).

$$(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$$
$$(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$$

Law 4 (De Morgan's Laws).

$$\overline{A_1 \cup \cdots \cup A_n} = \overline{A_1} \cap \cdots \cap \overline{A_n}$$
$$\overline{A_1 \cap \cdots \cap A_n} = \overline{A_1} \cup \cdots \cup \overline{A_n}$$

Proof.

$$\omega \in \overline{A_a \cup \cdots \cup A_n}$$
$$\iff \omega \notin A_{\cup} \cdots \cup A_n$$
$$\iff \omega \notin A_1 \text{ and } \ldots \text{ and } \omega \notin A_n$$
$$\iff \omega \in \overline{A_1} \text{ and } \ldots \text{ and } \omega \in \overline{A_n}$$
$$\iff \omega \in \overline{A_1} \cap \cdots \cap \overline{A_n}$$

Similarly,

$$\omega \in \overline{A_a \cap \cdots \cap A_n}$$
$$\iff \omega \notin A_{\cap} \cdots \cap A_n$$
$$\iff \omega \notin A_1 \text{ or } \ldots \text{ or } \omega \notin A_n$$
$$\iff \omega \in \overline{A_1} \text{ or } \ldots \text{ or } \omega \in \overline{A_n}$$
$$\iff \omega \in \overline{A_1} \cup \cdots \cup \overline{A_n}$$

□

1.3 Axioms of Probability

Definition 8 (Probability)**.** The probability of an event E is defined to be a function which satisfies the three basic axioms. It is denoted by $P(E)$.

Axiom 1.

$$0 \leq \mathrm{P}(E) \leq 1$$

Axiom 2.

$$\mathrm{P}(\Omega) = 1$$

Axiom 3. *For any sequence of mutually exclusive events* $A_1, \ldots,$

$$\mathrm{P}\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mathrm{P}(A_i)$$

Theorem 1.

$$\mathrm{P}(\emptyset) = 0$$

Theorem 2. *For a finite collection of mutually exclusive event* $A_1, \ldots, A_n,$

$$\mathrm{P}\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} \mathrm{P}(A_i)$$

Theorem 3.

$$\mathrm{P}\left(\overline{A}\right) = 1 - \mathrm{P}(A)$$

Proof.

$$A \cap \overline{A} = \emptyset$$

Therefore, A and $\overline{A}$ are mutually exclusive. Therefore,

$$\begin{aligned} \mathrm{P}(A) + \mathrm{P}\left(\overline{A}\right) &= \mathrm{P}\left(A \cup \overline{A}\right) \\ &= \mathrm{P}(\Omega) \\ &= 1 \\ \therefore \mathrm{P}\left(\overline{A}\right) &= 1 - \mathrm{P}(A) \end{aligned}$$

□

Theorem 4.

$$\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B)$$

Definition 9 (Symmetric sample spaces)**.** A sample space is said to be symmetric if the probabilities of all $\omega \in \Omega$ are the same.

1.4 Basics of Combinatorics

Theorem 5. *The number of combinations of k objects out of n, without repetition is*

$$\begin{aligned} \binom{n}{k} &= {}^n\mathrm{C}_k \\ &= \frac{n!}{(n-k)!k!} \end{aligned}$$

Theorem 6.

$$\begin{aligned} {}^n\mathrm{P}_k &= k!\,{}^n\mathrm{C}_k \\ &= \frac{n!}{(n-k)!} \end{aligned}$$

The number of permutations of k objects out of n, without repetition is

Exercise 1.
8 books are to be arranged on 2 shelves, of capacities 3 and 5 respectively. Out of the 8 books, 2 books are special. Find the probability that the two special books end up on the same shelf.

Exercise 2.
A lady crosses three traffic signals, with red and green lights only, on the way to her dog's hairdresser.
The probabilities of encountering 0, 1, and 2 red lights are 0.4, 0.1, 0.2 respectively. Find the probabilities of

1. Encountering at least one red light.
2. Encountering at least one green light.
3. Encountering an odd number of red lights.

Exercise 3.
5 cards are taken out randomly from a 52 card deck. Consider the following events.

1. A: All cards are with numbers higher than 10.
2. B: All cards are hearts.
3. C: All cards have different numbers.

4. D: All cards are consecutive numbers.

Assuming ace to have value 1, find the probabilities of A, B, C, and D.

Exercise 4.
A die is tossed 3 times. Consider the following events.

1. A: The sum of all three numbers is even.

Find the probability of A.

Exercise 5.
There are n students in a classroom. Assuming 365 days in a year, what is the probability that at least two of them share the same birthday, ignoring the year?

Exercise 6.
A president, a treasurer, and a secretary, all different, are to be chosen from a club consisting of 10 people. How many different choices of office bearers are possible if

1. There are no restrictions.
2. Emily and Jesse cannot serve together.
3. Frodo and Smeagol can serve together or not at all.
4. Kate must be an officer.
5. Bilbo can serve only if he is the president.

Exercise 7.
a different balls are divided randomly into n different cells. Find the probability that all cells are non-empty when

1. $a = n$
2. $a = n + 1$

1.5 Conditional Probability

Definition 10. For two events A and B in sample space Ω, where $\mathrm{P}(B) > 0$, the conditional probability, i.e. the probability that A will occur after B has already occurred is defined as

$$\mathrm{P}(A|B) = \frac{\mathrm{P}(A \cap B)}{\mathrm{P}(B)}$$

Exercise 8.
A die is rolled once. Consider the following events.

1. A: The result is even.

2. B: The result is higher than 3.

What is the probability that the result is even, if it is known that result is higher than 3?

Exercise 9.
A coin is flipped twice. What is the probability of getting 'Heads' on both flips, given that the first flip results in 'Heads'.

Exercise 10.
A coin is flipped twice. What is the probability of getting 'Heads' on both flips, given that at least one flip results in 'Heads'.

Theorem 7.

$$\mathrm{P}(A_1 \cap \cdots \cap A_n) = \prod_{k=1}^{n} \mathrm{P}\left(A_k | \bigcap_{l=1}^{k-1} A_l\right)$$

Exercise 11.
A deck of cards is randomly divided into four stacks of 13 cards each. Find the probability that each stack has exactly one ace.

Exercise 12.
A deck of cards is randomly divided into four stacks of 13 cards each. Find the probability, using combinatorics, that each stack has exactly one ace.

1.6 Bayes' Theorem

Definition 11 (Division)**.** A set of events $A_1, \ldots, A_n$ is called a division of the sample space Ω, if

$$\bigcup_{k=1}^{n} A_k = \Omega$$

and

$$A_i \cap A_j = \emptyset$$

for $i \neq j$.

Theorem 8 (Bayes' Theorem)**.** *Given a division $A_1, \dots, A_n$ of the sample space Ω, and an event B in Ω,*

$$\mathrm{P}(B) = \sum_{i=1}^{n} \mathrm{P}(A_i)\,\mathrm{P}(B|A_i)$$

Exercise 13.
A chocolate factory has three production lines.
50% of the production is milk chocolate, out of which 1% is defective.
30% of the production is dark chocolate, out of which 2% is defective.
20% of the production is white chocolate, out of which 0.5% is defective.
If a chocolate bar is picked randomly, what is the probability that it is defective?

Exercise 14.
In a certain stage of a criminal investigation, the inspector in charge is 60% convinced of the guilt of a certain suspect. Suppose that a new piece of evidence which shows that the criminal is bald, is uncovered. If 20% of the population is bald, and if the suspect is bald, how certain should that inspector be of the guilt of the suspect?

Exercise 15.
What is the probability that when a deck of cards is dealt in a game of bridge, the $\heartsuit$s will be dealt such that Emily gets 3, Jesse gets 4, Frodo gets 2, Smeagol gets 4.

1.7 Independent Events

Definition 12 (Two independent events)**.** Two events, A and B, are said to be independent if

$$\mathrm{P}(A \cap B) = \mathrm{P}(A)\,\mathrm{P}(B)$$

Theorem 9.

$$\mathrm{P}(A|B) = \mathrm{P}(A)$$

if and only if A and B are independent.

Theorem 10. *If A and B are independent, then so are $\overline{A}$ and B, A and $\overline{B}$, $\overline{A}$ and $\overline{B}$*

Exercise 16.
Two fair dice are rolled.
Let A be the event that the sum of the results of the dice is 6.
Let B be the event that the result of the first die is 4.
Let C be the event that the sum of the results of the dice is 7.
Which of the possible pairs of the events are independent?

Exercise 17.
There are 10 books in a library, 8 of type A and 2 of type B. On a day when one of the books is missing, Emily enters the library, randomly takes a book, reads it and returns it. After Emily returns the book, Jesse enters the library and randomly takes a book.
Let E be the event that Emily took a book of type A.
Let F be the event that Jesse took a book of type A.
Are E and F independent?

Definition 13. Three events, A, B, and C, are said to be independent if

$$\begin{aligned} \mathrm{P}(A \cap B \cap C) &= \mathrm{P}(A)\,\mathrm{P}(B)\,\mathrm{P}(C) \\ \mathrm{P}(A \cap B) &= \mathrm{P}(A)\,\mathrm{P}(B) \\ \mathrm{P}(B \cap C) &= \mathrm{P}(B)\,\mathrm{P}(C) \\ \mathrm{P}(C \cap A) &= \mathrm{P}(C)\,\mathrm{P}(A) \end{aligned}$$

Chapter 2

Discrete Random Variables

2.1 Discrete Random Variables

Definition 14 (Random variable)**.** A function $X : \Omega \to \mathbb{R}$ which maps points from the sample space to the real line is called a random variable.

Exercise 18.
Three balls are to be randomly selected, without replacement, from an urn containing 20 balls numbered 1 to 20. If Emily bets that at least one of the balls drawn has a number as large as or larger than 17, what is the probability that Emily wins the bet?

2.2 Probability Mass Function

Definition 15 (Discrete random variable)**.** A random variable that can have at most a countable number of possible values is said to be discrete.

Definition 16 (Probability mass function)**.** A function which gives the probability of a discrete random variable X having value x is called the probability mass function of X. It is denoted as $\mathrm{P}(X = x)$.

Exercise 19.
The probability mass function of a random variable X is given by

$$\mathrm{P}(X = i) = \frac{c\lambda^i}{i!}$$

where $i \in \mathbb{W}$, and $\lambda > 0$.
Find

1. $\mathrm{P}(X = 0)$
2. $\mathrm{P}(X > 2)$

Definition 17. If X is a discrete random variable having a probability mass function $P(X = x)$, then the expectation or the expected value of X is defined to be

$$\mathrm{E}[X] = \sum_{\left\{x \mid \mathrm{P}(X=x)>0\right\}} x\,\mathrm{P}(X = x)$$

Exercise 20.
Find $\mathrm{E}[X]$ where X is the outcome of rolling a fair die.

Exercise 21.
A class of 120 students is driven in 3 buses to a symphonic perfomance. There are 36 students in the first bus, 14 in the second, and 44 in the third bus. When the buses arrive, a student is randomly chosen. Let X denote the number of students on the bus of the chosen student. Find $\mathrm{E}[X]$.

Exercise 22.
X has the following distribution.

$$\begin{aligned} \mathrm{P}(X = -1) &= 0.2 \\ \mathrm{P}(X = 0) &= 0.5 \\ \mathrm{P}(X = 1) &= 0.3 \end{aligned}$$

Find $\mathrm{E}\left[x^2\right]$.

Theorem 11. *If X is a discrete random variable that takes on one of the values x_i, where $i \in \mathbb{N}$, with probability mass function* $\mathrm{P}(X = x_i)$. *Then, for any real valued function g,*

$$\mathrm{E}\left[g(x)\right] = \sum_i g(x_i)\,\mathrm{P}(x = x_i)$$

Exercise 23.
A product that is sold seasonally yeilds a net profit of b for each unit sold and a net loss of l for each unit left unsold when the season ends. The number of units of the product that are sold at a specific store during any season is a random variable X, with probability mass function such that

$$\mathrm{P}(X = i) = \mathrm{P}(i)$$

where $i \in \mathbb{N}$. If the store must stock this product in advance, determine the number of units the store should stock, so as to maximize its profit.

2.3 Variance

Definition 18 (Variance)**.** The variance of a random variable X is defined to be

$$\begin{aligned} \mathrm{V}(x) &= \mathrm{E}\left[(X - \mathrm{E}[X])^2\right] \\ &= \mathrm{E}\left[X^2\right] - \mathrm{E}[X]^2 \end{aligned}$$

Exercise 24.
Calculate $\mathrm{V}(X)$ where X represents the outcome of rolling a fair die.

Theorem 12.

$$\mathrm{V}(aX + b) = a^2\,\mathrm{V}(X)$$

Proof.

$$\begin{aligned} \mathrm{V}(aX + b) &= \mathrm{E}\left[(aX + b - \mathrm{E}[aX + b])^2\right] \\ &= \mathrm{E}\left[(aX + b - a\,\mathrm{E}[X] - b)^2\right] \\ &= \mathrm{E}\left[a^2\,(X - \mathrm{E}[X])^2\right] \\ &= a^2\,\mathrm{E}\left[(x - \mathrm{E}[x])^2\right] \\ &= a^2\,\mathrm{V}(x) \end{aligned}$$

□

Exercise 25.
You have a coin with probability p of getting 'Heads'. You flip this coin twice. For each flip, if the result is 'Heads', you win \$30. If the result is 'Tails', you lose \$20.
Let X be your profit in the game.

1. What is the sample space?
2. Describe the probability mass function of X.
3. Describe the cumulative distribution function of X.
4. What is the expected value of X?
5. What is the value of p upto which you would agree to participate in the game?
6. What is $\mathrm{V}(X)$?
7. What is $\sigma(X)$?

Chapter 3

Continuous Random Variables

3.1 Cumulative Distribution Function

Definition 19 (Cumulative distribution function)**.** The function

$$\mathrm{F}_X = \mathrm{P}(X \leq x)$$

for $-\infty < x \leq x$, is called the cumulative distribution function of the variable X.

Exercise 26.
Let X be the result of a die roll. Plot the cumulative distribution function of X.

Theorem 13. *Let* F_X *be the cumulative distribution function of a random variable* X*. Then,*

1. F_X *is a non-decreasing function.*

2. $\lim\limits_{b \to \infty} \mathrm{F}_X(b) = 1.$

3. $\lim\limits_{b \to -\infty} \mathrm{F}_X(b) = 0.$

4. F_X *is right continuous, i.e., the function is equal to its right hand limit.*

3.2 Continuous Random Variable

Definition 20 (Continuous random variable)**.** A random variable X is said to be a continuous random variable if there exists a function f such that

$$\mathrm{F}_X(x) = \int\limits_{-\infty}^{x} \mathrm{f}(t)\,\mathrm{d}t$$

Definition 21. Let

$$\mathrm{F}_X(x) = \int\limits_{-\infty}^{x} \mathrm{f}(t)\,\mathrm{d}t$$

Then, $\mathrm{f}(t)$ is called the probability density function of x.

Exercise 27.
Let X be a continuous variable whose probability density function is

$$\mathrm{f}(x) = \begin{cases} c\left(4x - 2x^2\right) & ; \quad 0 < x < 2 \\ 0 & ; \quad \text{otherwise} \end{cases}$$

1. Find c.
2. Find $\mathrm{P}(X > 1)$.

Exercise 28.
The amount of time in hours that a computer functions before breaking down has the distribution

$$\mathrm{f}(x) = \begin{cases} \lambda e^{-\frac{x}{100}} & ; \quad x \geq 0 \\ 0 & ; \quad \text{otherwise} \end{cases}$$

What is the probability that the computer functions for more than 50 but less than 150 hours?

3.3 Expectation and Variance

Definition 22. Let X be a continuous random variable. Then, the expectation is defined as

$$\mathrm{E}[X] = \int\limits_{-\infty}^{\infty} x f(x)\,\mathrm{d}x$$

Exercise 29.
Find the expectation of X is the probability density function is given to be

$$\mathrm{f}(x) = \begin{cases} 2x & ; \quad 0 < x < 1 \\ 0 & ; \quad \text{otherwise} \end{cases}$$

Exercise 30.
The probability density function of X is given by

$$\mathrm{f}(x) = \begin{cases} 1 & ; \quad 0 \leq x \leq 1 \\ 0 & ; \quad \text{otherwise} \end{cases}$$

Find $\mathrm{E}\left[e^X\right]$.

Theorem 14. *If X is a continuous random variable, then for any real function g,*

$$\mathrm{E}\left[g(x)\right] = \int_{-\infty}^{\infty} g(x) f(x) \, \mathrm{d}x$$

Chapter 4

Special Distributions

4.1 Bernoulli and Binomial Random Variables

Definition 23 (Bernoulli random variable)**.** A random variable X is said to be a Bernoulli random variable if its probability mass function is given by

$$\begin{aligned} \mathrm{P}(X = 0) &= 1 - p \\ \mathrm{P}(X = 1) &= p \end{aligned}$$

Theorem 15. *For a Bernoulli random variable,*

$$\begin{aligned} \mathrm{E}[x] &= p \\ \mathrm{V}(x) &= p(1 - p) \end{aligned}$$

Definition 24 (Binomial random variable)**.** Consider n independent trials, each of which has a probability of success p, and probability of failure $1-p$. If X represents the number of successes that occur in the n trials, then X is said to be a binomial random variable with parameters (n, p). It is denoted as

$$X \sim \mathrm{Bin}(n, p)$$

Theorem 16. *For a binomial random variable,*

$$X \sim \mathrm{Bin}(n, p)$$

the probability mass function is

$$\mathrm{P}(X = i) = \binom{n}{i} p^i (1 - p)^{n-i}$$

Exercise 31.
A die is rolled 5 times. What is the probability that the result is 6, 3 times?

Exercise 32.
A player bets on a number from 1 to 6, both including. Three dice are then rolled. If the number bet on by the player appears i times where $i = 1, 2, 3$, he wins i units. If the number bet on by the player does not appear on any of the dice, he loses 1 unit.
A game is considered to be fair if the expected value for the player is at least 0. Is this game fair towards the player?

Theorem 17. *Let*

$$\begin{aligned} X &\sim \mathrm{Bin}(n, p) \\ Y &\sim \mathrm{Bin}(n-1, p) \end{aligned}$$

Then,

$$\mathrm{E}\left[X^k\right] = np\,\mathrm{E}\left[(Y+1)^{k-1}\right]$$

Proof.

$$\begin{aligned} \mathrm{E}\left[X^k\right] &= \sum_{i=0}^{n} i_k \binom{n}{i} p^i (1-p)^{n-i} \\ &= \sum_{i=1}^{n} i_k \binom{n}{i} p^i (1-p)^{n-i} \\ &= np \sum_{i=1}^{n} i^{k-1} \binom{n-1}{i-1} p^{i-1} (1-p)^{n-1} \end{aligned}$$

Let

$$j = i - 1$$

Therefore,

$$\begin{aligned} \mathrm{E}\left[X^k\right] &= np \sum_{j=0}^{n-1} (j+1)^{k-1} \binom{n-1}{j} p^j (1-p)^{n-1-j} \\ &= \mathrm{P}(Y = j) \\ &= np\,\mathrm{E}\left[(Y+1)^{k-1}\right] \end{aligned}$$

□

4.2 Poisson Random Variables

Definition 25 (Poisson Random Variables)**.** A random variable X that takes on whole number values is said to be a Poisson random variable with parameter λ if for some $\lambda > 0$,

$$\mathrm{P}(X = i) = \frac{e^{-\lambda}\lambda^i}{i!}$$

where $i \in \mathbb{W}$.
It is denoted as

$$X \sim \mathrm{Poi}(p)$$

Theorem 18. *Let*

$$X \sim \mathrm{Bin}(n, p)$$

If $n \to \infty$, the probability distribution of X is a Poisson distribution.

Proof. Let

$$X \sim \mathrm{Bin}(n, p)$$

Therefore,

$$\begin{aligned}\mathrm{P}(X = i) &= \binom{n}{i} p^i (1-p)^{n-i} \\ &= \frac{n!}{(n-i)!i!} p^i (1-p)^{n-i}\end{aligned}$$

Let

$$\lambda = np$$

Therefore,

$$\begin{aligned}\mathrm{P}(X = i) &= \frac{n!}{(n-i)!i!} \left(\frac{\lambda}{n}\right)^i \left(1 - \frac{\lambda}{n}\right)^{n-i} \\ &= \frac{(n)(n-1)\dots(n-i+1)}{n^i} \frac{\lambda^i}{i!} \frac{\left(1-\frac{\lambda}{n}\right)^2}{\left(1-\frac{\lambda}{n}\right)^i} \\ \therefore \lim_{n\to\infty} \mathrm{P}(X = i) &= \frac{\lambda^i e^{-\lambda}}{i!}\end{aligned}$$

□

Theorem 19. *Let*

$$X \sim \mathrm{Poi}(\lambda)$$

Then,

$$\mathrm{E}[X] = \lambda$$

Proof.

$$\begin{aligned}\mathrm{E}[X] &= \sum_{i=0}^{\infty} i \frac{e^{-\lambda}\lambda^i}{i!} \\ &= \lambda \sum_{i=1}^{\infty} \frac{e^{-\lambda}\lambda^{i-1}}{(i-1)!}\end{aligned}$$

Let

$$j = i - 1$$

Therefore,

$$\mathrm{E}[X] = \lambda \sum_{j=0}^{\infty} \frac{e^{-\lambda}\lambda^j}{j!}$$

As $\sum\limits_{j=0}^{\infty} \frac{e^{-\lambda}\lambda^j}{j!}$ represents the total probability of X,

$$\sum_{j=0}^{\infty} \frac{e^{-\lambda}\lambda^j}{j!} = 1$$

Therefore,

$$\mathrm{E}[X] = \lambda$$

□

Theorem 20. *Let*

$$X = \mathrm{Poi}(\lambda)$$

Then,

$$\mathrm{V}(X) = \lambda$$

Proof.

$$\begin{aligned}\mathrm{E}\left[X^2\right] &= \sum_{i=0}^{\infty} i^2 \frac{e^{-\lambda}\lambda^i}{i!} \\ &= \lambda \sum_{i=1}^{\infty} i \frac{e^{-\lambda}\lambda^{i-1}}{(i-1)!}\end{aligned}$$

Let

$$j = i - 1$$

Therefore,

$$\begin{aligned}\mathrm{E}\left[X^2\right] &= \lambda \sum_{j=0}^{\infty} (j+1) \frac{e^{-\lambda}\lambda^j}{j!} \\ &= \lambda \left(\sum_{j=0}^{\infty} j \frac{e^{-\lambda}\lambda^j}{j!} + \sum_{j=0}^{\infty} \frac{e^{-\lambda}\lambda^j}{j!} \right) \\ &= \lambda \left(\mathrm{E}\left[\mathrm{Poi}(\lambda)\right] + \sum_{j=0}^{\infty} \mathrm{P}\left(\mathrm{Poi}(\lambda) = j\right) \right) \\ &= \lambda(\lambda + 1)\end{aligned}$$

Therefore,

$$\begin{aligned}\mathrm{V}(X) &= \mathrm{E}\left[X^2\right] - \mathrm{E}[X]^2 \\ &= \lambda\end{aligned}$$

□

Exercise 33.
Consider an experiment that consists of counting the number of α particles given off in a second by a gram of radioactive material. If it is known that on average, 3.2 such α particles are emitted, what is the probability that no more than 2 α particles will be emitted?

4.2.1 Assumptions for Poisson Distributions for Events Over a Period of Time

1. The probability that an event occurs in an interval of length h is $\lambda h + \mathrm{o}(h)$.
2. For a small enough h, the probability that two or more events occur in an interval of length h is small, i.e. is $\mathrm{o}(h)$.

3. The number of events in intervals that are not overlapping are independent.

Exercise 34.
Consider that earthquakes occur with the assumptions of Poisson distributions, with $\lambda = 2$, and with a week as a unit of time.

1. Find the probability that at least three earthquakes occur during the next two weeks.

2. Given that three earthquakes occurred in the last four weeks, what is the probability that exactly one of them occurred in the last week.

4.3 Geometric Random Variables

Definition 26 (Geometric random variable)**.** Suppose that independent trials, each having probability of success $0 < p < 1$, are performed until a success occurs. If X is the number of trials required, then X is said to have a geometric distribution. It is denoted as

$$X \sim \text{Geo}(p)$$

The probability distribution if X is

$$\text{P}(X = n) = (1 - p)^{n-1} p$$

Exercise 35.
Emily eats cookies one after another until she finds and a chocolate cookie. For each cookie, the probability of the cookie being a chocolate cookie is $\frac{1}{10}$.

1. What is the probability that Emily eats more than 3 cookies?

2. Given that Emily has already eaten 5 cookies, and has not found a chocolate cookie, what is the probability that she will eat at least 8 more cookies?

Theorem 21. *Let*

$$X \sim \text{Geo}(p)$$

Then,

$$\text{E}[X] = \frac{1}{p}$$

Proof.

$$\mathrm{E}[X] = \sum_{i=1}^{\infty} i(1-p)^{i-1}p$$

Let

$$q = 1 - p$$

Therefore,

$$\begin{aligned}
\mathrm{E}[X] &= \sum_{i=1}^{\infty} iq^{i-1}p \\
&= \sum_{i=1}^{\infty}(i-1+1)q^{i-1}p \\
&= \sum_{i=1}^{\infty}(i-1)q^{i-1}p + \sum_{i=1}^{\infty} q^{i-1}p \\
&= \sum_{i=1}^{\infty}(i-1)q^{i-1}p + \frac{p}{1-q} \\
&= \sum_{i=1}^{\infty}(i-1)q^{i-1}p + \frac{p}{1-(1-p)} \\
&= \sum_{i=1}^{\infty}(i-1)q^{i-1}p + 1 \\
&= \sum_{j=0}^{\infty} jq^{j}p + 1 \\
&= q\sum_{j=0}^{\infty} jq^{j-1}p + 1 \\
&= q\,\mathrm{E}[X] + 1 \\
\therefore \mathrm{E}[X](1-q) &= 1 \\
\therefore \mathrm{E}[X] &= \frac{1}{1-q} \\
&= \frac{1}{p}
\end{aligned}$$

$\square$

Theorem 22. *Let*

$$X \sim \mathrm{Geo}(p)$$

Then,

$$\mathrm{V}(X) = \frac{1-p}{p^2}$$

Proof. Let

$$q = 1 - p$$

$$\begin{aligned}
\mathrm{E}\left[X^2\right] &= \sum_{i=1}^{\infty} i^2 q^{i-1} p \\
&= \sum_{i=1}^{\infty} (i - 1 + i)^2 q^{i-1} p \\
&= \sum_{i=1}^{\infty} (i-1)^2 q^{i-1} p + \sum_{i=1}^{\infty} 2(i-1) q^{i-1} p + \sum_{i=1}^{\infty} q^{i-1} p \\
&= \sum_{j=0}^{\infty} j^2 q^j p + 2 \sum_{j=1}^{\infty} j q^j p + 1 \\
&= q\,\mathrm{E}\left[X^2\right] + 2q\,\mathrm{E}[X] + 1 \\
\therefore p\,\mathrm{E}\left[X^2\right] &= \frac{2q}{p} + 1 \\
\therefore \mathrm{E}\left[X^2\right] &= \frac{2q+p}{p^2} \\
&= \frac{q+1}{p^2}
\end{aligned}$$

Therefore,

$$\begin{aligned}
\mathrm{V}(X) &= \mathrm{E}\left[X^2\right] - \mathrm{E}[X]^2 \\
&= \frac{q+1}{p^2} - \frac{1}{p^2} \\
&= \frac{q}{p^2} \\
&= \frac{1-p}{p^2}
\end{aligned}$$

□

4.4 Negative Binomial Random Variable

Definition 27 (Negative binomial random variable)**.** Suppose that independent trials, each having probability of success $0 < p < 1$, are performed until a total of

r successes are accumulated. If X is the number of trials required, then X is said to be a negative binomial random variable.
It is denoted as

$$X \sim \mathrm{NB}(r, p)$$

The last trial must necessarily result in a success, and there must be $r-1$ more success in the first $n-1$ trials. Therefore, the probability distribution of X is

$$\begin{aligned}\mathrm{P}(X = n) &= \binom{n-1}{r-1} p^{r-1}(1-p)^{n-r} p \\ &= \binom{n-1}{r-1} p^{r}(1-p)^{n-r}\end{aligned}$$

Theorem 23.

$$n\binom{n-1}{r-1} = r\binom{n}{r}$$

Theorem 24. *Let*

$$X \sim \mathrm{NB}(r, p)$$

Then,

$$\mathrm{E}[X] = \frac{r}{p}$$

Proof.

$$\begin{aligned}\mathrm{E}\left[X^k\right] &= \sum_{n=r}^{\infty} n^k \binom{n-1}{r-1} p^r (1-p)^{n-r} \\ &= \frac{r}{p}\sum_{n=r}^{\infty} n^{k-1}\binom{n}{r} p^{r+1}(1-p)^{n-r} \\ &= \frac{r}{p}\sum_{m=r+1}^{\infty} (m-1)^{k-1}\binom{m-1}{r} p^{r+1}(1-p)^{m-(r+1)}\end{aligned}$$

Let

$$Y \sim \mathrm{NB}(r+1, p)$$

Therefore,

$$\mathrm{E}\left[X^k\right] = \frac{r}{p}\,\mathrm{E}\left[(Y-1)^{k-1}\right]$$

Therefore,

$$\mathrm{E}[X] = \frac{r}{p}$$

□

Theorem 25. *Let*

$$X \sim \mathrm{NB}(r, p)$$

Then,

$$\mathrm{E}[X] = \frac{r}{p}$$

Proof.

$$\begin{aligned}
\mathrm{E}\left[X^k\right] &= \sum_{n=r}^{\infty} n^k \binom{n-1}{r-1} p^r (1-p)^{n-r} \\
&= \frac{r}{p} \sum_{n=r}^{\infty} n^{k-1} \binom{n}{r} p^{r+1} (1-p)^{n-r} \\
&= \frac{r}{p} \sum_{m=r+1}^{\infty} (m-1)^{k-1} \binom{m-1}{r} p^{r+1} (1-p)^{m-(r+1)}
\end{aligned}$$

Let

$$Y \sim \mathrm{NB}(r+1, p)$$

Therefore,

$$\mathrm{E}\left[X^k\right] = \frac{r}{p} \mathrm{E}\left[(Y-1)^{k-1}\right]$$

Therefore,

$$\begin{aligned}
\mathrm{E}[X] &= \frac{r}{p} \\
\mathrm{E}\left[X^2\right] &= \frac{r}{p} \mathrm{E}[Y-1] \\
&= \frac{r}{p} \left(\frac{r+1}{p} - 1\right)
\end{aligned}$$

Therefore,

$$\begin{aligned} \mathrm{V}(X) &= \mathrm{E}\left[X^2\right] - \mathrm{E}[X]^2 \\ &= \frac{r(1-p)}{p^2} \end{aligned}$$

□

Exercise 36.
Find the expected value and variance of the number of time one must throw a die until the outcome 1 has occurred four times.

4.5 Hypergeometric Random Variable

Definition 28 (Hypergeometric random variable)**.** Suppose that a sample of size n is to be chosen randomly and without replacement from a population of N, of which m possess a particular characteristic, and the other $N - m$ do not. If X is the number of individuals in the selected sample, then X is said to be a hypergeometric random variable.
It is denoted as

$$X \sim \mathrm{HG}(n, N, m)$$

The probability distribution of X is

$$\mathrm{P}(X = i) = \frac{\binom{n}{i}\binom{N-m}{n-i}}{\binom{N}{n}}$$

Theorem 26. *Let*

$$X \sim \mathrm{HG}(n, N, m)$$

Then,

$$\mathrm{E}[X] = \frac{nm}{N}$$

Theorem 27. *Let*

$$X \sim \mathrm{HG}(n, N, m)$$

Then,

$$\mathrm{V}(X) = n\frac{m}{N}\left(1 - \frac{m}{N}\right)\left(\frac{N-n}{N-1}\right)$$

Exercise 37.
A buyer of electrical component buys electrical components in lots of size 10. It is his policy to inspect 3 components randomly from a lot, and to accept the lot only if all 3 are non-defective. If 30% of the lots have 4 defective components and 70% of the lots have 1 defective components, what is the proportion of the lots that the purchaser rejects.

4.6 Uniform Random Variable

Definition 29 (Uniform Random Variable)**.** A random variable X is said to be a uniform random variable over the interval (a, b) if its probability density function is

$$\mathrm{f}(x) = \begin{cases} \frac{1}{b-a} & ; \quad a < x < b \\ 0 & ; \quad \text{otherwise} \end{cases}$$

It is denoted as

$$X \sim \mathrm{U}(a, b)$$

Theorem 28. *The cumulative distribution function of a uniform random variable X is*

$$\mathrm{F}_X(x) = \begin{cases} 0 & ; \quad x < a \\ \frac{x-a}{b-a} & ; \quad a \leq x \leq b \\ 1 & ; \quad b < x \end{cases}$$

Theorem 29. *Let*

$$X \sim \mathrm{U}(a, b)$$

Then,

$$\mathrm{E}[X] = \frac{a+b}{2}$$

Proof.

$$\begin{aligned} \mathrm{E}[X] &= \int_{-\infty}^{\infty} x \, \mathrm{f}(x) \, \mathrm{d}x \\ &= \int_{a}^{b} x \frac{1}{b-a} \, \mathrm{d}x \\ &= \frac{a+b}{2} \end{aligned}$$

□

Theorem 30. *Let*

$$X \sim \mathrm{U}(a, b)$$

Then,

$$\mathrm{V}(X) = \frac{(b-a)^2}{12}$$

Proof.

$$\begin{aligned}
\mathrm{E}\left[X^2\right] &= \int\limits_{-\infty}^{\infty} x^2\, \mathrm{f}\left(x^2\right) \mathrm{d}x \\
&= \int\limits_{a}^{b} x^2 \frac{1}{b-a} \\
&= \frac{b^3 - a^3}{3(b-a)} \\
&= \frac{a^2 + ab + b^2}{3}
\end{aligned}$$

Therefore,

$$\begin{aligned}
\mathrm{V}(X) &= \mathrm{E}\left[X^2\right] - \mathrm{E}[X]^2 \\
&= \frac{a^2 + ab + b^2}{3} - \left(\frac{a+b}{2}\right)^2 \\
&= \frac{(b-a)^2}{12}
\end{aligned}$$

□

4.7 Exponential Random Variable

Definition 30 (Uniform Random Variable)**.** A random variable X is said to be a exponential random variable over the interval (a, b) if its probability density function is

$$\mathrm{f}(x) = \begin{cases} \lambda e^{-\lambda x} & ; \quad x \geq 0 \\ 0 & ; \quad x < 0 \end{cases}$$

It is denoted as

$$X \sim \mathrm{Exp}(\lambda)$$

Theorem 31. *The cumulative distribution function of a exponential random variable X with parameter λ is*

$$\mathrm{F}_X(x) = \begin{cases} 1 - e^{-\lambda x} & ; \quad x \geq 0 \\ 0 & ; \quad x < 0 \end{cases}$$

Theorem 32. *Let*

$$X \sim \mathrm{Exp}(\lambda)$$

Then,

$$\mathrm{E}[X] = \frac{1}{\lambda}$$

Theorem 33. *Let*

$$X \sim \mathrm{Exp}(\lambda)$$

Then,

$$\mathrm{E}\left[X^n\right] = \frac{n!}{\lambda^n}$$

Theorem 34. *Let*

$$X \sim \mathrm{Exp}(\lambda)$$

Then,

$$\mathrm{V}(X) = \frac{1}{\lambda^2}$$

4.8 Normal Distribution

Definition 31 (Normal Random Variable)**.** A random variable X is said to be a normal random variable if its probability density function is

$$\mathrm{f}(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

where μ and σ^2 are parameters.
If

$$\mu = 0$$
$$\sigma^2 = 1$$

then X is said to be a standard normal random variable.
The cumulative distribution function of a standard normal X is denoted by $\Phi(x)$.

Theorem 35. *Let X be a standard normal random variable. Then,*

$$\Phi(-x) = 1 - \Phi(x)$$

Theorem 36. *If X is normally distributed with parameters μ and σ^2, then $Y = ax + b$ where $a > 0$ is normally distributed with parameters $a\mu + b$ and $a^2\sigma^2$.*

Proof.

$$\begin{aligned}
\mathrm{F}_Y(y) &= \mathrm{P}(Y \le y) \\
&= \mathrm{P}(ax + b \le y) \\
&= \mathrm{P}\left(x \le \frac{y-b}{a}\right) \\
&= \mathrm{F}_X\left(\frac{y-b}{a}\right)
\end{aligned}$$

Therefore, differentiating,

$$\begin{aligned}
\mathrm{f}_Y(y) &= \frac{1}{a}\mathrm{f}_X\left(\frac{y-b}{a}\right) \\
&= \frac{1}{\sqrt{2\pi}a\sigma}e^{-\frac{\left(\frac{y-b}{a}-\mu\right)^2}{2\sigma^2}} \\
&= \frac{1}{\sqrt{2\pi}a\sigma}e^{-\frac{(y-b-a\mu)^2}{2(a\sigma)^2}}
\end{aligned}$$

□

Theorem 37. *If X is normally distributed with parameters μ and σ^2, then $Z = \frac{X-\mu}{\sigma}$ is normally distributed with parameters 0 and 1.*

Exercise 38.
Let X be a normal random variable with parameters

$$\mu = 3$$
$$\sigma^2 = 9$$

Find

1. $\mathrm{P}(2 < X < 5)$

2. $\mathrm{P}(X > 3)$

Theorem 38. *Let Z be a standard normal random variable. Then,*

$$\mathrm{E}[X] = 0$$

Proof.

$$\begin{aligned}
\mathrm{E}[Z] &= \int\limits_{-\infty}^{\infty} z\,\mathrm{f}(z)\,\mathrm{d}z \\
&= \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} z e^{-\frac{z^2}{2}}\,\mathrm{d}x \\
&= 0
\end{aligned}$$

□

Theorem 39. *Let Z be a standard normal random variable. Then,*

$$\mathrm{V}(X) = 1$$

Proof.

$$\begin{aligned}
\mathrm{E}\left[Z^2\right] &= \frac{1}{2\pi} \int\limits_{-\infty}^{\infty} z^2 e^{-\frac{z^2}{2}}\,\mathrm{d}z \\
&= 1
\end{aligned}$$

Therefore,

$$\begin{aligned}
\mathrm{V}(X) &= \mathrm{E}\left[X^2\right] - \mathrm{E}[X]^2 \\
&= 1 - 0 \\
&= 1
\end{aligned}$$

□

Theorem 40. *Let X be a normal random variable. Then,*

$$\mathrm{E}[X] = \mu$$

Proof. Let

$$Z = \frac{X - \mu}{\sigma}$$

Therefore,

$$X = Z\sigma + \mu$$

Therefore,

$$\begin{aligned} \mathrm{E}[X] &= \mathrm{E}[Z]\sigma + \mu \\ &= \mu \end{aligned}$$

□

Theorem 41. *Let X be a normal random variable. Then,*

$$\mathrm{V}(X) = \sigma^2$$

Proof. Let

$$Z = \frac{X - \mu}{\sigma}$$

Therefore,

$$X = Z\sigma + \mu$$

Therefore,

$$\begin{aligned} \mathrm{V}(X) &= \sigma^2\,\mathrm{V}(Z) \\ &= \sigma^2 \end{aligned}$$

□

4.9 de Moivre-Laplace Limit Theorem

Theorem 42 (de Moivre-Laplace Limit Theorem)**.** *Let S_n be the number of successes that occur when n independent trials, each with probability of success p, are performed. Then, for any $a < b$,*

$$\lim_{n\to\infty} \mathrm{P}\left(a < \frac{S_n - np}{\sqrt{np(1-p)}} < b\right) = \Phi(b) - \Phi(a)$$

Exercise 39.

To determine the effectiveness of a certain diet in reducing the amount of cholesterol in the bloodstream, 100 people are put on the diet. A month later, the cholesterol is measured.

The nutritionist in charge of this experiment has decided to endorse this diet if at least 65% of the people have lower cholesterol levels than before.

What is the probability to endorse the diet if it has no effect on cholesterol levels?

Chapter 5

Combination of Random Variables

5.1 Expected Value of Sums of Random Variables

Theorem 43. *Let*

$$X = \sum_{i=1}^{n} X_i$$

where each X_i is a random variable, possibly of different distributions.
Then,

$$\mathrm{E}[X] = \sum_{i=1}^{n} \mathrm{E}[X_i]$$

Proof.

$$\begin{aligned}
\mathrm{E}[X] &= \sum_{\omega \in \Omega} X(\omega)\,\mathrm{P}(\omega) \\
&= \sum_{\omega \in \Omega} \left(\sum_{i=1}^{n} X_i(\omega) \right) \mathrm{P}(\omega) \\
&= \sum_{i=1}^{n} \sum_{\omega}^{\Omega} X_i(\omega)\,\mathrm{P}(\omega) \\
&= \sum_{i=1}^{n} \mathrm{E}[X_i]
\end{aligned}$$

□

Exercise 40.
n people came to a winter party, carrying a coat each. While leaving the party, each person took a coat randomly. Let X be the number of people that left with their own coat. Find $\mathrm{E}[X]$.

Chapter 6

Jointly Distributed Random Variables

6.1 Joint Cumulative Probability Distribution Function

Definition 32. For any two random variables X and Y, the joint cumulative probability distribution function of X and Y is defined to be

$$\mathrm{F}_{X,Y}(a,b) = \mathrm{P}(X \leq a, Y \leq b)$$

where $a \in \mathbb{R}$ and $b \in \mathbb{R}$.

Theorem 44. *If the joint cumulative probability distribution function of X and Y is $\mathrm{F}_{X,Y}(a,b)$, then the probability distribution function of X is*

$$\mathrm{F}_X(a) = \lim_{b\to\infty} \mathrm{F}_{X,Y}(a,b)$$

Exercise 41.
3 balls are randomly selected from an urn containing 3 red, 4 white, and 5 blue balls.
Let X be the number of red balls chosen.
Let Y be the number of white balls chosen.
Find the joint probability mass function of X and Y.

6.2 Joint Continuous Variables

Definition 33 (Joint continuity and joint probability density function)**.** X and Y are said to be jointly continuous if there exists a function $\mathrm{f}(x,y)$ defined for all real

x and y, such that for every set C of pairs of real numbers,

$$\mathrm{P}\left((x,y)\in C\right) = \iint\limits_{(x,y)\in C} \mathrm{f}(x,y)\,\mathrm{d}x\,\mathrm{d}y$$

The function $\mathrm{f}(x,y)$ is called the joint probability density function of X and Y. The joint cumulative probability distribution function is

$$\begin{aligned}\mathrm{F}_{X,Y}(a,b) &= \mathrm{P}(-\infty \le X \le a, -\infty \le Y \le b)\\ &= \int\limits_{-\infty}^{b}\int\limits_{-\infty}^{a} \mathrm{f}(x,y)\,\mathrm{d}x\,\mathrm{d}y\end{aligned}$$

Therefore,

$$\mathrm{f}(a,b) = \frac{\partial^2}{\partial a \partial b} F(a,b)$$

Exercise 42.
The joint density function of X and Y is given by

$$\mathrm{f}(x,y) = \begin{cases} 2e^{-x}e^{-2y} & ; \quad 0 < x < \infty, 0 < y < \infty \\ 0 & ; \quad \text{otherwise} \end{cases}$$

Compute

1. $\mathrm{P}(X > 1, Y < 1)$
2. $\mathrm{P}(X < Y)$
3. $\mathrm{P}(X < a)$

Exercise 43.
The joint density function of X and Y is given by

$$\mathrm{f}(x,y) = \begin{cases} e^{-(x+y)} & ; \quad 0 < x < \infty, 0 < y < \infty \\ 0 & ; \quad \text{otherwise} \end{cases}$$

Find the density function of the random variable $\frac{X}{Y}$.

6.3 Independent Random Variables

Definition 34 (Independent random variables)**.** Two random variables X and Y are said to be independent if

$$\mathrm{P}(X \in A, Y \in B) = \mathrm{P}(X \in A)\,\mathrm{P}(Y \in B)$$

where $A \subseteq \mathbb{R}$, $B \subseteq \mathbb{R}$.

Theorem 45. *X and Y are independent random variables if and only if*

$$\mathrm{F}_{X,Y}(a,b) = \mathrm{F}_X(a)\mathrm{F}_Y(a)$$

Proof. Let

$$\begin{aligned} A &= (-\infty, a] \\ B &= (-\infty, b] \end{aligned}$$

X and Y are independent if and only if

$$\begin{aligned} &\mathrm{P}(X \in A, Y \in B) = \mathrm{P}(X \in A)\,\mathrm{P}(Y \in B) \\ \iff &\mathrm{P}(X \le a, Y \le b) = \mathrm{P}(X \le a)\,\mathrm{P}(Y \le b) \\ \iff &\mathrm{F}_{X,Y}(a,b) = \mathrm{F}_X(a)\mathrm{F}_Y(b) \end{aligned}$$

□

Theorem 46. *If X and Y are discrete, then*

$$\mathrm{P}(X = x, Y = y) = \mathrm{P}(X = x)\,\mathrm{P}(Y = y)$$

for all (x, y), if and only if X and Y are independent.

Theorem 47. *If X and Y are continuous, then*

$$\mathrm{f}(x,y) = \mathrm{f}_X(x)\,\mathrm{f}_Y(y)$$

for all (x, y), if and only if X and Y are independent.

Exercise 44.
Let

$$\begin{aligned} X &\sim \mathrm{U}(0,1) \\ Y &\sim \mathrm{U}(0,1) \end{aligned}$$

Calculate the probability density function of $X + Y$

Theorem 48. *If* X_i*, for* $i \in \mathbb{N}$ *are independent normal random variables, with parameters* μ_i *and* ${\sigma_i}^2$ *respectively, then* $\sum_{i=1}^{n} x_i$ *is a normal random variable with parameters* $\sum_{i=1}^{n} \mu_i$ *and* $\sum_{i=1}^{n} {\sigma_i}^2$*.*

Theorem 49. *Let*

$$X = \mathrm{Poi}(\lambda_1)$$
$$Y = \mathrm{Poi}(\lambda_2)$$

be independent random variables. Then,

$$X + Y \sim \mathrm{Poi}(\lambda_1 + \lambda_2)$$

Proof.

$$\begin{aligned}
\mathrm{P}(X+Y=n) &= \sum_{k=0}^{n} \mathrm{P}(X=k, Y=n-k) \\
&= \sum_{k=0}^{n} \mathrm{P}(X=k)\,\mathrm{P}(Y=n-k) \\
&= \sum_{k=0}^{n} \frac{e^{-\lambda_1}{\lambda_1}^k}{k!} \frac{e^{-\lambda_2}{\lambda_2}^{n-k}}{(n-k)!} \\
&= e^{-\lambda_1-\lambda_2} \sum_{k=0}^{n} \frac{{\lambda_1}^k {\lambda_2}^{n-k}}{k!(n-k)!} \\
&= \frac{e^{-(\lambda_1+\lambda_2)}}{n!} \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} {\lambda_1}^k {\lambda_2}^{n-k} \\
&= \frac{e^{-(\lambda_1+\lambda_2)}}{n!} \sum_{k=0}^{n} {}^n\mathrm{C}_k {\lambda_1}^k {\lambda_2}^{n-k} \\
&= \frac{e^{-(\lambda_1+\lambda_2)}}{n!} (\lambda_1+\lambda_2)^n
\end{aligned}$$

Therefore,

$$X + Y \sim \mathrm{Poi}(\lambda_1 + \lambda_2)$$

□

Theorem 50. *Let*

$$X \sim \mathrm{Bin}(n, p)$$
$$Y \sim \mathrm{Bin}(m, p)$$

be independent random variables. Then,

$$X + Y \sim \mathrm{Bin}(n + m, p)$$

Proof.

$$\begin{aligned}
\mathrm{P}(X+Y=k) &= \sum_{i=0}^{n} \mathrm{P}(X=i, Y=k-i) \\
&= \sum_{i=0}^{n} \mathrm{P}(X=i)\,\mathrm{P}(X=i)\,\mathrm{P}(Y=k-i) \\
&= \sum_{i=0}^{n} \binom{n}{i} p^i (1-p)^{n-i} \binom{m}{k-i} p^{k-i} (1-p)^{m-k+i} \\
&= p^k (1-p)^{n+m-k} \sum_{i=0}^{n} \binom{n}{i}\binom{m}{k-i} \\
&= p^k (1-p)^{n+m-k} \binom{n+m}{k}
\end{aligned}$$

Therefore,

$$X+Y \sim \mathrm{Bin}(n+m, p)$$

□

Theorem 51. *Let*

$$X_i \sim \mathrm{Geo}(p)$$

be independent random variables, for $i \in \mathbb{N}$*. Then,*

$$\sum_{i=1}^{n} X_i \sim \mathrm{NB}(n, p)$$

6.4 Conditional Random Variables

Theorem 52. *The conditional probability mass function of two discrete random variables* X *and* Y *is*

$$\mathrm{P}(X=x|Y=y) = \frac{\mathrm{P}(X=x, Y=y)}{\mathrm{P}(Y=y)}$$

Theorem 53. *The conditional probability density function of two continuous random variables* X *and* Y *is*

$$\mathrm{f}_{X|Y}(x|y) = \frac{\mathrm{f}(x,y)}{\mathrm{f}_Y(y)}$$

and the cumulative distribution function is

$$\mathrm{F}_{X|Y}(a|y) = \int_{-\infty}^{a} \mathrm{f}_{X|Y}(x|y)\,\mathrm{d}x$$

Exercise 45.

$$\mathrm{f}(x,y) = \begin{cases} \frac{e^{-\frac{x}{y}}e^{-y}}{y} & ; \quad 0 < x < \infty, 0 < y < \infty \\ 0 & ; \quad \text{otherwise} \end{cases}$$

Find $\mathrm{P}(X > 1 | Y = y)$.

6.5 Properties of Expectation

Theorem 54. *If X and Y have a joint probability mass function* $\mathrm{P}(X = x, Y = y)$*, then*

$$\mathrm{E}\left[g(X,Y)\right] = \sum_{x=-\infty}^{\infty} \sum_{y=-\infty}^{\infty} g(x,y)\, \mathrm{P}(X = x, Y = y)$$

Theorem 55. *If X and Y have a joint probability density function* $\mathrm{f}(x,y)$*, then*

$$\mathrm{E}\left[g(X,Y)\right] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x,y)\, \mathrm{f}(x,y)$$

Exercise 46.
An accident occurs at a point X that is uniformly distributed on a road of length L. At the time of the accident, and ambulance is at a location Y that is also uniformly distributed on the same road.
Assuming that X and Y are independent, find the expected distance between the point of occurrence of the accident, and the position of the ambulance.

Definition 35 (Sample)**.** Let $X_1, \ldots, X_n$ be independent and identically distributed random variables having distribution function function F, and expected value μ. Such a sequence of random variables is said to constitute a sample from the distribution F.

Definition 36 (Sample mean)**.** Let $X_1, \ldots, X_n$ be a sample with F and μ.

$$\overline{X} = \sum_{i=1}^{n} \frac{X_i}{n}$$

is called the sample mean of the sample.

Theorem 56. *Let $X_1, \ldots, X_n$ be a sample with* F *and μ. Then*

$$\mathrm{E}\left[\overline{X}\right] = \mu$$

Proof.

$$\begin{aligned}
\mathrm{E}\left[\overline{X}\right] &= \mathrm{E}\left[\frac{\sum_{i=1}^{n} X_i}{n}\right] \\
&= \frac{1}{n}\,\mathrm{E}\left[\sum_{i=1}^{n} X_i\right] \\
&= \frac{1}{n}\sum_{i=1}^{n} \mathrm{E}[X_i] \\
&= \frac{1}{n} n\mu \\
&= \mu
\end{aligned}$$

□

Exercise 47.
Suppose that there are N different types of coupons, and each time one obtains a coupon, it is equally likely to be any one of the N types. Find the expected number of coupons that one needs to collect before obtaining a complete set.

Exercise 48.
A sequence of n 1s and m 0s is randomly permuted. Any consecutive string of 1s is said to constitute a run of 1s. Compute the mean number of such runs.

6.6 Properties of Variance

Definition 37 (Covariance)**.** Let X and Y be random variables. Then

$$\mathrm{Cov}(X, Y) = \mathrm{E}\left[(X - \mathrm{E}[X])\,(Y - \mathrm{E}[Y])\right]$$

is defined to be the covariance of X and Y.

Theorem 57.

$$\mathrm{Cov}(X, Y) = \mathrm{E}[XY] - \mathrm{E}[X]\,\mathrm{E}[Y]$$

Proof.

$$\begin{aligned}
\mathrm{Cov}(X, Y) &= \mathrm{E}\left[(X - \mathrm{E}[X])\,(Y - \mathrm{E}[Y])\right] \\
&= \mathrm{E}\left[XY - \mathrm{E}[X]Y - X\,\mathrm{E}[Y] + \mathrm{E}[X]\,\mathrm{E}[Y]\right] \\
&= \mathrm{E}[XY] - \mathrm{E}[X]\,\mathrm{E}[Y] - \mathrm{E}[X]\,\mathrm{E}[Y] + \mathrm{E}[X]\,\mathrm{E}[Y] \\
&= \mathrm{E}[XY] - \mathrm{E}[X]\,\mathrm{E}[Y]
\end{aligned}$$

□

Theorem 58.

$$\mathrm{V}(X+Y) = \mathrm{V}(X) + \mathrm{Cov}(X,Y) + \mathrm{V}(Y)$$

Proof.

$$\mathrm{V}(X) = \mathrm{E}\left[(X - \mathrm{E}[X])^2\right]$$

Therefore,

$$\begin{aligned}
\mathrm{V}(X+Y) &= \mathrm{E}\left[(X+Y-\mathrm{E}[X+Y])^2\right] \\
&= \mathrm{E}\left[(X+Y-\mathrm{E}[X]-\mathrm{E}[Y])^2\right] \\
&= \mathrm{E}\left[(X-\mathrm{E}[X])^2 + 2\,(X-\mathrm{E}[X])\,(Y-\mathrm{E}[Y]) + (Y-\mathrm{E}[Y])^2\right] \\
&= \mathrm{E}\left[(X-\mathrm{E}[X])^2\right] + 2\,\mathrm{E}\left[(X-\mathrm{E}[X])\,(Y-\mathrm{E}[Y])\right] + \mathrm{E}\left[(Y-\mathrm{E}[Y])^2\right] \\
&= \mathrm{V}(X) + \mathrm{Cov}(X,Y) + \mathrm{V}(Y)
\end{aligned}$$

□

Definition 38 (Pearson corellation coefficient)**.**

$$\rho(X,Y) = \frac{\mathrm{Cov}(X,Y)}{\sqrt{\mathrm{V}(X)\,\mathrm{V}(Y)}}$$

Theorem 59. *If X and Y are independent, then, for any function g and h,*

$$\mathrm{E}\left[g(X)h(Y)\right] = \mathrm{E}\left[g(X)\right]\mathrm{E}\left[h(Y)\right]$$

Theorem 60. *Let X and Y be jointly continuous with joint density function $\mathrm{f}\,x,y$. Then,*

$$\mathrm{E}\left[g(X)h(Y)\right] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x)h(y)\,\mathrm{f}_{X,Y}(x,y)\,\mathrm{d}x\,\mathrm{d}y$$

As X and Y are independent,

$$\begin{aligned}
\mathrm{E}\left[g(X)h(Y)\right] &= \int_{-\infty}^{\infty} g(x)h(y)\,\mathrm{f}_X(x)\,\mathrm{f}_Y(y)\,\mathrm{d}x\,\mathrm{d}y \\
&= \int_{-\infty}^{\infty} g(x)\,\mathrm{f}_X(x)\,\mathrm{d}x \int_{-\infty}^{\infty} h(y)\,\mathrm{f}_Y(y)\,\mathrm{d}y \\
&= \mathrm{E}\left[g(X)\right]\mathrm{E}\left[h(Y)\right]
\end{aligned}$$

Theorem 61. *If X and Y are independent, then*

$$\mathrm{Cov}(X, Y) = 0$$

However, the converse is not true.

Proof.

$$\mathrm{Cov}(X, Y) = \mathrm{E}[XY] - \mathrm{E}[X]\,\mathrm{E}[Y]$$

As X and Y are independent,

$$\begin{aligned}\mathrm{Cov}(X, Y) &= \mathrm{E}[X]\,\mathrm{E}[Y] - \mathrm{E}[X]\,\mathrm{E}[Y] \\ &= 0\end{aligned}$$

□

Definition 39 (Uncorrelated random variables)**.** X and Y are said to uncorrelated if and only if

$$\mathrm{Cov}(X, Y) = 0$$

Theorem 62.

$$\mathrm{Cov}(X, Y) = \mathrm{Cov}(Y, X)$$

Theorem 63.

$$\mathrm{Cov}(X, X) = \mathrm{V}(X)$$

Theorem 64.

$$\mathrm{Cov}(aX, Y) = \mathrm{V}(X)$$

Theorem 65.

$$\mathrm{Cov}\left(\sum_{i=1}^{n} X_i, \sum_{j=1}^{m} Y_j\right) = \sum_{i=1}^{n}\sum_{j=1}^{m} \mathrm{Cov}(X_i, Y_j)$$

Theorem 66.

$$\mathrm{V}\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} \mathrm{V}(X_i) + 2\sum_{i=1}^{n}\sum_{j=1}^{i} \mathrm{Cov}(X_i, X_j)$$

Proof.

$$
\begin{aligned}
\mathrm{V}\left(\sum_{i=1}^{n} X_i\right) &= \mathrm{Cov}(\sum_{i=1}^{n} X_i, \sum_{j=1}^{n} X_j) \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n} \mathrm{Cov}\left(X_i, X_j\right) \\
&= \sum_{i=1}^{n} \mathrm{V}(X_i) + \sum_{i=1}^{n}\sum_{j\neq i} \mathrm{Cov}(X_i, X_j) \\
&= \mathrm{V}\left(\sum_{i=1}^{n} X_i\right) \\
&= \sum_{i=1}^{n} \mathrm{V}(X_i) + 2\sum_{i=1}^{n}\sum_{j=1}^{i} \mathrm{Cov}(X_i, X_j)
\end{aligned}
$$

□

Definition 40 (Sample variance)**.** Let $X_1, \ldots, X_n$ be a sample with expected value μ and variation σ^2.

$$S^2 = \sum_{i=1}^{n} \frac{\left(X_i - \overline{X}\right)^2}{n-1}$$

is called the sample variance of the sample.

Exercise 49.
Let $X_1, \ldots, X_n$ be independent and identically distributed random variables having expected value μ, and variance σ^2.
Let

$$\overline{X} = \frac{\sum_{i=1}^{n} X_i}{n}$$

Find $V\left(\overline{X}\right)$ and $\mathrm{E}\left[S^2\right]$.

Exercise 50.
n people came to a winter party, carrying a coat each. While leaving the party, each person took a coat randomly. Let X be the number of people that left with their own coat. Find $\mathrm{V}(X)$.

6.7 Conditional Expectation

Definition 41 (Conditional expectation for discrete random variables)**.** Let X and Y be discrete random variables.
The conditional expectation of X, given that $Y = y$ is

$$\mathrm{E}[X|Y = y] = \sum_x \mathrm{P}(X = x|Y = y)$$

$$\mathrm{f}_{X|Y}(X|Y = y) = \frac{\mathrm{f}_{X,Y}(x, y)}{\mathrm{f}_Y(y)}$$

Definition 42 (Conditional expectation for continuous random variables)**.** Let X and Y be continuous random variables.
The conditional expectation of X, given that $Y = y$ is

$$\mathrm{f}_{X|Y}(X|Y = y) = \frac{\mathrm{f}_{X,Y}(x, y)}{\mathrm{f}_Y(y)}$$

Exercise 51.
If X and Y are independent binomial random variables with identical parameters n and p, calculate the conditional expected value of X, given that $X + Y = m$.

6.8 Central Limit Theorem

Theorem 67 (Central Limit Theorem)**.** *Let $X_1, \dots, X_n$ be a sequence of independent and identically distributed random variables, each having mean μ and variance σ^2. Then, as $n \to \infty$, the distribution of $\frac{X_1+\cdots+X_n+n\mu}{\sigma\sqrt{n}}$ tends to the standard normal distribution, i.e. for $a \in \mathbb{R}$,*

$$\lim_{n\to\infty} \left(\frac{X_1 + \cdots + X_n + n\mu}{\sigma\sqrt{n}} \leq a \right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{a} e^{-\frac{x^2}{2}} \,\mathrm{d}x$$

Exercise 52.
The number of students who enroll in a psychology course is a Poisson random variable with mean 100. The professor in charge of the course has decided that if the number of students enroll in the course if 120 or more, he will teach the course in two separate sections, whereas if it is less than 120, he will teach all of the students in a single section. What is the probability that the professor will have to teach two sections?

Chapter 7

Hypothesis Testing

7.1 Types of Errors

Definition 43 (Type I error)**.** An error in which a correct null hypothesis is rejected is called a type I error.

Definition 44 (Type II error)**.** An error in which an incorrect null hypothesis is accepted is called a type II error.

Definition 45 (Test statistic)**.** A function of a sample of observations which provides a basis for testing the validity of the null hypothesis is called a test statistic.

Definition 46 (Critical region)**.** The region such that the null hypothesis is rejected when a caluclated value of the test statistic lies within this region is called the critical region.

Definition 47 (Critical value)**.** The value which determines the boundary of the critical region is called the critical value.

Definition 48 (Significance level)**.** The size of the critical region, i.e. the probability of type I error is called the significance level. It is denoted by α.

Definition 49 (p-value)**.** The smallest significance value which allows the null hypothesis to be rejected is called the p-value.

Definition 50 (Sample variance)**.** Let $\overline{X}$ be the sample mean. The sample variance is defined to be

$$\begin{aligned} s^2 &= \frac{\sum_{i=1}^{n} \left(X_i - \overline{X}\right)^2}{n-1} \\ &= \frac{\sum_{i=1}^{n} {X_i}^2 - n\overline{X}^2}{n-1} \end{aligned}$$

Exercise 53.
A consumer group, concerned about the mean fat content of a certain grade of steakburger submits to an independent laboratory, a random sample of 12 steakburgers for analysis. The percentage of fat in each of the steakburgers is 21%, 18%, 19%, 16%, 18%, 24%, 22%, 19%, 24%, 14%, 18%, 15%. The manufacturer claims that the mean fat content of this grade of steakburger is less than 20%. Assuming that the percentage of fat content is normally distributed with standard deviation 3, test the hypothesis. Assume the significance level to be 5%.

Definition 51.

$$t = \frac{\overline{X} - \mu_0}{\frac{s}{\sqrt{n}}}$$

Exercise 54.
A consumer group, concerned about the mean fat content of a certain grade of steakburger submits to an independent laboratory, a random sample of 12 steakburgers for analysis. The percentage of fat in each of the steakburgers is 21%, 18%, 19%, 16%, 18%, 24%, 22%, 19%, 24%, 14%, 18%, 15%. The manufacturer claims that the mean fat content of this grade of steakburger is less than 20%. Assuming that the percentage of fat content is normally distributed with standard deviation 3, test the hypothesis. Assume the significance level to be 5%.

7.2 Confidence Interval

Definition 52 (Confidence interval)**.** An interval such that the parameter lies inside the interval with probability $1 - \alpha$ is called a confidence interval. Therefore,

$$\mathrm{P}\left(Z_{\frac{\alpha}{2}} \leq \frac{\overline{X} - \mu}{\frac{\sigma}{\sqrt{n}}} \leq X_{1-\frac{\alpha}{2}}\right) = 1 - \alpha$$

$$\therefore \mathrm{P}\left(\frac{\sigma}{\sqrt{n}} Z_{1-\frac{\alpha}{2}} \leq \overline{X} - \mu \leq \frac{\sigma}{\sqrt{n}} Z_{1-\frac{\alpha}{2}}\right) = 1 - \alpha$$

$$\therefore \mathrm{P}\left(\left|\overline{X} - \mu\right| \leq \frac{\sigma}{\sqrt{n}} Z_{1-\frac{\alpha}{2}}\right) = 1 - \alpha$$

Therefore, the confidence interval is $\overline{X} \pm Z_{1-\frac{\alpha}{2}} \frac{\sigma}{\sqrt{n}}$.
Similarly, for t, the confidence interval is $\overline{X} \pm t_{n-1,1-\frac{\alpha}{2}} \frac{s}{\sqrt{n}}$.

Chapter 8

Miscellaneous Exercises

8.1 probability concepts

Exercise 55.
In a certain class, the following groups were defined.

1. A: Hebrew speaking students
2. B: English speaking students
3. C: Yiddish speaking students

Use the union, intersection, and complement actions to describe the following groups.

1. Students who speak all three languages.
2. Students who speak exactly one language.
3. Students who speak at least one of the languages.
4. Students who do not speak Hebrew.
5. Students who speak exactly two languages.
6. Students who speak at least two languages.

Which of the following groups are mutually exclusive?

Exercise 56.
A series of ten coin flips is performed. The possible results of each flip are 'Head' or 'Tail'. Describe with words the complement of each of the following events.

1. 'Head' appeared at least 6 times.
2. 'Head' appeared no more than 4 times.
3. 'Head' never appeared.
4. In the first two flips, 'Head' appeared.
5. Number of 'Head's is greater than the number of 'Tail's.
6. Less than 8 'Tails' appeared.

Exercise 57.
Let A and B be events such that

$$\mathrm{P}(A) = \frac{3}{5}$$
$$\mathrm{P}(B) = \frac{1}{3}$$
$$\mathrm{P}(A \cap B) = \frac{1}{10}$$

Calculate

1. $\mathrm{P}(A \cup B)$
2. $\mathrm{P}\left(A^{\mathrm{c}} \cap B^{\mathrm{c}}\right)$
3. $\mathrm{P}\left(A^{\mathrm{c}} \cup B^{\mathrm{c}}\right)$
4. $\mathrm{P}\left(A \cap B^{\mathrm{c}}\right)$

Exercise 58.
A cafeteria offers a three-course meal consisting of an entree, a starch, and a dessert. The possible choices are given in the following table. A person is to choose one course from each category.

1. How many outcomes are in the sample space?
2. Let A be the event that ice cream is chosen. How many outcomes are in A?

Course	Choices
Entree	Chicken or roast beef
Starch	Pasta or rice or potatoes
Dessert	Ice cream or jello or apple pie or a peach

3. Let B be the event that chicken is chosen. How many outcomes are in B?
4. List all the outcomes in the event AB.
5. Let C be the event that rice is chosen. How many outcomes are in C?
6. List all the outcomes in the event ABC.

Exercise 59.
A customer visiting the suit department of a certian store will purchase a suit with probability 0.22, a shirt with probability 0.30, and a tie with probability 0.28. The customer wil purchase both a suit and a shirt with probability 0.11, both a suit an a tie with probability 0.14, and both a shirt and a tie with probability 0.10. A customer will purchase all three items with probability 0.06.
Find the probability that a customer purchases

1. None of these items.
2. Exactly one of these items?

Answer the questions both with the inclusion-exclusion identity, and a Venn diagram.

8.2 counting

Exercise 60.
If 4 Americans, 3 French people, and 3 British people are to be seated in a row, how many seating arrangements are possible when people of the same nationality must sit next to each other? Assume that people of the same nationality are not identical.

Exercise 61.
From 10 married couples, we want to select a group of 6 people that is not allowed to contain a married couple.

1. How many choices exist?

2. How many choices exist if the group must also consist of 3 men and 3 women?

Exercise 62.
If there are no restrictions on where the digits and letters are placed, how many many 8-character license plates, consisting of 5 letters and 3 digits are possible if no repetitions of letters or digits are allowed? What if the 3 digits must be consecutive?

Exercise 63.
An urn contains 20 balls: 8 black, 7 red, and 5 white. If 8 balls are randomly selected, calculate the following probabilities.

1. 6 black and 2 white balls were selected.

2. 3 black, 2 red, and 3 white balls were selected.

3. At least 1 white ball was selected.

Exercise 64.
In a state lottery, a player must choose 8 of the numbers from 1 to 40. The lottery commission then performs an experiment that selects 8 of these 40 numbers. Assuming that the choice of the lottery commission is equally likely to be any of the $\binom{40}{8}$ combinations, find the probabilities that a player has

1. all 8 of the numbers selected by the lottery commission.

2. 7 of the numbers selected by the lottery commission.

3. at least 6 of the numbers selected by the lottery commission.

Exercise 65.
10 married couples randomly select their seats around a round table. Find the following probabilities.

1. All couples sit next to each other.

2. Only 9 couples sit next to each other.

Exercise 66.
In a factory, each worker was randomly allocated a phone extension with 5 digits. The first digit in a manager's number is 0, and in a non-manager worker is different than 0. Find the probabilities of the following events for a manager, and for a non-manager worker.

1. E_1: The number has a digit which repeats.
2. E_2: The digit 3 appears in the number.
3. E_3: The number is divisible by 5.
4. E_4: The number is a palindrome.

8.3 independence

Exercise 67.
You ask your neighbour to water a sickly plant while you are on vacation.
Without water, it will die with probability 0.8.
With water, it will die with probability 0.15.
You are 90% certain that your neighbour will remember to water the plant.

1. What is the probability that the plant will be alive when you return?
2. If the plant is dead upon your return, what is the probability that your neighbour forgot to water it?

Exercise 68.
Six balls are to be randomly chosen from an urn containing 8 red, 10 green, and 12 blue balls.

1. What is the probability that at least one red ball is chosen?
2. Given that no red balls are chosen, what is the conditional probability that there are exactly 2 green balls among the 6 chosen balls?

Exercise 69.
A coin having probability 0.8 of landing on 'Heads' is flipped. Emily observes the result, and rushes off to tell Jesse. However, with probability 0.4, Emily will have forgotten the result by the time she reaches Jesse. If Emily has forgotten, then rather than admitting this to Jesse, she is equally likely to tell Jesse that the coin landed on 'Heads' or that it landed on 'Tails'. If she does remember, she tells Jesse the correct result.

1. What is the probability that Jesse is told that the coin landed on heads?
2. What is the probability that Jesse is told the correct result?
3. Given that Jesse is told that the coin landed on heads, what is the probability that it did in fact land on 'Heads'?

Exercise 70.
In the urn, there are 5 red and 4 white balls. You sequentially draw 3 balls randomly without replacement. What is the probability that all balls are white?

Exercise 71.
Let A and B be events having positive probability. State whether each of the following statements is necessarily true, necessarily false, or possibly true.

1. If A and B are mutually exclusive, then they are independent.
2. If A and B are independent, then they are mutually exclusive.
3. $\mathrm{P}(A) = \mathrm{P}(B) = 0.6$, and A and B are mutually exclusive.
4. $\mathrm{P}(A) = \mathrm{P}(B) = 0.6$, and A and B are independent.

Exercise 72.
Arrange the following from most likely to occur to least likely to occur.

1. A fair coin lands on 'Heads'.
2. Three independent trials, each of which is a success with probability 0.8, all result in successes.
3. Seven independent trials, each of which is a success with probability 0.9, all result in successes.

8.4 random variables

Exercise 73.
Suppose that a random variable X is equal to the number of hits obtained by a certain baseball player in his next 3 bats. If

$$\begin{aligned} \mathrm{P}(X = 1) &= 0.3 \\ \mathrm{P}(X = 2) &= 0.2 \\ \mathrm{P}(X = 0) &= 3\,\mathrm{P}(X = 3) \end{aligned}$$

find $\mathrm{E}[X]$.

Exercise 74.
Suppose that X takes on one of the values 0, 1, or 2. If for some constant c,

$$\mathrm{P}(X = i) = c\,\mathrm{P}(X = i - 1)$$

for $i = 1$ and $i = 2$, find $\mathrm{E}[X]$ in terms of c.

Exercise 75.
Suppose that

$$\mathrm{P}(X = 0) = 1 - \mathrm{P}(X = 1)$$

If

$$\mathrm{E}[X] = 3\,\mathrm{V}(X)$$

find $\mathrm{P}(X = 0)$.

Exercise 76.
There are two coins in a bin. When one of them is flipped, it lands on 'Heads' with probability 0.6, and when the other is flipped, it lands on 'Heads' with probability 0.3.
One of these coins is to be randomly chosen and then flipped. Without knowing which coin is chosen, you can bet any amount up to \$10, and you then either win that amount if the coin comes up 'Heads' or lose it if it comes up 'Tails'.
Suppose, however, that an insider is willing to sell you, for an amount C, the information as to which coin was selected. What is your expected payoff if you buy this information? Note that if you but it and then bet x, you will end up wither winning $x - C$ or $-x - C$. Also, for what values of C does it pay to purchase the information?

Exercise 77.
A philanthropist writes a positive number x on a piece of red paper, shows the paper to an impartial observer, and then turns it face down on a table. The observer then flips a fair coin. If it shows he writes the value $2x$ if it shows 'Heads', and the value $\frac{x}{2}$ if it shows 'Tails', on a piece of blue paper, which he then turns face down on the table. Without knowing either the value x or the result of the coin flip, you have to option of turning over either the red or the blue piece of paper. After doing so and observing the number written on that paper, you may elect to receive as a reward either that amount or the unknown amount written on the other piece of paper.
Suppose that you would like your expected reward to be large, argue that there is no reason to turn over the red paper first, because if you do so, then no matter what value you observe, it is always better to switch to the blue paper.

8.5 distributions

Exercise 78.
If X is a binomial random variable with expected value 6 and variance 2.4, find $\mathrm{P}(X = 5)$.

Exercise 79.
Each member of a seven judge panel independently makes a correct decision with probability 0.7. If the panel's decision is made by majority rule, what is the probability that the panel makes the correct decision? Given that four of the judges agreed, what is the probability that the panel made the correct decision?

Exercise 80.
A player bets on a number from 1 to 6. Three dice are then rolled, and if the number bet by the player appears i times, where $i = 1, 2, 3$, then the player wins i units. If the number bet by the player does not appear on any of the dice, then the player loses 1 unit. Is this game fair to the player?

Exercise 81.
Surveys indicated that 40% of the Zulu tribe members use the toothpaste 'Zebra'. A marketing company wants to interview a user of that toothpaste. Let X be the number of persons that the company would contact until a person who uses 'Zebra' will be found.

1. What is the distribution of X?

2. What is the expectation and variance of X?

3. What is the probability that the company will interview at most 3 persons until a 'Zebra' user is found?
4. Let Y be the number of clients that the company would contact until 2 persons who 'Zebra' will be found. What is the distribution of Y?
5. What is the expectation and variance of Y?

Exercise 82.
In the Israeli Parliament, the Knesset, 24 out of 120 MPs are women.

1. If you randomly choose chairpersons for the 12 parliamentary committees, what are the chances that exactly 3 of them are women, given that
   (a) Each MP can be a chairperson of any number of committees.
   (b) Each MP can be a chairperson of at most one committee.
2. Which method is better for promoting gender equality?

Exercise 83.
Emily likes hippopotamuses and walks daily with Jesse to the zoo. The walking time to the zoo is distributed uniformly between 15 and 20 minutes with intervals of 1 minute.

1. What is the chance that on a random day, Emily and Jesse would not walk more than 17 minutes?
2. What is the expectation and variance of the walking time to the zoo?
3. What is the probability that today and tomorrow the walking time would exceed 19 minutes?

8.6 probability density functions

Exercise 84.
On average, 5.2 hurricanes hit a certain region in a year. What is the probability that there will be 3 or fewer hurricanes hitting this year?

Exercise 85.
The number of eggs laid on a tree leaf by an insect of a certain type is a Poisson random variable with parameter λ. However, such a random variable can be observed only if it is positive, since if it is zero, then we cannot know that such an insect was on the leaf. Let Y denote the observed number of eggs, then

$$\mathrm{P}(Y = i) = \mathrm{P}(X = i | X > 0)$$

where X is a Poisson random variable with parameter λ. Find $\mathrm{E}[Y]$.

Exercise 86.
A professor with ADHD never ends his classes on time. Let T be the time in minutes that the class continues beyond its scheduled end. The probability distribution function of T is

$$\mathrm{f}(t) = \begin{cases} ct(5-t)^2 & ; \quad 0 \leq t \leq 5 \\ 0 & ; \quad \text{otherwise} \end{cases}$$

1. Find c.
2. Find the cumulative distribution function of T.
3. A student bets with her friends that in 12 out of 13 independent classes, the class will continue at least 2 minutes after its scheduled end. What are her chances to win the bet?

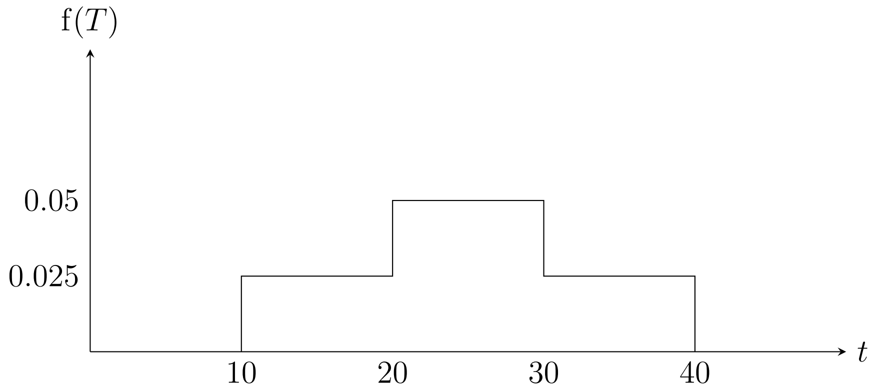
Exercise 87.
X, the percentage of correct answers of a student in a test is distributed according to

$$\mathrm{f}(t) = \begin{cases} ct(100-t) & ; \quad 0 \leq t \leq 100 \\ 0 & ; \quad \text{otherwise} \end{cases}$$

1. What is the value of c, given that $\mathrm{f}(t)$ is a legitimate probability distribution function?
2. Find $\mathrm{F}_X(t)$.
3. Calculate the probability that a student will fail, i.e. get less than 55.
4. Calculate the probability of failure in the test if
   (a) The student's grade is higher than 54.

(b) The student's grade is higher than 40.

Exercise 88.
The number of minutes of playing time of a certain high school basketball player in a randomly chosen game is a random variable whose probability density function is given.



Find the probability that the player plays

1. over 15 minutes.
2. between 20 and 35 minutes.
3. less than 30 minutes.
4. more than 36 minutes.

Exercise 89.
For some constant c, the random variable X has the following probability density function.

$$\mathrm{f}(x) = \begin{cases} ax + bx^2 & ; \quad 0 < x < 1 \\ 0 & ; \quad \text{otherwise} \end{cases}$$

If

$$\mathrm{E}[X] = 0.6$$

find

1. $\mathrm{P}(X < 12)$.

2. V(X).

Exercise 90.

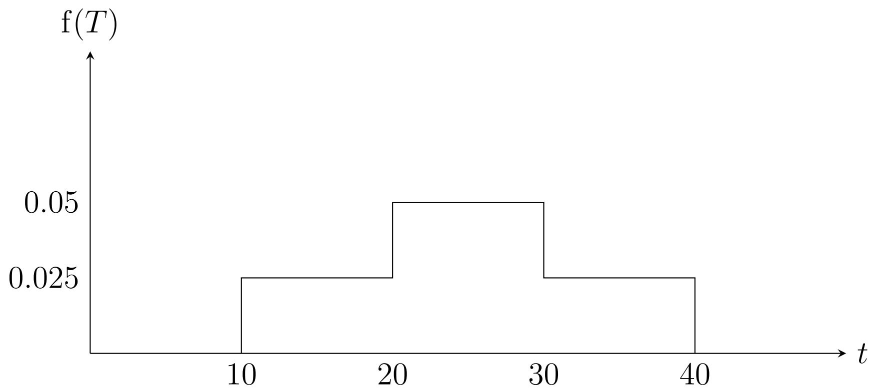
Your company must make a scaled bid for a construction project. If you succeed in winning the contract by having the lowest bid, then you plan to pay another firm $\$100,000$ to do the work. If you believe that the minimum bid of the other participating companies can be modelled as the value of a random variable that is uniformly distributed on $(70, 140)$, how much should you bid to maximize your profit?

Exercise 91.

At a certain bank, the amount of time that a customer spends being served by a teller is an exponential random variable with mean of 5 minutes. If there is a customer in service when you enter the bank, what is the probability that he will still be with the teller after an additional 4 minutes?

Exercise 92.

The number of minutes of playing time of a certain high school basketball player in a randomly chosen game is a random variable whose probability density function is given.



Find the probability that the player plays

1. over 15 minutes.
2. between 20 and 35 minutes.
3. less than 30 minutes.
4. more than 36 minutes.

Exercise 93.

For some constant c, the random variable X has probability density function

$$\mathrm{f}(x) = \begin{cases} cx^4 & ; \quad 0 < x < 2 \\ 0 & ; \quad \text{otherwise} \end{cases}$$

Find $\mathrm{E}[X]$ and $\mathrm{V}(X)$.

Exercise 94.

Your company must make a scaled bid for a construction project. If you succeed in winning the contract by having the lowest bid, then you plan to pay another firm \$100,000 to do the work. If you believe that the minimum bid of the other participating companies can be modelled as the value of a random variable that is uniformly distributed on $(70, 140)$, how much should you bid to maximize your profit?

Exercise 95.

At a certain bank, the amount of time that a customer spends being served by a teller is an exponential random variable with mean of 5 minutes. If there is a customer in service when you enter the bank, what is the probability that he will still be with the teller after an additional 4 minutes?

Exercise 96.

Emily is leaving work in a uniformly distributed time between 7 and 9, i.e.

$$X \sim \mathrm{U}(7, 9)$$

The time it takes her to commute home is

$$Y = 1 + \frac{1}{X}$$

Find the density and distribution of Y.

Exercise 97.

A randomly chosen IQ test taker obtains a score that is approximately a normal random variable with mean 100 and standard deviation 15. Find the probability that the score of such a person is

1. above 125.

2. between 90 and 110.

Exercise 98.
The life of a certain type of automobile tyre is normally distributed with mean $34,000$ miles and standard deviation $4,000$ miles.

1. What is the probability that such a tyre lasts over $40,000$ miles?
2. What is the probability that such a tyre lasts between $30,000$ and $35,000$ miles?
3. Given that such a tyre has survived $30,000$ miles, what is the conditional probability that the tyre survives another $10,000$ miles?

Exercise 99.
The annual rainfall in Cleveland, Ohio, is approximately a normal random variable with mean 40.2 inches and standard deviation 8.4 inches. Let A_i be the event that the rainfall in the next ith year exceeds 44 inches, and fume that all A_i are independent. Find the probability that

1. next year's rainfall will exceed 44 inches?
2. the yearly rainfall in exactly 3 of the next 7 years will exceed 44 inches?

Appendix A

Answers to selected exercises

Solution 1.

1. Let A be the event that he would go to pub A, and let B be the event that he goes to pub B. Therefore,

$$S = \{A \cap B^{\mathrm{C}}, A^{\mathrm{C}} \cap B, A \cap B, A^{\mathrm{C}} \cap B^{\mathrm{C}}\}$$

2. The probability that he would go to both pubs is

$$\begin{aligned} \mathrm{P}(A \cap B) &= \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cup B) \\ &= 0.5 + 0.4 - 0.8 \\ &= 0.1 \end{aligned}$$

3. The probability that he would go to exactly one pub is

$$\begin{aligned} \mathrm{P}\left((A \cup B) \setminus (A \cap B)\right) &= \mathrm{P}(A \cup B) - \mathrm{P}(A \cap B) \\ &= 0.8 - 0.1 \\ &= 0.7 \end{aligned}$$

Solution 2.

1. The required probability is

$$\begin{aligned} \mathrm{P}(1 \text{ red}) + \mathrm{P}(2 \text{ red}) + \mathrm{P}(3 \text{ red}) &= 1 - \mathrm{P}(0 \text{ red}) \\ &= 1 - 0.4 \\ &= 0.6 \end{aligned}$$

2. The required probability is

$$\begin{aligned} \mathrm{P}(1 \text{ green}) + \mathrm{P}(2 \text{ green}) + \mathrm{P}(3 \text{ green}) &= \mathrm{P}(0 \text{ red}) + \mathrm{P}(1 \text{ red}) + \mathrm{P}(2 \text{ red}) \\ &= 0.4 + 0.1 + 0.2 \\ &= 0.7 \end{aligned}$$

3. The required probability is

$$\begin{aligned} \mathrm{P}(1 \text{ red}) + \mathrm{P}(3 \text{ red}) &= 1 - \mathrm{P}(0 \text{ red}) - \mathrm{P}(2 \text{ red}) \\ &= 1 - 0.4 - 0.2 \\ &= 0.4 \end{aligned}$$

Solution 3.

$$|\Omega| = \binom{52}{5}$$

There are 12 cards with numbers higher than 10. Therefore,

$$|A| = \binom{12}{5}$$

Therefore,

$$\begin{aligned} \mathrm{P}(A) &= \frac{|A|}{|\Omega|} \\ &= \frac{\binom{12}{5}}{\binom{52}{5}} \end{aligned}$$

There are 13 heart cards. Therefore,

$$|B| = \binom{13}{5}$$

Therefore,

$$\begin{aligned} \mathrm{P}(B) &= \frac{|B|}{|\Omega|} \\ &= \frac{\binom{13}{5}}{\binom{52}{5}} \end{aligned}$$

The number of ways of selecting 5 different numbers out of 13 is $\binom{13}{5}$. For each of the selected number, there are 4 cards, of which exactly one has to be selected. Therefore,

$$|C| = \binom{13}{5} 4^5$$

Therefore,

$$\mathrm{P}(C) = \frac{\binom{13}{5} 4^5}{\binom{52}{5}}$$

There are 9 sequences of consecutive numbers. Each of the numbers have 4 corresponding cards each. Therefore,

$$|D| = 9 \cdot 4^5$$

Therefore,

$$\mathrm{P}(D) = \frac{9 \cdot 4^5}{\binom{52}{3}}$$

Solution 4.
Every time the die is rolled, there as 6 possible outcomes. Therefore,

$$|\Omega| = 6^3$$

For the sum of three numbers to be even, exactly 0 or 2 of them must be odd.
There is 1 combination for all three numbers to be even. Each of these even numbers has 3 options. Therefore, the total number of combinations following the restriction in 3^3.
There are 3 combinations for exactly 2 numbers to be odd. Each of the odd numbers has 3 option, and the even number has 3 options. Therefore, the total number of combinations following the restriction in 3^3.
Therefore,

$$|A| = 4 \cdot 3^3$$

Therefore,

$$\mathrm{P}(A) = \frac{4 \cdot 3^3}{6^3}$$

Solution 5.
There are $\binom{365}{1}$ options for each person's birthday. Therefore,

$$|\Omega| = 365^n$$

Let A be the event that everyone has distinct birthdays.
If $n > 365$, at least two persons must share a birthday.
If $n \leq 365$,

$$|A| = {}^{365}\mathrm{P}_n$$

Therefore,

$$\begin{aligned}\mathrm{P}(A) &= \frac{{}^{365}\mathrm{P}_n}{365^n} \\ &= \frac{\frac{365!}{(365-n)!}}{365^n}\end{aligned}$$

Therefore, the probability of at least two persons sharing a birthday is

$$\begin{aligned} \mathrm{P}\left(\overline{A}\right) &= 1 - \mathrm{P}(A) \\ &= 1 - \frac{\frac{365!}{(365-n)!}}{365^n} \end{aligned}$$

Solution 6.

1. The total possible combinations are ${}^{10}\mathrm{P}_3$.

2. If neither Emily nor Jesse are office bearers, there are ${}^{8}\mathrm{P}_3$ possible combinations.
   If one of Emily and Jesse is an office bearer, there are three possible posts for the selected person. The number of combinations for the rest of the posts are ${}^{8}\mathrm{P}_2$. Therefore, the total number of combinations are ${}^{8}\mathrm{P}_3 + 3{}^{8}\mathrm{P}_2$.

3. If both Frodo and Smeagol are chosen, the number of combinations are $3 \cdot 2 \cdot {}^{8}\mathrm{P}_1$. If neither Frodo nor Smeagol are chosen, the number of combinations are ${}^{8}\mathrm{P}_3$. Therefore, the total number of combinations are $3 \cdot 2 \cdot {}^{8}\mathrm{C}_1 + {}^{8}\mathrm{P}_3$.

4. There are three possible posts for Kate. Therefore, the total number of combinations are $3 \cdot {}^{9}\mathrm{P}_2$.

5. If Bilbo is the president, the number of combinations are ${}^{9}\mathrm{P}_2$. If Bilbo is not the president, the number of combinations are ${}^{9}\mathrm{P}_3$. Therefore, the total number of combinations are ${}^{9}\mathrm{P}_2 + {}^{9}\mathrm{P}_3$.

Solution 7.

1.

$$\begin{aligned} |S| &= n^a \\ &= n^n \end{aligned}$$

   If all cells are to be non-empty, the number of combinations are $n!$. Therefore, the probability is $\frac{n!}{n^n}$.

2.

$$\begin{aligned} |S| &= n^a \\ &= n^{n+1} \end{aligned}$$

   The number of combinations to select 2 balls is $\binom{n+1}{2}$. Let these two balls be glued together and be treated as one.
   The number of arrangements of these n balls are $n!$.
   Therefore, the total number of combinations are $\binom{n+1}{2}n!$.
   Therefore, the probability is $\frac{\binom{n+1}{2}}{n!}$.

Solution 8.

$$\begin{aligned} \mathrm{P}(A|B) &= \frac{\mathrm{P}(A \cap B)}{\mathrm{P}(B)} \\ &= \frac{\mathrm{P}\left(\{2,4,6\} \cap \{4,5,6\}\right)}{\mathrm{P}\left(\{4,5,6\}\right)} \\ &= \frac{\mathrm{P}\left(\{4,6\}\right)}{\mathrm{P}\left(\{4,5,6\}\right)} \\ &= \frac{\frac{1}{3}}{\frac{1}{2}} \\ &= \frac{2}{3} \end{aligned}$$

Solution 9.

$$\Omega = \{(H,H),(H,T),(T,H),(T,T)\}$$

Let A be the event of getting two 'Heads'. Therefore,

$$A = \{(H,H)\}$$

Let B be the event that the first flip results in 'Heads'. Therefore,

$$B = \{(H,T),(H,H)\}$$

Therefore,

$$\begin{aligned} \mathrm{P}(A|B) &= \frac{\mathrm{P}(A \cap B)}{\mathrm{P}(B)} \\ &= \frac{\frac{1}{4}}{\frac{1}{2}} \\ &= \frac{1}{2} \end{aligned}$$

Solution 10.

$$\Omega = \{(H,H),(H,T),(T,H),(T,T)\}$$

Let A be the event of getting two 'Heads'. Therefore,

$$A = \{(H,H)\}$$

Let B be the event that at least one flip results in 'Heads'. Therefore,

$$B = \{(H,T),(T,H),(H,H)\}$$

Therefore,

$$\begin{aligned} \mathrm{P}(A|B) &= \frac{\mathrm{P}(A \cap B)}{\mathrm{P}(B)} \\ &= \frac{\frac{1}{4}}{\frac{3}{4}} \\ &= \frac{1}{3} \end{aligned}$$

Solution 11.

Let A_1 be the event that A♠ is in any one of the stacks. Therefore,

$$\mathrm{P}(A_1) = 1$$

Let A_2 be the event that A♠ and A♡ are in different stacks. Therefore,

$$\begin{aligned} \mathrm{P}(A_2|A_1) &= \frac{\mathrm{P}(A_1 \cap A_2)}{\mathrm{P}(A_1)} \\ &= \frac{39}{51} \end{aligned}$$

Let A_3 be the event that A♠, A♡, and A♢ are in different stacks. Therefore,

$$\begin{aligned} \mathrm{P}(A_3|A_1 \cap A_2) &= \frac{\mathrm{P}(A_1 \cap A_2 \cap A_3)}{\mathrm{P}(A_1 \cap A_2)} \\ &= \frac{26}{50} \end{aligned}$$

Let A_4 be the event that A♠, A♡, A♢, and A♣ are in different stacks. Therefore,

$$\begin{aligned} \mathrm{P}(A_4|A_1 \cap A_2 \cap A_3) &= \frac{\mathrm{P}(A_1 \cap A_2 \cap A_3 \cap A_4)}{\mathrm{P}(A_1 \cap A_2 \cap A_3)} \\ &= \frac{13}{49} \end{aligned}$$

Therefore,

$$\begin{aligned} \mathrm{P}(A_1 \cap A_2 \cap A_3 \cap A_4) &= \mathrm{P}(A_1)\,\mathrm{P}(A_2|A_1)\,\mathrm{P}(A_3|A_1 \cap A_2)\,\mathrm{P}(A_4|A_1 \cap A_2 \cap A_3) \\ &= (1)\left(\frac{39}{51}\right)\left(\frac{26}{50}\right)\left(\frac{13}{49}\right) \end{aligned}$$

Solution 12.

$$|\Omega| = \binom{52}{13}\binom{39}{13}\binom{26}{13}\binom{13}{13}$$

Let A be the event that each stack has exactly one ace. Therefore, each stack has one ace, and 12 non-ace cards. Therefore,

$$|A| = \left(\binom{4}{1}\binom{48}{12}\right)\left(\binom{3}{1}\binom{36}{12}\right)\left(\binom{2}{1}\binom{24}{12}\right)\left(\binom{1}{1}\binom{12}{12}\right)$$

Therefore,

$$\begin{aligned} \mathrm{P}(A) &= \frac{|A|}{|\Omega|} \\ &= \frac{\left(\binom{4}{1}\binom{48}{12}\right)\left(\binom{3}{1}\binom{36}{12}\right)\left(\binom{2}{1}\binom{24}{12}\right)\left(\binom{1}{1}\binom{12}{12}\right)}{\binom{52}{13}\binom{39}{13}\binom{26}{13}\binom{13}{13}} \end{aligned}$$

Solution 13.

Let A_1, A_2, A_3 be the events that selected chocolate bar is made of milk, dark, white chocolate, respectively.
Let B be the event that the selected chocolate bar is defective.
Therefore,

$$\begin{aligned} \mathrm{P}(B) &= \sum_{i=1}^{3} \mathrm{P}(A_i)\,\mathrm{P}(B|A_1) \\ &= ((0.5)(0.01)) + ((0.3)(0.02)) + ((0.2)(0.005)) \\ &= 0.005 + 0.006 + 0.001 \\ &= 0.012 \end{aligned}$$

Solution 14.

Let G be the event that the suspect is guilty.
Let B be the event that the suspect is bald.
Therefore, the event that the suspect is both bald and guilty, is $G|C$.
Therefore,

$$\begin{aligned} \mathrm{P}(G|B) &= \frac{\mathrm{P}(G \cap B)}{\mathrm{P}(B)} \\ &= \frac{0.6}{(1)(0.6) + (0.4)(0.2)} \end{aligned}$$

Solution 15.

Let E_{Emily} be the event of Emily getting the required number of $\heartsuit$s.
Let E_{Jesse} be the event of Jesse getting the required number of $\heartsuit$s.
Let E_{Frodo} be the event of Frodo getting the required number of $\heartsuit$s.
Let E_{Smeagol} be the event of Smeagol getting the required number of $\heartsuit$s.
Therefore,

$$\begin{aligned}
\mathrm{P}(E_{\text{Emily}}) &= \frac{{}^{13}\mathrm{C}_3\,{}^{39}\mathrm{C}_{10}}{{}^{52}\mathrm{C}_{13}} \\
\mathrm{P}(E_{\text{Jesse}}|E_{\text{Emily}}) &= \frac{{}^{10}\mathrm{C}_4\,{}^{29}\mathrm{C}_9}{{}^{39}\mathrm{C}_{13}} \\
\mathrm{P}(E_{\text{Frodo}}|E_{\text{Emily}} \cap E_{\text{Jesse}}) &= \frac{{}^{6}\mathrm{C}_2\,{}^{20}\mathrm{C}_{11}}{{}^{26}\mathrm{C}_{13}} \\
\mathrm{P}(E_{\text{Smeagol}}|E_{\text{Emily}} \cap E_{\text{Jesse}} \cap E_{\text{Frodo}}) &= \frac{{}^{4}\mathrm{C}_4\,{}^{9}\mathrm{C}_9}{{}^{13}\mathrm{C}_{13}}
\end{aligned}$$

Therefore,

$$\begin{aligned}
\mathrm{P}(E_{\text{Emily}} \cap E_{\text{Jesse}} \cap E_{\text{Frodo}} \cap E_{\text{Smeagol}}) = {}& \mathrm{P}(E_{\text{Emily}}) \\
&\times \mathrm{P}(E_{\text{Jesse}}|E_{\text{Emily}}) \\
&\times \mathrm{P}(E_{\text{Frodo}}|E_{\text{Emily}} \cap E_{\text{Jesse}}) \\
&\times \mathrm{P}(E_{\text{Smeagol}}|E_{\text{Emily}} \cap E_{\text{Jesse}} \cap E_{\text{Frodo}}) \\
={}& \frac{{}^{13}\mathrm{C}_3\,{}^{39}\mathrm{C}_{10}}{{}^{52}\mathrm{C}_{13}} \frac{{}^{10}\mathrm{C}_4\,{}^{29}\mathrm{C}_9}{{}^{39}\mathrm{C}_{13}} \frac{{}^{6}\mathrm{C}_2\,{}^{20}\mathrm{C}_{11}}{{}^{26}\mathrm{C}_{13}} \frac{{}^{4}\mathrm{C}_4\,{}^{9}\mathrm{C}_9}{{}^{13}\mathrm{C}_{13}}
\end{aligned}$$

Solution 16.

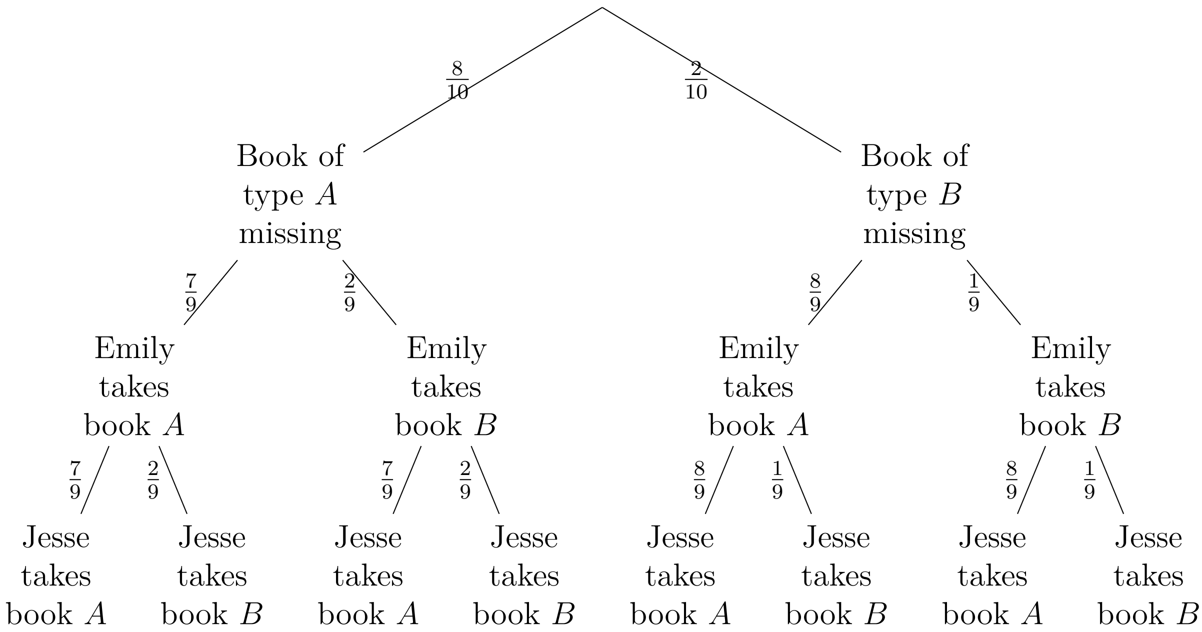
$$\begin{aligned}
\mathrm{P}(A) &= \frac{5}{36} \\
\mathrm{P}(B) &= \frac{1}{6} \\
\mathrm{P}(C) &= \frac{6}{36} \\
\mathrm{P}(A \cap B) &= \frac{1}{36} \\
\mathrm{P}(B \cap C) &= \frac{1}{36} \\
\mathrm{P}(A \cap C) &= 0
\end{aligned}$$

Therefore,

$$\begin{aligned}
\mathrm{P}(A \cap B) &\neq \mathrm{P}(A)\,\mathrm{P}(B) \\
\mathrm{P}(B \cap C) &= \mathrm{P}(B)\,\mathrm{P}(C) \\
\mathrm{P}(C \cap A) &\neq \mathrm{P}(C)\,\mathrm{P}(A)
\end{aligned}$$

Therefore, only B and C are independent.

Solution 17.



$$\begin{aligned}
\mathrm{P}(E) &= \left(\frac{8}{10}\right)\left(\frac{7}{9}\right) + \left(\frac{2}{10}\right)\left(\frac{8}{9}\right) \\
&= \frac{8}{10} \\
\mathrm{P}(F) &= \left(\frac{2}{10}\right)\left(\frac{8}{9}\right)\left(\frac{8}{9}\right) + \left(\frac{2}{10}\right)\left(\frac{1}{9}\right)\left(\frac{8}{9}\right) + \left(\frac{8}{10}\right)\left(\frac{7}{9}\right)\left(\frac{7}{9}\right) + \left(\frac{8}{10}\right)\left(\frac{2}{9}\right)\left(\frac{7}{9}\right) \\
&= \frac{8}{10}
\end{aligned}$$

Therefore,

$$\begin{aligned}
\mathrm{P}(E \cap F) &= \left(\frac{2}{10}\right)\left(\frac{8}{9}\right)\left(\frac{8}{9}\right) + \left(\frac{8}{10}\right)\left(\frac{7}{9}\right)\left(\frac{7}{9}\right) \\
&\neq \mathrm{P}(E)\,\mathrm{P}(F)
\end{aligned}$$

¡++¿ Therefore, the events are not independent.

Solution 18.

Let X be the largest number selected.
Therefore, X is a random variable which has a value from $\{3, \ldots, 20\}$.
Let the value of the highest valued ball be i. Therefore, the number of ways to select the

remaining two balls is ${}^{i-1}\mathrm{C}_2$.
Therefore, the probability of the value of the highest valued ball being i is

$$\mathrm{P}(X = i) = \frac{{}^{i-1}\mathrm{C}_2}{{}^{20}\mathrm{C}_3}$$

Therefore,

$$\begin{aligned}\mathrm{P}(X \geq 17) &= \mathrm{P}(X = 17) + \mathrm{P}(X = 18) + \mathrm{P}(X = 19) + \mathrm{P}(X = 20)\\ &= \frac{{}^{16}\mathrm{C}_2 + {}^{17}\mathrm{C}_2 + {}^{18}\mathrm{C}_2 + {}^{19}\mathrm{C}_2}{{}^{20}\mathrm{C}_3}\end{aligned}$$

Solution 19.

1.

$$\begin{aligned}\sum_{i=0}^{\infty} \mathrm{P}(X = i) &= 1\\ \therefore \sum_{i=0}^{\infty} \frac{c\lambda^i}{i!} &= 1\\ \therefore c\sum_{i=0}^{\infty} \frac{\lambda^i}{i!} &= 1\\ \therefore ce^{\lambda} &= 1\\ \therefore c &= e^{-\lambda}\end{aligned}$$

Therefore,

$$\mathrm{P}(X = i) = \frac{e^{-\lambda}\lambda^i}{i!}$$

Therefore,

$$\begin{aligned}\mathrm{P}(X = 0) &= \frac{e^{-\lambda}\lambda^0}{0!}\\ &= e^{-\lambda}\end{aligned}$$

2.

$$\begin{aligned}\sum_{i=0}^{\infty} \mathrm{P}(X = i) &= 1\\ \therefore \sum_{i=0}^{\infty} \frac{c\lambda^i}{i!} &= 1\\ \therefore c\sum_{i=0}^{\infty} \frac{\lambda^i}{i!} &= 1\\ \therefore ce^{\lambda} &= 1\\ \therefore c &= e^{-\lambda}\end{aligned}$$

Therefore,

$$P(X = i) = \frac{e^{-\lambda}\lambda^i}{i!}$$

Therefore,

$$\begin{aligned} P(X > 2) &= 1 - P(X \leq 2) \\ &= 1 - P(X = 2) - P(X = 1) - P(X = 0) \\ &= 1 - \frac{e^{-\lambda}\lambda^2}{2!} - \frac{e^{-\lambda}\lambda^1}{1!} - \frac{e^{-\lambda}\lambda^0}{0!} \end{aligned}$$

Solution 20.

$$\begin{aligned} E[x] &= \sum_{x=1}^{6} \frac{1}{6}x \\ &= 3.5 \end{aligned}$$

Solution 21.

$$\begin{aligned} P(X = 36) &= \frac{36}{120} \\ P(X = 40) &= \frac{40}{120} \\ P(X = 44) &= \frac{44}{120} \end{aligned}$$

Therefore,

$$\begin{aligned} E[X] &= 36\left(\frac{36}{120}\right) + 40\left(\frac{40}{120}\right) + 44\left(\frac{44}{120}\right) \\ &= 40.2667 \end{aligned}$$

Solution 22.
Let

$$Y = X^2$$

Therefore,

$$\begin{aligned} P(Y = 0) &= P(X = 0) \\ &= 0.5 \\ P(Y = 1) &= P(X = -1) + P(X = 1) \\ &= 0.5 \end{aligned}$$

Therefore,

$$\begin{aligned} \mathrm{E}[Y] &= \mathrm{E}\left[X^2\right] \\ &= (0)(0.5) + (1)(0.5) \\ &= 0.5 \end{aligned}$$

Solution 23.

Let the number of units sold by X.
Let the number of units stocked by s.
Therefore, the total profit is

$$\pi(s) = \begin{cases} bX - (s-X)l & ; \quad X \leq s \\ bs & ; \quad X > s \end{cases}$$

Therefore,

$$\begin{aligned} \mathrm{E}\left[\pi(s)\right] &= \sum_{i=0}^{s} \big(bi - (s-i)l\big) \, \mathrm{P}(X=i) + \sum_{i=s+1}^{\infty} sb \, \mathrm{P}(X=i) \\ &= (b+l) \sum_{i=0}^{s} i \, \mathrm{P}(X=i) - sl \sum_{i=0}^{s} \mathrm{P}(X=i) + sb \left(1 - \sum_{i=0}^{s} \mathrm{P}(X=i)\right) \\ &= (b+l) \sum_{i=0}^{s} i \, \mathrm{P}(X=i) - (b+l)s \sum_{i=0}^{s} \mathrm{P}(X=i) + sb \\ &= sb + (b+l) \sum_{i=0}^{s} (i-s) \, \mathrm{P}(X=i) \end{aligned}$$

Therefore,

$$\begin{aligned} \mathrm{E}\left[\pi(s+1)\right] &= b(s+1) + (b+l) \sum_{i=0}^{s+1} (i-s-1) \, \mathrm{P}(X=i) \\ &= b(s+1) + (b+l) \sum_{i=0}^{s} (i-s-1) \, \mathrm{P}(X=i) \end{aligned}$$

Therefore,

$$\mathrm{E}\left[\pi(s+1)\right] - \mathrm{E}\left[\pi(s)\right] = b - (b+l) \sum_{i=0}^{s} \mathrm{P}(X=i)$$

Therefore, increasing the stock is profitable when

$$\sum_{i=0}^{s} \mathrm{P}(X=i) < \frac{b}{b+l}$$

Solution 24.

$$\begin{aligned}\mathrm{E}[X] &= \sum_{x=1}^{6} \frac{1}{6}x \\ &= \frac{7}{2} \\ \mathrm{E}\left[X^2\right] &= \sum_{x=1}^{6} \frac{1}{6}x^2 \\ &= \frac{91}{6}\end{aligned}$$

Therefore,

$$\begin{aligned}\mathrm{V}(X) &= \mathrm{E}\left[X^2\right] - \mathrm{E}[X]^2 \\ &= \frac{91}{6} - \frac{49}{4} \\ &= \frac{35}{12}\end{aligned}$$

Solution 25.

1.

$$\begin{aligned}S &= \{(30+30), (30-20), (-20-20)\} \\ &= \{60, 10, -40\}\end{aligned}$$

2.

$$\mathrm{P}(X = x) = \begin{cases} p^2 & ; \quad x = 60 \\ 2p(1-p) & ; \quad x = 10 \\ (1-p)^2 & ; \quad x = -40 \end{cases}$$

3.

$$\mathrm{F}_(X) = \begin{cases} 0 & ; \quad x < -40 \\ (1-p)^2 & ; \quad -40 \leq x < 10 \\ (1-p)^2 + 2p(1-p) & ; \quad 10 \leq x < 60 \\ (1-p)^2 + 2p(1-p) + p^2 & ; \quad x \leq 60 \end{cases}$$

4.

$$\begin{aligned}\mathrm{E}[X] &= (-40)(1-p)^2 + (10)\left(2p(1-p)\right) + (60)\left(p^2\right) \\ &= 100p - 40\end{aligned}$$

5. The game should be played as long as the expected value of X is non negative. Therefore,

$$\begin{aligned} \mathrm{E}[X] &\geq 0 \\ \therefore 100p - 40 &\geq 0 \\ \therefore p &\geq 0.4 \end{aligned}$$

6.

$$\begin{aligned} \mathrm{E}\left[x^2\right] &= (-40)^2(1-p)^2 + (10)^2 2p(1-p) + (60)^2 p^2 \\ &= 5000p^2 - 3000p + 1600 \\ \mathrm{E}[x]^2 &= (100p - 40)^2 \\ &= 10000p^2 - 8000p + 1600 \end{aligned}$$

Therefore,

$$\begin{aligned} \mathrm{V}(X) &= \mathrm{E}\left[X^2\right] - \mathrm{E}[X]^2 \\ &= 5000p(1-p) \end{aligned}$$

7.

$$\begin{aligned} \sigma(X) &= \sqrt{\mathrm{V}(X)} \\ &= \sqrt{5000p(1-p)} \end{aligned}$$

Solution 26.

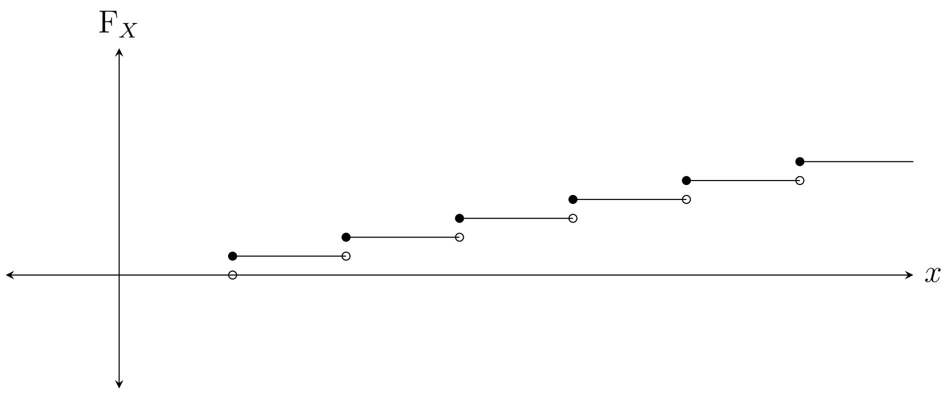


Figure A.1: Cumulative Distribution Function for a Die Roll

Solution 27.

1. As $\mathrm{f}(x)$ is a probability density function,

$$\begin{aligned} 1 &= \int\limits_{-\infty}^{\infty} \mathrm{f}(x)\,\mathrm{d}x \\ &= \int\limits_{0}^{2} c\left(4x^2 - 2x^2\right)\mathrm{d}x \\ &= \frac{8c}{3} \end{aligned}$$

Therefore,

$$c = \frac{3}{8}$$

2.

$$\begin{aligned} \mathrm{P}(X > 1) &= \int\limits_{1}^{\infty} \mathrm{f}(x)\,\mathrm{d}x \\ &= \int\limits_{1}^{2} \frac{3}{8}\left(4x - 2x^2\right)\mathrm{d}x \\ &= \frac{1}{2} \end{aligned}$$

Solution 28.

$$\begin{aligned} 1 &= \int\limits_{-\infty}^{\infty} \mathrm{f}(x)\,\mathrm{d}x \\ &= \lambda \int\limits_{0}^{\infty} e^{-\frac{x}{100}}\,\mathrm{d}x \\ &= 100\lambda \end{aligned}$$

Therefore,

$$\lambda = \frac{1}{100}$$

Therefore,

$$\begin{aligned} \mathrm{P}(50 < x < 150) &= \int\limits_{50}^{150} \lambda e^{-\frac{x}{100}} \, \mathrm{d}x \\ &= \int\limits_{50}^{150} \frac{1}{100} e^{-\frac{x}{100}} \, \mathrm{d}x \\ &\approx 0.384 \end{aligned}$$

Solution 29.

$$\begin{aligned} \mathrm{E}[X] &= \int\limits_{-\infty}^{\infty} x \, \mathrm{f}(x) \, \mathrm{d}x \\ &= \int\limits_{0}^{1} 2x \, \mathrm{d}x \\ &= \frac{2}{3} \end{aligned}$$

Solution 30.
Let

$$Y = e^X$$

Therefore,

$$\begin{aligned} \mathrm{F}_Y(y) &= \mathrm{P}(Y \leq y) \\ &= \mathrm{P}\left(e^X \leq y\right) \\ &= \mathrm{P}(X \leq \ln y) \\ &= \int\limits_{-\infty}^{\ln y} \mathrm{f}_X(x) \, \mathrm{d}x \\ &= \int\limits_{0}^{\ln y} \mathrm{d}x \\ &= \ln y \end{aligned}$$

Therefore, differentiating,

$$\begin{aligned} \mathrm{f}_Y(y) &= \frac{\mathrm{d}\mathrm{F}_Y(y)}{\mathrm{d}y} \\ &= \frac{\mathrm{d}\ln y}{\mathrm{d}y} \\ &= \frac{1}{y} \end{aligned}$$

Therefore,

$$\begin{aligned} \mathrm{E}\left[e^X\right] &= \mathrm{E}[Y] \\ &= \int\limits_{-\infty}^{\infty} y\,\mathrm{f}_Y(y)\,\mathrm{d}y \\ &= \int\limits_{1}^{e} y\frac{1}{y}\,\mathrm{d}y \\ &= e-1 \end{aligned}$$

Solution 31.
Let X be the number of times 6 appears.
Therefore,

$$X \sim \mathrm{Bin}\left(5, \frac{1}{6}\right)$$

Therefore,

$$\begin{aligned} \mathrm{P}(X=i) &= {}^{n}\mathrm{C}_{i} p^{i}(1-p)^{n-i} \\ \therefore \mathrm{P}(X=3) &= {}^{5}\mathrm{C}_{3}\left(\frac{1}{6}\right)^{3}\left(\frac{5}{6}\right)^{3} \end{aligned}$$

Solution 32.
Let X be the player's winnings.
Let Y be the number of times the number the player bet on appeared. Therefore,

$$Y \sim \mathrm{Bin}\left(3, \frac{1}{6}\right)$$

Therefore,

$$\begin{aligned}
\mathrm{P}(X=-1) &= \mathrm{P}(Y=0) \\
&= {}^{3}\mathrm{C}_0 \left(\frac{1}{6}\right)^0 \left(\frac{5}{6}\right)^3 \\
&= \frac{125}{216} \\
\mathrm{P}(X=1) &= \mathrm{P}(Y=1) \\
&= {}^{3}\mathrm{C}_1 \left(\frac{1}{6}\right)^1 \left(\frac{5}{6}\right)^2 \\
&= \frac{75}{216} \\
\mathrm{P}(X=2) &= \mathrm{P}(Y=2) \\
&= {}^{3}\mathrm{C}_2 \left(\frac{1}{6}\right)^2 \left(\frac{5}{6}\right)^1 \\
&= \frac{15}{216} \\
\mathrm{P}(X=3) &= \mathrm{P}(Y=3) \\
&= {}^{3}\mathrm{C}_3 \left(\frac{1}{6}\right)^3 \left(\frac{5}{6}\right)^0 \\
&= \frac{1}{216}
\end{aligned}$$

Therefore,

$$\begin{aligned}
\mathrm{E}[X] &= (-1)\left(\frac{125}{216}\right) + (1)\left(\frac{75}{216}\right) + (2)\left(\frac{15}{216}\right) + (3)\left(\frac{1}{216}\right) \\
&= -\frac{17}{216}
\end{aligned}$$

Therefore, as the expected value of the winnings is less than 0, the game is not fair towards the player.

Solution 33.

Let X be the number of α particles emitted. Therefore,

$$X \sim \mathrm{Poi}(3.2)$$

Therefore,

$$\begin{aligned}
\mathrm{P}(X \leq 2) &= \mathrm{P}(X=0) + \mathrm{P}(X=1) + \mathrm{P}(X=2) \\
&= \frac{e^{-3.2}\lambda^0}{0!} + \frac{e^{-3.2}\lambda^1}{1!} + \frac{e^{-3.2}\lambda^2}{2!} \\
&\approx 0.3799
\end{aligned}$$

Solution 34.

1. Let X be the number of earthquakes occurring in two weeks. Therefore,

$$X \sim \text{Poi}(4)$$

Therefore,

$$\begin{aligned}\text{P}(X \geq 3) &= 1 - (\text{P}(X = 0) + \text{P}(X = 1) + \text{P}(X = 2)) \\ &= 1 - \left(\frac{e^{-4}4^0}{0!} + \frac{e^{-4}4^1}{1!} + \frac{e^{-4}4^2}{21}\right) \\ &= 1 - 13e^{-4}\end{aligned}$$

2. Let Y be the number of earthquakes in the last four weeks.
Let Z_1 be the number of earthquakes in the first three of the last four weeks.
Let Z_2 be the number of earthquakes in the last week.
Therefore,

$$\begin{aligned}Y &\sim \text{Poi}(8) \\ Z_1 &\sim \text{Poi}(6) \\ Z_2 &\sim \text{Poi}(2)\end{aligned}$$

Therefore,

$$\begin{aligned}\text{P}(Z_2 = 1|Y = 3) &= \frac{\text{P}(Z_2 = 1 \cap Y = 3)}{\text{P}(Y = 3)} \\ &= \frac{\text{P}(Z_2 = 1 \cap Z_1 = 2)}{\text{P}(Y = 3)}\end{aligned}$$

Therefore, as Z_1 and Z_2 belong to non-overlapping intervals, the events $Z_2 = 1$ and $Z_1 = 2$ are independent. Therefore,

$$\begin{aligned}\text{P}(Z_2 = 1|Y = 3) &= \frac{\text{P}(Z_2 = 1)\,\text{P}(Z_1 = 2)}{\text{P}(Y = 3)} \\ &= \frac{\frac{e^{-2}2^1}{1!}\frac{e^{-6}6^2}{2!}}{\frac{e^{-8}8^3}{3!}} \\ &= \left({}^3\text{C}_1\right)\left(\frac{2}{8}\right)^1\left(\frac{6}{8}\right)^2\end{aligned}$$

Solution 35.

1.

$$\begin{aligned}
\mathrm{P}(X > 3) &= \sum_{k=4}^{\infty} \mathrm{P}(X = k) \\
&= \sum_{k=4}^{\infty} \left(1 - \frac{1}{10}\right)^{k-1} \left(\frac{1}{10}\right) \\
&= \left(1 - \frac{1}{10}\right)^{3} \sum_{j=1}^{\infty} \left(1 - \frac{1}{10}\right)^{j-1} \left(\frac{1}{10}\right) \\
&= \left(\frac{9}{10}\right)^{3} \left(\frac{1}{10}\right) \left(\frac{1}{1 - \frac{9}{10}}\right) \\
&= \left(\frac{9}{10}\right)^{3}
\end{aligned}$$

2.

$$\begin{aligned}
\mathrm{P}(X \geq 13 | X > 5) &= \mathrm{P}(X > 12 | X > 5) \\
&= \frac{\mathrm{P}(X > 12 \cap X > 5)}{\mathrm{P}(X > 5)} \\
&= \frac{\mathrm{P}(X > 12)}{\mathrm{P}(X > 5)} \\
&= \frac{\left(\frac{9}{10}\right)^{12}}{\left(\frac{9}{10}\right)^{5}} \\
&= \left(\frac{9}{10}\right)^{7} \\
&= \mathrm{P}(X > 7)
\end{aligned}$$

Therefore, the fact that Emily has already eaten 5 cookies does not affect the probability of her eating at least 8 more cookies.

Solution 36.

Let X be the number of times the die must be thrown for 1 to occur four times. Therefore,

$$X \sim \mathrm{NB}\left(4, \frac{1}{6}\right)$$

Therefore,

$$
\begin{aligned}
\mathrm{E}[X] &= \frac{r}{p} \\
&= \frac{4}{\frac{1}{6}} \\
&= 24 \\
\mathrm{V}(X) &= \frac{r(1-p)}{p^2} \\
&= \frac{4\left(1-\frac{1}{6}\right)}{\left(\frac{1}{6}\right)^2} \\
&= 120
\end{aligned}
$$

Solution 37.
Let A be the event that the buyer accepts the lot. Therefore,

$$
\begin{aligned}
\mathrm{P}(A) &= \mathrm{P}(4\text{ defective})\,\mathrm{P}(A|4\text{ defective}) + \mathrm{P}(1\text{ defective})\,\mathrm{P}(A|1\text{ defective}) \\
&= \left(\frac{3}{10}\right)\left(\frac{{}^{4}C_{0}\,{}^{6}C_{3}}{{}^{10}C_{3}}\right) + \left(\frac{7}{10}\right)\left(\frac{{}^{1}C_{0}\,{}^{9}C_{3}}{{}^{10}C_{3}}\right) \\
&= \frac{54}{100}
\end{aligned}
$$

Therefore, the buyer rejects 54% of the lots, i.e., he rejects 46% of the lots.

Solution 38.
1. Let

$$
Z = \frac{X-\mu}{\sigma}
$$

Therefore, Z is a standard normal random variable.
Therefore,

$$
\begin{aligned}
\mathrm{P}(2 < X < 5) &= \mathrm{P}\left(\frac{2-3}{3} < \frac{X-3}{3} < \frac{5-3}{3}\right) \\
&= \mathrm{P}\left(-\frac{1}{3} < Z < \frac{2}{3}\right) \\
&= \Phi\left(\frac{2}{3}\right) - \Phi\left(-\frac{1}{3}\right) \\
&= \Phi\left(\frac{2}{3}\right) - \left(1 - \Phi\left(\frac{1}{3}\right)\right) \\
&\approx 0.3779
\end{aligned}
$$

2. Let

$$Z = \frac{X - \mu}{\sigma}$$

Therefore, Z is a standard normal random variable.
Therefore,

$$\begin{aligned} \mathrm{P}(X > 3) &= \mathrm{P}\left(\frac{X-3}{3} > \frac{3-3}{3}\right) \\ &= \mathrm{P}(Z > 0) \\ &= 0.5 \end{aligned}$$

Solution 39.
If the diet has no effect on cholesterol levels,

$$X \sim \mathrm{Bin}\left(100, \frac{1}{2}\right)$$

Therefore,

$$\begin{aligned} \mathrm{P}(X \geq 65) &= \mathrm{P}\left(\frac{X - (100)\left(\frac{1}{2}\right)}{\sqrt{(100)\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)}} \geq \frac{65 - (100)\left(\frac{1}{2}\right)}{\sqrt{(100)\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)}}\right) \\ &\approx (Z \geq 2.9) \\ &\approx 0.0019 \end{aligned}$$

Solution 40.
Let

$$X_i = \begin{cases} 1 & ; \quad \text{person } i \text{ got own coat} \\ 0 & ; \quad \text{otherwise} \end{cases}$$

Therefore,

$$\mathrm{P}(X_i = 1) = \frac{1}{n}$$

Therefore,

$$\begin{aligned} \mathrm{E}[X] &= \sum_{i=1}^{n} \mathrm{E}[X_i] \\ &= \sum_{i=1}^{n} \frac{1}{n} \\ &= 1 \end{aligned}$$

Solution 41.

$$\mathrm{P}(X=x,Y=y)=\frac{{}^{3}\mathrm{C}_x\,{}^{4}\mathrm{C}_y\,{}^{5}\mathrm{C}_{3-x-y}}{{}^{12}\mathrm{C}_3}$$

Therefore,

	0	1	2	3	
0	$\frac{10}{220}$	$\frac{40}{220}$	$\frac{30}{220}$	$\frac{4}{220}$	$\frac{10}{220}$
1	$\frac{30}{220}$	$\frac{60}{220}$	$\frac{18}{220}$	0	$\frac{10}{220}$
2	$\frac{15}{220}$	$\frac{12}{220}$	0	0	$\frac{10}{220}$
3	$\frac{1}{220}$	0	0	0	$\frac{10}{220}$
	$\frac{56}{220}$	$\frac{112}{220}$	$\frac{48}{220}$	$\frac{4}{220}$	

Solution 42.

1.

$$\begin{aligned}\mathrm{P}(X>1,Y<1)&=\int_{-\infty}^{1}\int_{1}^{\infty}\mathrm{f}(x,y)\,\mathrm{d}x\,\mathrm{d}y\\&=\int_{0}^{1}\int_{1}^{\infty}2e^{-x}e^{-2y}\,\mathrm{d}x\,\mathrm{d}y\\&=\int_{0}^{1}2e^{-2y}\left.-e^{-x}\right|_{x=1}^{x=\infty}\mathrm{d}y\\&=e^{-1}\int_{0}^{1}2e^{-2y}\,\mathrm{d}y\\&=e^{-1}\left(1-e^{-2}\right)\end{aligned}$$

2.

$$
\begin{aligned}
\mathrm{P}(X < Y) &= \iint\limits_{(x,y):x<y} \mathrm{f}(x,y)\,\mathrm{d}x\,\mathrm{d}y \\
&= \iint\limits_{(x,y):x<y} 2e^{-x}e^{-2y}\,\mathrm{d}x\,\mathrm{d}y \\
&= \int\limits_0^\infty \int\limits_0^y 2e^{-x}e^{-2y}\,\mathrm{d}x\,\mathrm{d}y \\
&= \int\limits_0^\infty 2e^{-2y}\left(1 - e^{-y}\right)\mathrm{d}y \\
&= \int\limits_0^\infty 2e^{-2y}\,\mathrm{d}y - \int\limits_0^\infty 2e^{-3y}\,\mathrm{d}y \\
&= 1 - \frac{2}{3} \\
&= \frac{1}{3}
\end{aligned}
$$

3.

$$
\begin{aligned}
\mathrm{P}(X < a) &= \int\limits_{-\infty}^{a} \int\limits_{-\infty}^{\infty} \mathrm{f}(x,y)\,\mathrm{d}y\,\mathrm{d}x \\
&= \int\limits_0^a \int\limits_0^\infty 2e^{-2y}e^{-x}\,\mathrm{d}y\,\mathrm{d}x \\
&= \int\limits_0^a e^{-x}\,\mathrm{d}x \\
&= 1 - e^{-a}
\end{aligned}
$$

Solution 43.

$$\begin{aligned}
F_{\frac{X}{Y}}(a) &= P\left(\frac{X}{Y} \le a\right) \\
&= \iint\limits_{(x,y):\frac{x}{y}\le a} f(x,y)\,dx\,dy \\
&= \iint\limits_{(x,y):\frac{x}{y}\le a} e^{-(x+y)}\,dx\,dy \\
&= \int_0^\infty \int_0^{ay} e^{-(x+y)}\,dx\,dy \\
&= \int_0^\infty \left(1 - e^{-ay}\right) e^{-y}\,dy \\
&= -e^{-y} + \left.\frac{e^{-(a+1)y}}{a+1}\right|_0^\infty \\
&= 1 - \frac{1}{a+1}
\end{aligned}$$

Therefore,

$$\begin{aligned}
f_{\frac{X}{Y}}(a) &= \frac{dF_{\frac{X}{Y}}(a)}{da} \\
&= \frac{1}{(a+1)^2}
\end{aligned}$$

Solution 44.
Therefore,

$$\begin{aligned}
F_{X+Y}(t) &= P(X+Y \le t) \\
&= \begin{cases} \frac{t^2}{2} & ; \quad 0 \le t \le 1 \\ 1 - \frac{(2-t)^2}{2} & ; \quad 1 \le t \le 2 \end{cases}
\end{aligned}$$

Therefore,

$$f_{X+Y}(t) = \begin{cases} t & ; \quad 0 \le t \le 1 \\ 2-t & ; \quad 1 < t < 2 \\ 0 & ; \quad 2 \le t \end{cases}$$

Solution 45.

$$
\begin{aligned}
\mathrm{f}_{X|Y}(x|y) &= \frac{\mathrm{f}(x,y)}{\mathrm{f}_Y(y)} \\
&= \frac{\frac{e^{-\frac{x}{y}}e^{-y}}{y}}{e^{-y}\int\limits_0^{\infty}\left(\frac{1}{y}\right)e^{-\frac{x}{y}}\,\mathrm{d}x} \\
&= \frac{e^{-\frac{x}{y}}}{y}
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\mathrm{P}(X>1|Y=y) &= \int\limits_1^{\infty}\frac{1}{y}e^{-\frac{x}{y}}\,\mathrm{d}x \\
&= e^{-\frac{1}{y}}
\end{aligned}
$$

Solution 46.

$$
\begin{aligned}
X &\sim \mathrm{U}(0,L) \\
Y &\sim \mathrm{U}(0,L)
\end{aligned}
$$

Therefore,

$$
\mathrm{f}_X(x) = \begin{cases} \frac{1}{L} & ; \quad 0<x<L \\ 0 & ; \quad \text{otherwise} \end{cases}
$$

$$
\mathrm{f}_Y(y) = \begin{cases} \frac{1}{L} & ; \quad 0<x<L \\ 0 & ; \quad \text{otherwise} \end{cases}
$$

Therefore, as the variables are independent,

$$
\begin{aligned}
\mathrm{f}_{X,Y}(x,y) &= \mathrm{f}_X(x)\,\mathrm{f}_Y(y) \\
&= \begin{cases} \frac{1}{L^2} & ; \quad 0<x<L, 0<y<L \\ 0 & ; \quad \text{otherwise} \end{cases}
\end{aligned}
$$

Therefore,

$$\begin{aligned}
\mathrm{E}\left[|X-Y|\right] &= \frac{1}{L^2}\int_0^L\int_0^L |x-y|\,\mathrm{d}y\,\mathrm{d}x \\
&= \frac{1}{L^2}\int_0^L \left(\int_0^x (x-y)\,\mathrm{d}y + \int_x^L (y-x)\,\mathrm{d}y\right)\mathrm{d}x \\
&= \frac{1}{L^2}\int_0^L \left(\frac{L^2}{2} + x^2 - xL\right)\mathrm{d}x \\
&= \frac{L}{3}
\end{aligned}$$

Solution 47.

Let X be the number of coupons collected.
Let X_i be the number of additional coupons that need to be collected after i distinct types have been collected in order to obtain another distinct type.
The probability that a coupon will be of a new type, when i distinct types have already been collected is $\frac{N-i}{N}$. Therefore,

$$\mathrm{P}(X_i = k) = \left(\frac{i}{N}\right)^{k-1}\frac{N-i}{k}$$

Therefore,

$$X_i \sim \mathrm{Geo}\left(\frac{N-i}{N}\right)$$

Therefore,

$$\mathrm{E}[X_i] = \frac{N}{N-i}$$

Therefore,

$$\begin{aligned}
\mathrm{E}[X] &= \sum_{i=0}^{n} \mathrm{E}[X_i] \\
&= \sum_{i=0}^{n} \frac{N}{N-i} \\
&= 1 + \frac{N}{N-1} + \cdots + \frac{N}{1} \\
&= N\left(1 + \cdots + \frac{1}{N-1} + \frac{1}{N}\right)
\end{aligned}$$

Solution 48.
Let

$$I_i = \begin{cases} 1 & ; \quad \text{run of 1s starts at the } i\text{th position} \\ 0 & ; \quad \text{otherwise} \end{cases}$$

Let

$$R(1) = \sum_{i=1}^{n+m} I_i$$

Therefore,

$$\begin{aligned} \mathrm{E}\left[R(1)\right] &= \mathrm{E}\left[\sum_{i=1}^{n+m} I_i\right] \\ &= \sum_{i=1}^{n+m} \mathrm{E}[I_i] \end{aligned}$$

Therefore,

$$\begin{aligned} \mathrm{E}[I_1] &= \mathrm{P}(1 \text{ in position } 1) \\ &= \frac{n}{n+m} \\ \mathrm{E}[I_i] &= \mathrm{P}(0 \text{ in position } i-1,\ 1 \text{ in position } i) \\ &= \frac{m}{n+m}\frac{n}{n+m-1} \end{aligned}$$

Therefore,

$$\begin{aligned} \mathrm{E}\left[R(1)\right] &= \mathrm{E}[I_1] + (n+m-1)\,\mathrm{E}[I_i] \\ &= \frac{n}{n+m} + (n+m-1)\left(\frac{nm}{(n+m)(n+m-1)}\right) \end{aligned}$$

Solution 49.

$$\begin{aligned} \mathrm{V}\left(\overline{X}\right) &= \mathrm{V}\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) \\ &= \left(\frac{1}{n}\right)^2 \mathrm{V}\left(\sum_{i=1}^{n} X_i\right) \end{aligned}$$

As all X_i are independent,

$$\begin{aligned}
\mathrm{V}\left(\overline{X}\right) &= \left(\frac{1}{n}\right)^2 \sum_{i=1}^{n} \mathrm{V}(X_i) \\
&= \left(\frac{1}{n}\right)^2 n\sigma^2 \\
&= \frac{\sigma^2}{n}
\end{aligned}$$

Therefore,

$$\begin{aligned}
(n-1)S^2 &= \sum_{i=1}^{n} \left(X_i - \mu + \mu + \overline{X}\right)^2 \\
&= \sum_{i=1}^{n}(X_i-\mu)^2 + \sum_{i=1}^{n}\left(\overline{X}-\mu\right)^2 - 2\left(\overline{X}-\mu\right)\sum_{i=1}^{n}(X_i-\mu) \\
&= \sum_{i=1}^{n}(X_i-\mu)^2 + \sum_{i=1}^{n}\left(\overline{X}-\mu\right)^2 - 2\left(\overline{X}-\mu\right)\sum_{i=1}^{n}\left(n\overline{X}-n\mu\right) \\
&= \sum_{i=1}^{n}(X_i-\mu)^2 + \sum_{i=1}^{n}\left(\overline{X}-\mu\right)^2 - 2\left(\overline{X}-\mu\right)\sum_{i=1}^{n} n\left(\overline{X}-\mu\right) \\
&= \sum_{i=1}^{n}(X_i-\mu)^2 - n\left(\overline{X}-\mu\right)^2
\end{aligned}$$

Therefore,

$$\begin{aligned}
\mathrm{E}\left[(n-1)S^2\right] &= \mathrm{E}\left[\sum_{i=1}^{n}(X_i-\mu)^2\right] - n\,\mathrm{E}\left[\left(\overline{X}-\mu\right)^2\right] \\
&= \sum_{i=1}^{n}\mathrm{V}(X_i) - n\,\mathrm{V}\left(\overline{X}\right) \\
&= n\sigma^2 - n\,\mathrm{V}\left(\overline{X}\right) \\
&= n\sigma^2 - n\frac{\sigma^2}{n} \\
&= (n-1)\sigma^2
\end{aligned}$$

Therefore,

$$\begin{aligned}
\mathrm{V}\left(S^2\right) &= \mathrm{E}\left[\left(S^2 - \mathrm{E}\left[S^2\right]\right)^2\right] \\
&= \sigma^2
\end{aligned}$$

Solution 50.
Let

$$X_i = \begin{cases} 1 & ; \quad \text{person } i \text{ got own coat} \\ 0 & ; \quad \text{otherwise} \end{cases}$$

Therefore,

$$X = \sum_{i=1}^{n} I_i$$

Therefore,

$$\begin{aligned} V(I_i) &= \mathrm{E}\left[{X_i}^2\right] - \mathrm{E}[X_i]^2 \\ &= \left(\frac{1}{n}\right)\left(1^2\right) + \left(1 - \frac{1}{n}\right)\left(0^2\right) - \left(\left(\frac{1}{n}\right)(1) + \left(1 - \frac{1}{n}\right)(0)\right)^2 \\ &= \frac{1}{n} - \frac{1}{n^2} \\ &= \left(\frac{1}{n}\right)\left(1 - \frac{1}{n}\right) \end{aligned}$$

Therefore,

$$\begin{aligned} \mathrm{Cov}(X_i, X_j) &= \mathrm{E}[X_i X_j] - \mathrm{E}[X_i]\,\mathrm{E}[X_j] \\ &= \mathrm{P}(X_i = 1 \cap X_j = 1)(1) - \left(\frac{1}{n}\right)\left(\frac{1}{n}\right) \\ &= \left(\frac{1}{n}\right)\left(\frac{1}{n-1}\right) - \left(\frac{1}{n}\right)^2 \\ &= \frac{1}{n}\left(\frac{1}{n-1} - \frac{1}{n}\right) \\ &= \frac{1}{n^2(n-1)} \end{aligned}$$

Therefore,

$$\begin{aligned} \mathrm{V}(X) &= \mathrm{V}\left(\sum_{i=1}^{n} X_i\right) \\ &= \sum_{i=1}^{n} \mathrm{V}(X_i) + \sum_{i=1}^{n}\sum_{j=i}^{n} 2\,\mathrm{Cov}(X_i, X_j) \\ &= (n)\left(\frac{1}{n}\right)\left(1 - \frac{1}{n}\right) + 2\binom{n}{2}\left(\frac{1}{n^2(n-1)}\right) \\ &= 1 - \frac{1}{n} + \frac{1}{n} \\ &= 1 \end{aligned}$$

Solution 51.

$$
\begin{aligned}
P(X = k|X + Y = m) &= \frac{P(X = k, X + Y = m)}{P(X + Y = m)} \\
&= \frac{P(X = k, Y = m - k)}{P(X + Y = m)} \\
&= \frac{P(X = k)\, P(Y = m - k)}{P(X + Y = m)} \\
&= \frac{\left(\binom{n}{k} p^k (1-p)^{n-k}\right)\left(\binom{n}{m-k} p^{m-k} (1-p)^{n-m+k}\right)}{\binom{2n}{m} p^m (1-p)^{2n-m}} \\
&= \frac{\binom{n}{k}\binom{n}{m-k}}{\binom{2n}{m}}
\end{aligned}
$$

Therefore, the conditional distribution of X, given that $X + Y = m$ is hypergeometric, i.e.

$$(X|X + Y) \sim \mathrm{HG}(m, 2n, n)$$

Therefore, the expectation is

$$E[X|X + Y = m] = \frac{m}{2}$$

Solution 52.

The Poisson random variable with mean 100 is the sum of 100 independent Poisson random variables with mean 1 each. Also,

$$
\begin{aligned}
E[X] &= \lambda \\
&= 100 \\
V(X) &= \lambda \\
&= 100
\end{aligned}
$$

Therefore, by Central Limit Theorem,

$$X \sim N(100, 100)$$

Therefore,

$$
\begin{aligned}
P(X \geq 120) &= P\left(\frac{X - 100}{\sqrt{100}} \geq \frac{120 - 100}{\sqrt{100}}\right) \\
&=\approx 1 - \Phi(2)
\end{aligned}
$$

Solution 53.
Let X be the percentage of fat content in the particular grade of steakburger. Therefore,

$$X \sim \mathrm{N}(\mu, 9)$$

Let the null hypothesis, H_0, be that the mean fat content is 20%.
Let the new hypothesis, H_1, be that the mean fat content is less than 20%.
Therefore,

$$\overline{X} \sim \mathrm{N}\left(\mu, \frac{9}{12}\right)$$

Standardizing X and solving for the critical value,

$$\begin{aligned} \frac{k - 20}{\frac{3}{\sqrt{12}}} &= \Phi(0.05) \\ &= -1.645 \end{aligned}$$

Therefore,

$$\begin{aligned} k &= 20 - 1.645 \frac{3}{\sqrt{12}} \\ &= 18.575 \end{aligned}$$

Therefore, as the $\overline{X}$ is 19, it is not in the critical region. Therefore, the null hypothesis can be accepted.

Solution 54.
Let X be the percentage of fat content in the particular grade of steakburger. Therefore,

$$X \sim \mathrm{N}(\mu, 9)$$

Let the null hypothesis, H_0, be that the mean fat content is 20%.
Let the new hypothesis, H_1, be that the mean fat content is less than 20%.
Therefore,

$$\overline{X} \sim \mathrm{N}\left(\mu, \frac{9}{12}\right)$$

Therefore, the sample variance is

$$\begin{aligned} s^2 = \frac{\sum {x_i}^2 - n\overline{x}^2}{n - 1} \\ &= \frac{4448 - 12.192}{11} \\ &= 10.545 \end{aligned}$$

Therefore, the test statistic is

$$\begin{aligned} t &= \frac{\overline{x} - \mu}{\frac{s}{\sqrt{n}}} \\ &= \frac{19 - 20}{\frac{\sqrt{10.545}}{\sqrt{12}}} \\ &= -1.07 \end{aligned}$$

Therefore, as t is not in the critical region, the null hypothesis cannot be rejected for $\alpha = 0.05$.

Solution 55.

1. $A \cap B \cap C$
2. $(A \cap B^{\mathrm{C}} \cap C^{\mathrm{C}}) \cup (A^{\mathrm{C}} \cap B \cap C^{\mathrm{C}}) \cup (A^{\mathrm{C}} \cap B^{\mathrm{C}} \cap C)$
3. $A \cup B \cup C$
4. A^{C}
5. $(A^{\mathrm{C}} \cap B \cap C) \cup (A \cap B^{\mathrm{C}} \cap C) \cup (A \cap B \cap C^{\mathrm{C}})$
6. $(A \cap B) \cup (B \cap C) \cup (C \cap A)$

The following pairs of groups are mutually exclusive.

1. 1, 2
2. 1, 5
3. 1, 4
4. 2, 5
5. 2, 6

Solution 56.

1. ‘Head’ appeared at most 5 times.
2. ‘Head’ appeared more than 4 times.
3. ‘Head’ appeared at least once.
4. In the first two flips, ‘Tail’ appeared at least once.
5. Number of ‘Head’s is less than or equal to the number of ‘Tail’s.
6. 8 or more ‘Tails’ appeared.

Solution 57.

1.

$$\begin{aligned} P(A \cup B) &= P(A) + P(B) - P(A \cap B) \\ &= \frac{3}{5} + \frac{1}{3} - \frac{1}{10} \\ &= \frac{5}{6} \end{aligned}$$

2.

$$\begin{aligned} P\left(A^{C} \cap B^{C}\right) &= P\left((A \cup B)^{C}\right) \\ &= 1 - P(A \cup B) \\ &= 1 - \frac{5}{6} \\ &= \frac{1}{6} \end{aligned}$$

3.

$$\begin{aligned} P\left(A^{C} \cup B^{C}\right) &= P\left((A \cap B)^{C}\right) \\ &= 1 - P(A \cap B) \\ &= 1 - \frac{1}{10} \\ &= \frac{9}{10} \end{aligned}$$

4.

$$\begin{aligned} P\left(A \cap B^{C}\right) &= P(A) - P(A \cap B) \\ &= \frac{3}{5} - \frac{1}{10} \\ &= \frac{1}{2} \end{aligned}$$

Solution 58.

1.

$$\begin{aligned} |S| &= {}^{2}C_1 \cdot {}^{3}C_1 \cdot {}^{4}C_1 \\ &= 2 \cdot 3 \cdot 4 \\ &= 24 \end{aligned}$$

2.

$$\begin{aligned} |A| &= {}^{2}C_1 \cdot {}^{3}C_1 \\ &= 2 \cdot 3 \\ &= 6 \end{aligned}$$

Entree	Starch	Dessert
Chicken	Pasta	Ice cream
Chicken	Rice	Ice cream
Chicken	Potatoes	Ice cream

Entree	Starch	Dessert
Chicken	Rice	Ice cream

3.

$$\begin{aligned} |B| &= {}^3\mathrm{C}_1 \cdot {}^4\mathrm{C}_1 \\ &= 3 \cdot 4 \\ &= 12 \end{aligned}$$

4. $A \cap B$ is the event in which chicken and ice cream are chosen. Therefore, the outcomes are

5.

$$\begin{aligned} |C| &= {}^2\mathrm{C}_1 \cdot {}^4\mathrm{C}_1 \\ &= 2 \cdot 4 \\ &= 8 \end{aligned}$$

6. $A \cap B \cap C$ is the event in which chicken, rice, and ice cream are chosen. Therefore, the only outcome is

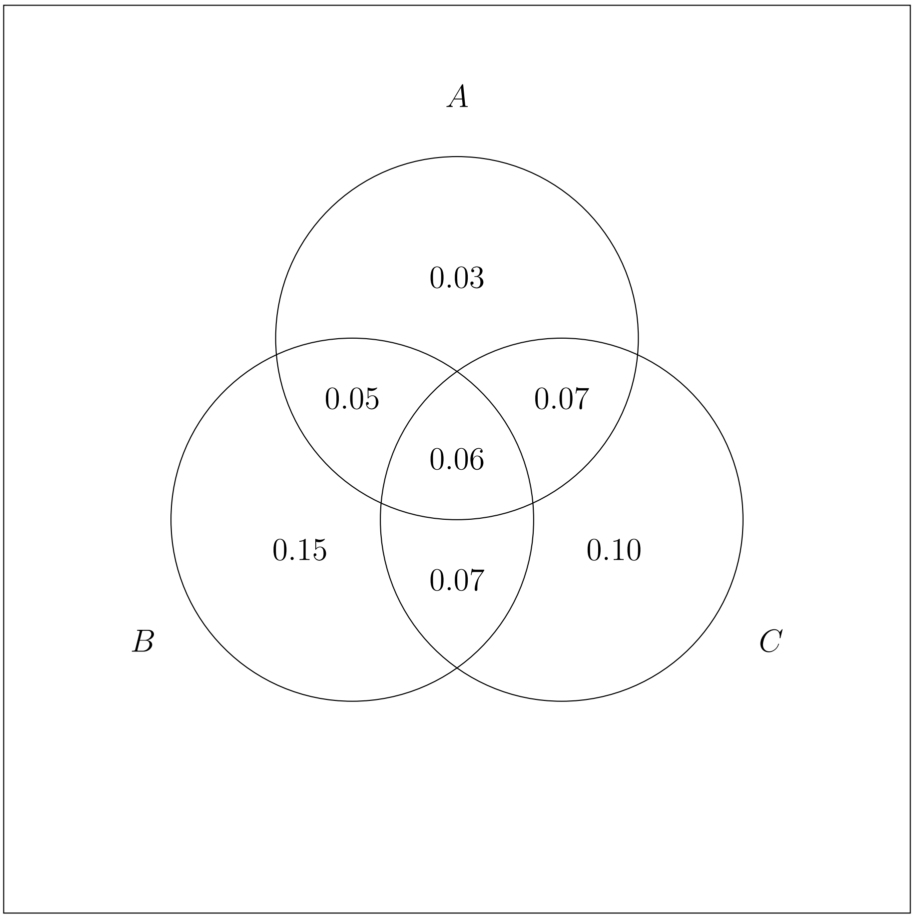
Solution 59.

1. Let A be the event that a customer buys a suit.
   Let B be the event that a customer buys a shirt
   Let C be the event that a customer buys a tie.
   Therefore,

$$\begin{aligned} \mathrm{P}\left((A \cup B \cup C)^{\mathrm{C}}\right) = &\quad 1 \\ &- (\mathrm{P}(A) + \mathrm{P}(B) + \mathrm{P}(C)) \\ &- (-\mathrm{P}(A \cap B) - \mathrm{P}(B \cap C) - \mathrm{P}(C \cap A)) \\ &- (\mathrm{P}(A \cap B \cap C)) \\ = &\ 1 - (0.22 + 0.30 + 0.28 - 0.11 - 0.14 - 0.10 + 0.06) \\ = &\ 1 - (0.51) \\ = &\ 0.49 \end{aligned}$$

2. Let the required outcome be D. Therefore,

$$\begin{aligned} \mathrm{P}(D) &= \mathrm{P}\left((A \setminus (B \cup C)) \cup (B \setminus (C \cup A)) \cup (C \setminus (A \cup B))\right) \\ &= \mathrm{P}(A \cup B \cup C) - \mathrm{P}(A \cap B) - \mathrm{P}(B \cap C) - \mathrm{P}(C \cap A) + 2\,\mathrm{P}(A \cap B \cap C) \\ &= 0.51 - 0.11 - 0.14 - 0.10 + 2(0.06) \\ &= 0.28 \end{aligned}$$



Solution 60.

The number of ways to seat 4 Americans together is 4!.
The number of ways to seat 3 French people together is 3!.
The number of ways to seat 3 British people together is 3!.
The number of ways to arrange these groups is 3!. Therefore, the total number of seating arrangements is 4!3!3!3!.

Solution 61.

1. The number of choices to select 6 couples from 10 is ${}^{10}\mathrm{C}_6$.
   The number of choices to select 1 person from each of these 6 couples is 2^6.
   Therefore, the total number of choices are ${}^{10}\mathrm{C}_6 2^6$

2. The number of choices to select 6 couples from 10 is ${}^{10}\mathrm{C}_6$.
   The number of choices to select 3 couples from which the men will be selected is ${}^{6}\mathrm{C}_3$.
   Therefore, the total number of choices are ${}^{10}\mathrm{C}_6{}^{6}\mathrm{C}_3$.

Solution 62.

The number of options to select the 5 letters is ${}^{26}\mathrm{C}_5$.
The number of options to select the 3 letters is ${}^{10}\mathrm{C}_3$.
If there are no restrictions on the positions, the number of arrangements of these 8 characters is 8!. Therefore, the total number of possible license plates is ${}^{26}\mathrm{C}_5{}^{10}\mathrm{C}_3 8!$.
If the 3 digits must be consecutive, the number of arrangements of the digits is 3!. The number of ways of arrangements of the letters, and the consecutive three digits, is 6!. Therefore, the total number of possible license plates is ${}^{26}\mathrm{C}_5{}^{10}\mathrm{C}_3 3!6!$.

Solution 63.

1. The total number of ways to select 8 balls is ${}^{20}\mathrm{C}_8$.
   The possible number of ways of selecting 6 black balls from 8 is ${}^{8}\mathrm{C}_6$.
   The possible number of ways of selecting 2 white balls from 5 is ${}^{8}\mathrm{C}_6$.
   Therefore, the probability is $\frac{{}^{8}\mathrm{C}_6{}^{5}\mathrm{C}_2}{{}^{20}\mathrm{C}_8}$.

2. The total number of ways to select 8 balls is ${}^{20}\mathrm{C}_8$.
   The possible number of ways of selecting 3 black balls from 8 is ${}^{8}\mathrm{C}_3$.
   The possible number of ways of selecting 2 red balls from 7 is ${}^{7}\mathrm{C}_2$.
   The possible number of ways of selecting 3 white balls from 5 is ${}^{5}\mathrm{C}_3$.
   Therefore, the probability is $\frac{{}^{8}\mathrm{C}_3{}^{7}\mathrm{C}_2{}^{5}\mathrm{C}_3}{{}^{20}\mathrm{C}_8}$.

3. The total number of ways to select 8 balls is ${}^{20}\mathrm{C}_8$.
   The possible number of ways of selecting 8 balls from 8 black and 7 red balls is ${}^{15}\mathrm{C}_8$.
   Therefore, the probability of selecting 0 white balls is $\frac{{}^{15}\mathrm{C}_8}{{}^{20}\mathrm{C}_8}$.
   Therefore, the probability of selecting at least 1 white ball is $1 - \frac{{}^{15}\mathrm{C}_8}{{}^{20}\mathrm{C}_8}$.

Solution 64.

1. The number of combinations with all of the 8 selected numbers is 1.
   Therefore the probability is $\frac{1}{{}^{40}\mathrm{C}_8}$.

2. The number of ways to choose 1 of the non-selected numbers is $\frac{32}{1}$.
   The number of ways to choose 7 of the selected numbers is ${}^{8}\mathrm{C}_7$.
   Therefore, the number of ways to choose 7 of the selected numbers is ${}^{32}\mathrm{C}_1{}^{8}\mathrm{C}_7$.
   Therefore, the probability is $\frac{{}^{32}\mathrm{C}_1{}^{8}\mathrm{C}_7}{{}^{40}\mathrm{C}_8}$.

3. The number of ways to choose 2 of the non-selected numbers is $\frac{32}{2}$.
   The number of ways to choose 6 of the selected numbers is ${}^{8}\mathrm{C}_{6}$.
   Therefore, the number of ways to choose 6 of the selected numbers is ${}^{32}\mathrm{C}_{2}{}^{8}\mathrm{C}_{6}$.
   The number of ways to choose 7 of the selected numbers is ${}^{32}\mathrm{C}_{1}{}^{8}\mathrm{C}_{7}$.
   The number of ways to choose 8 of the selected numbers is 1.
   Therefore the number of ways to choose at least 6 of the selected numbers is ${}^{32}\mathrm{C}_{2}{}^{8}\mathrm{C}_{6} + {}^{32}\mathrm{C}_{1}{}^{8}\mathrm{C}_{7} + 1$.
   Therefore, the probability is $\frac{{}^{32}\mathrm{C}_{2}{}^{8}\mathrm{C}_{6} + {}^{32}\mathrm{C}_{1}{}^{8}\mathrm{C}_{7} + 1}{{}^{40}\mathrm{C}_{8}}$.

Solution 65.

1. The total possible number of arrangements is 19!.
   The number of ways to arrange each couple amongst themselves is 2!. Therefore, the number of ways to arrange all 10 couples, amongst themselves, is 2^{10}.
   The number of ways to arrange all couples, grouped together, is 9!.
   Therefore, the total number of arrangements is $2^{10}9!$.
   Therefore, the probability is $\frac{2^{10}9!}{19!}$.

2. The total possible number of arrangements is 19!.
   The number of ways to select the couple which would not sit together is ${}^{10}\mathrm{C}_{1}$. The number of ways to arrange the remaining 9 couples is $2^{9}8!$.
   The number of positions available for the first member of the selected couple, between two of the other couples, is 9. The number of positions available for the second member of the selected couple, between two of the other couples, is 8. Therefore, the total number of arrangements is $10 \cdot 2^{9}8! \cdot 9 \cdot 8$. Therefore, the probability is $\frac{10 \cdot 2^{9}8! \cdot 9 \cdot 8}{19!}$.

Solution 66.

1. (a) For a manager, the total possible number of extensions is 10^{4}.
   The number of extensions with no repeating digits is ${}^{9}\mathrm{P}_{4}$.
   Therefore, the probability of a manager's extension having no repeating digits is $\frac{{}^{9}\mathrm{P}_{4}}{10^{4}}$.
   Therefore, the probability of a manager's extension having at least one repeating digit is $1 - \frac{{}^{9}\mathrm{P}_{4}}{10^{4}}$.

   (b) For a worker, the total possible number of extensions is $9 \cdot 10^{4}$.
   The number of extensions with no repeating digits is ${}^{9}\mathrm{C}_{1}{}^{9}\mathrm{P}_{4}$.
   Therefore, the probability of a worker's extension having no repeating digits is $\frac{{}^{9}\mathrm{C}_{1}{}^{9}\mathrm{P}_{4}}{9 \cdot 10^{4}}$.
   Therefore, the probability of a worker's extension having at least one repeating digit is $1 - \frac{{}^{9}\mathrm{C}_{1}{}^{9}\mathrm{P}_{4}}{9 \cdot 10^{4}}$.

2. (a) For a manager, the total possible number of extensions is 10^{4}.
   The number of possible positions for the digit 3 are 4.
   The number of options for the remaining digits is 10^{3}.

Therefore, the total number of extensions with at least one 3 is $4 \cdot 10^3$.
Therefore, the probability of a manager's extension having 3 is $\frac{4 \cdot 10^3}{10^4}$.

(b) For a worker, the total possible number of extensions is $9 \cdot 10^4$.
The number of possible positions for the digit 3 are 4.
The number of options for the remaining digits is 10^3.
Therefore, the total number of extensions with at least one 3 is $9 \cdot 4 \cdot 10^3$.
Therefore, the probability of a worker's extension having 3 is $\frac{9 \cdot 4 \cdot 10^3}{9 \cdot 10^4}$.

3. (a) For a manager, the total possible number of extensions is 10^4.
For the extension number to be divisible by 5, the number of options for the last digit are 2.
The number of options for the remaining digits is 10^3.
Therefore, the total number of extension numbers divisible by 5 is $2 \cdot 10^3$.
Therefore, the probability of a manager's extension number being divisible is $\frac{2 \cdot 10^3}{10^4}$.

(b) For a worker, the total possible number of extensions is $9 \cdot 10^4$.
For the extension number to be divisible by 5, the number of options for the last digit are 2.
The number of options for the remaining digits is 10^3.
Therefore, the total number of extension numbers divisible by 5 is $9 \cdot 2 \cdot 10^3$.
Therefore, the probability of a worker's extension number being divisible is $\frac{9 \cdot 2 \cdot 10^3}{9 \cdot 10^4}$.

4. (a) For a manager, the total possible number of extensions is 10^4.
For the extension number to be a palindrome, the fourth digit and the fifth digit must be the same as the second digit and the first digit respectively. Therefore, the extension is determined by the first three digits only.
For a manager, the number of options for the first digit is 1.
The number of options for the second digit is 10.
The number of options for the third digit is 10.
Therefore, the number of possible extensions is $1 \cdot 10 \cdot 10$.
Therefore, the probability is $\frac{1 \cdot 10 \cdot 10}{10^4}$.

(b) For a worker, the total possible number of extensions is $9 \cdot 10^4$.
For the extension number to be a palindrome, the fourth digit and the fifth digit must be the same as the second digit and the first digit respectively. Therefore, the extension is determined by the first three digits only.
For a manager, the number of options for the first digit is 9.
The number of options for the second digit is 10.
The number of options for the third digit is 10.
Therefore, the number of possible extensions is $9 \cdot 10 \cdot 10$.
Therefore, the probability is $\frac{9 \cdot 10 \cdot 10}{9 \cdot 10^4}$.

Solution 67.

1. The probability of the plant being alive is $(0.9)(0.85) + (0.1)(0.2)$.
   Therefore, the probability of the plant being alive is 0.785.

2. The total probability that the plant will be dead is 0.215.
   The probability that the plant will be dead and the neighbour watered it is $(0.9)(0.15)$.
   The probability that the plant will be dead and the neighbour did not water it is $(0.1)(0.8)$. Therefore, if the plant is dead, the probability that the neighbour did not water it is $\frac{0.08}{0.215}$.

Solution 68.

1. The total number of ways to choose 6 balls is ${}^{30}\mathrm{C}_6$.
   The number of ways to choose 0 red balls is ${}^{22}\mathrm{C}_6$.
   Therefore, the probability of selecting 0 red balls is $\frac{{}^{22}\mathrm{C}_6}{{}^{30}\mathrm{C}_6}$. Therefore, the probability of selecting at least 1 red ball is $1 - \frac{{}^{22}\mathrm{C}_6}{{}^{30}\mathrm{C}_6}$.

2. If 0 red balls are chosen and exactly 2 green balls are chosen, the remaining 4 balls must be blue.
   Therefore, the number of ways of choosing 2 green balls and 4 blue balls is ${}^{10}\mathrm{C}_2{}^{12}\mathrm{C}_4$.
   The total number of ways to choose 6 non-red balls is ${}^{22}\mathrm{C}_6$.
   Therefore, the probability is $\frac{{}^{10}\mathrm{C}_2{}^{12}\mathrm{C}_4}{{}^{22}\mathrm{C}_6}$.

Solution 69.

1. If Emily forgets the result, the probability that Jesse is told that the result is 'Heads' is 0.5.
   If Emily remembers the result, the probability that Jesse is told that the result is 'Heads' is 0.8
   Therefore, the total probability that Jesse is told that the result is 'Heads' is $(0.4)(0.5)+(0.6)(0.8)$. Therefore, the probability is 0.68.

2. The probability that Jesse is told the correct result if Emily remembers the result is 1.
   The probability that Jesse is told the correct result if Emily forgets the result is 0.5.
   Therefore, the probability that Jesse is told the correct result is $(0.6)(1) + (0.4)(0.5)$
   Therefore, the probability is 0.8.

Solution 70.

The total number of ways to select 3 balls from the urn is ${}^{9}\mathrm{C}_3$.
The number of ways to select 3 white balls from 4 white balls is ${}^{4}\mathrm{C}_3$.
Therefore, the probability is $\frac{{}^{4}\mathrm{C}_3}{{}^{9}\mathrm{C}_3}$.

Solution 71.

1. If A and B are mutually exclusive,

$$\mathrm{P}(A \cap B) = 0$$

A and B can be independent if and only if

$$\begin{aligned} \mathrm{P}(A \cap B) &= \mathrm{P}(A)\,\mathrm{P}(B) \\ \iff 0 &= \mathrm{P}(A)\,\mathrm{P}(B) \end{aligned}$$

However, as A and B have positive probability, this statement is necessarily false.

2. If A and B are independent,

$$\begin{aligned} \mathrm{P}(A \cap B) &= \mathrm{P}(A)\,\mathrm{P}(B) \\ \therefore \mathrm{P}(A \cap B) &> 0 \end{aligned}$$

A and B can be mutually exclusive if and only if

$$\begin{aligned} \mathrm{P}(A)\,\mathrm{P}(B) &= 0 \\ \iff \mathrm{P}(A \cap B) &= 0 \end{aligned}$$

However, as A and B have positive probability, this statement is necessarily false.

3. A and B are mutually exclusive if and only if

$$\begin{aligned} \mathrm{P}(A) + \mathrm{P}(B) &\le 1 \\ \iff 0.6 + 0.6 &\le 1 \end{aligned}$$

Therefore, this statement is necessarily false.

4. A and B are independent if and only if

$$\begin{aligned} \mathrm{P}(A)\,\mathrm{P}(B) &= \mathrm{P}(A \cap B) \\ \iff (0.6)(0.6) &= \mathrm{P}(A \cap B) \\ \iff \mathrm{P}(A \cap B) &= 0.36 \end{aligned}$$

Therefore, this statement is possibly true.

Solution 72.

The probability that a fair coin lands on 'Heads' is 0.5.
The probability that three independent trials with probability of success 0.8 all result in successes is $(0.8)^3$. Therefore, the probability is 0.512.
The probability that seven independent trials with probability of success 0.9 all result in successes is $(0.9)^7$. Therefore, the probability is 0.4782969.
Therefore, the descending order of probabilities is

1. Three independent trials, each of which is a success with probability 0.8, all result in successes.
2. A fair coin lands on 'Heads'.
3. Seven independent trials, each of which is a success with probability 0.9, all result in successes.

Solution 73.

$$\begin{aligned}\sum_{i=0}^{3} \mathrm{P}(X=i) &= 1\\ \therefore 0.3+0.2+4\,\mathrm{P}(X=3) &= 1\\ \therefore 4\,\mathrm{P}(X=3) &= 0.5\\ \therefore \mathrm{P}(X=3) &= 0.125\\ \therefore \mathrm{P}(X=0) &= 0.375\end{aligned}$$

Therefore,

$$\begin{aligned}\mathrm{E}[E] &= \sum_{i=0}^{3} i\,\mathrm{P}(X=i)\\ &= (0)(0.375)+(1)(0.3)+(2)(0.2)+(3)(0.125)\\ &= 0+0.3+0.4+0.375\\ &= 1.075\end{aligned}$$

Solution 74.

$$\begin{aligned}\sum_{i=0}^{2} \mathrm{P}(X=i) &= 1\\ \therefore \mathrm{P}(X=0)+c\,\mathrm{P}(X=0)+c^2\,\mathrm{P}(X=0) &= 1\\ \therefore \mathrm{P}(X=0) &= \frac{1}{1+c+c^2}\end{aligned}$$

Therefore,

$$\begin{aligned}\mathrm{E}[E] &= \sum_{i=0}^{2} i\,\mathrm{P}(X=i)\\ &= (0)\left(\frac{1}{1+c+c^2}\right)+(1)\left(\frac{c}{1+c+c^2}\right)+(2)\left(\frac{c^2}{1+c+c^2}\right)\\ &= \frac{c+2c^2}{1+c+c^2}\end{aligned}$$

Solution 75.

$$\mathrm{E}[X] = 3\,\mathrm{V}(X)$$

Therefore X is a Bernoulli random variable.
Therefore, let the probability

$$\begin{aligned}\mathrm{P}(X = 0) &= 1 - p \\ \mathrm{P}(X = 1) &= p\end{aligned}$$

Therefore,

$$\begin{aligned}\mathrm{E}[X] &= 3\,\mathrm{V}(X) \\ \therefore p &= 3p(1 - p) \\ \therefore p &= 3p - 3p^2 \\ \therefore 3p^2 &= 2p\end{aligned}$$

Therefore, either $p = 0$ or $p = \frac{2}{3}$. Therefore, $\mathrm{P}(X = 0)$ is either 1 or $\frac{1}{3}$.

Solution 76.
Let the amount bet be x. Therefore, without knowing which coin is selected, the probability of the outcome being 'Heads' is $(0.5)(0.6) + (0.5)(0.3)$, i.e. 0.45.
Let the winnings be denoted by the random variable X. Therefore the expected winnings are

$$\begin{aligned}E[X] &= (x)(0.45) + (-x)(0.55) \\ &= -0.1x\end{aligned}$$

Therefore, as the expected winnings are negative, one would never bet anything, i.e. bet 0.
If the information is bought from the insider, one would bet the maximum allowed money if the coin with 0.6 chance of 'Heads' is selected, and no money if the other coin is selected. Therefore, the expected winnings are

$$\begin{aligned}E[X] &= (0.6)(10 - C) + (0.4)(-10 - C) + (0.3)(0 - C) + (0.7)(0 - C) \\ &= 6 - 0.6C - 4 - 0.4C - C \\ &= 2 - 2C\end{aligned}$$

Therefore, it pays to buy the information if and only if

$$\begin{aligned}2 - 2C &> 0 \\ \iff C &< 1\end{aligned}$$

Solution 77.
The probability of x being written on the red paper is 1.
The probability of $2x$ being written on the blue paper is 0.5, and that of $\frac{x}{2}$ being written on the blue paper is 0.5.
Therefore, if the red paper is turned over first, the expected value of the number written is x.
If the blue paper is turned over first, the expected value of the number written is $x + \frac{x}{4}$.
Therefore, as the expected value of the number written on the blue paper is always higher than that written on the red paper, there is no point in turning over the red paper first. Even if one does turn over the red paper first, it is better to then switch to the blue paper.

Solution 78.

$$\begin{aligned} \mathrm{E}[X] &= 6 \\ \therefore np &= 6 \\ \mathrm{V}(X) &= 2.4 \\ \therefore np(1-p) &= 2.4 \end{aligned}$$

Therefore,

$$\begin{aligned} 1-p &= 0.4 \\ \therefore p &= 0.6 \end{aligned}$$

Therefore,

$$\begin{aligned} \mathrm{P}(X=i) &= {}^{n}\mathrm{C}_{i}(p)^{i}(1-p)^{n-i} \\ \therefore (X=5) &= {}^{10}\mathrm{C}_{5}(0.6)^{5}(0.4)^{5} \end{aligned}$$

Solution 79.
Let A be the decision that the panel makes the correct decision.
Let B be the decision that four of the judges agree.
Therefore,

$$\mathrm{P}(A) = \sum_{i=4}^{7} {}^{7}\mathrm{C}_{i}(0.7)^{i}(0.3)^{7-i}$$

Therefore,

$$\begin{aligned} \mathrm{P}(A|B) &= \frac{\mathrm{P}(A \cap B)}{\mathrm{P}(B)} \\ &= \frac{{}^{7}\mathrm{C}_{4}(0.7)^{4}(0.3)^{3}}{{}^{7}\mathrm{C}_{4}(0.7)^{4}(0.3)^{3} + {}^{7}\mathrm{C}_{4}(0.3)^{4}(0.7)^{3}} \end{aligned}$$

Solution 80.
Let X be the player's winnings.
Let Y be the number of times the number the player bet on appeared. Therefore,

$$Y \sim \text{Bin}\left(3, \frac{1}{6}\right)$$

Therefore,

$$\begin{aligned}
\text{P}(X = -1) &= \text{P}(Y = 0) \\
&= {}^3\text{C}_0 \left(\frac{1}{6}\right)^0 \left(\frac{5}{6}\right)^3 \\
&= \frac{125}{216} \\
\text{P}(X = 1) &= \text{P}(Y = 1) \\
&= {}^3\text{C}_1 \left(\frac{1}{6}\right)^1 \left(\frac{5}{6}\right)^2 \\
&= \frac{75}{216} \\
\text{P}(X = 2) &= \text{P}(Y = 2) \\
&= {}^3\text{C}_2 \left(\frac{1}{6}\right)^2 \left(\frac{5}{6}\right)^1 \\
&= \frac{15}{216} \\
\text{P}(X = 3) &= \text{P}(Y = 3) \\
&= {}^3\text{C}_3 \left(\frac{1}{6}\right)^3 \left(\frac{5}{6}\right)^0 \\
&= \frac{1}{216}
\end{aligned}$$

Therefore,

$$\begin{aligned}
\text{E}[X] &= (-1)\left(\frac{125}{216}\right) + (1)\left(\frac{75}{216}\right) + (2)\left(\frac{15}{216}\right) + (3)\left(\frac{1}{216}\right) \\
&= -\frac{17}{216}
\end{aligned}$$

Therefore, as the expected value of the winnings is less than 0, the game is not fair towards the player.

Solution 81.
1. X is a geometric random variable, with probability 0.4 for success. Therefore,

$$X \sim \text{Geo}(0.4)$$

2.

$$X \sim \text{Geo}(0.4)$$

Therefore,

$$\begin{aligned}
\text{E}[X] &= \frac{1}{p} \\
&= \frac{1}{0.4} \\
&= 2.5 \\
\text{V}(X) &= \frac{1-p}{p^2} \\
&= \frac{1-0.4}{(0.4)^2} \\
&= \frac{0.6}{0.16} \\
&= 3.75
\end{aligned}$$

3.

$$X \sim \text{Geo}(0.4)$$

Therefore,

$$\begin{aligned}
\text{P}(X \leq 3) &= 1 - \text{P}(X > 3) \\
&= 1 - \sum_{k=4}^{\infty} (1-p)^{k-1} p \\
&= 1 - (1-p)^3 \\
&= 1 - (0.6)^3 \\
&= 1 - 0.216 \\
&= 0.784
\end{aligned}$$

4.

$$X \sim \text{NB}(2, 0.4)$$

5.

$$X \sim \text{NB}(2, 0.4)$$

Therefore,

$$\begin{aligned}
\mathrm{E}[X] &= \frac{r}{p} \\
&= \frac{2}{0.4} \\
&= 5 \\
\mathrm{V}(X) &= \frac{r(1-p)}{p^2} \\
&= \frac{(2)(0.6)}{(0.4)^2} \\
&= \frac{0.12}{0.16} \\
&= 7.5
\end{aligned}$$

Solution 82.

1. Let X be the number of parliamentary committees which have women as chairpersons such that a MP can chair any number of committees. Therefore,

$$\begin{aligned}
X &\sim \mathrm{Bin}\left(12, \frac{24}{120}\right) \\
&\sim \mathrm{Bin}\left(12, \frac{1}{5}\right)
\end{aligned}$$

Therefore,

$$\begin{aligned}
\mathrm{P}(X=3) &= {}^{12}\mathrm{C}_3 \left(\frac{1}{5}\right)^3 \left(\frac{4}{5}\right)^9 \\
&= 0.2362232013
\end{aligned}$$

2. Let X be the number of parliamentary committees which have women as chairpersons such that a MP can chair at most one committee. Therefore,

$$X \sim \mathrm{HG}\,(12, 120, 24)$$

Therefore,

$$\begin{aligned}
\mathrm{P}(X=3) &= \frac{{}^{12}\mathrm{C}_3\,{}^{120-24}\mathrm{C}_{12-3}}{{}^{120}\mathrm{C}_{12}} \\
&= \frac{{}^{12}\mathrm{C}_3\,{}^{96}\mathrm{C}_9}{{}^{120}\mathrm{C}_{12}} \\
&= 0.02705523278
\end{aligned}$$

Solution 84.
Let X be the number of hurricanes hitting the region in a year. Therefore,

$$X \sim \mathrm{Poi}(5.2)$$

Therefore,

$$\begin{aligned}
\mathrm{P}(X \leq 3) &= \sum_{i=0}^{3} \frac{e^{-\lambda}\lambda^i}{i!} \\
&= \sum_{i=0}^{3} \frac{e^{-5.2}(5.2)^i}{i!}
\end{aligned}$$

Solution 85.

$$\begin{aligned}
\mathrm{E}[Y] &= \sum_{i=0}^{\infty} i\,\mathrm{P}(X = i | X > 0) \\
&= \sum_{i=0}^{\infty} i \frac{\mathrm{P}(X = i \cap X > 0)}{\mathrm{P}(X > 0)} \\
&= \sum_{i=1}^{\infty} i \frac{\mathrm{P}(X = i)}{\mathrm{P}(X > 0)} \\
&= \frac{\mathrm{E}[X]}{\mathrm{P}(X > 0)} \\
&= \frac{\lambda}{1 - e^{-\lambda}}
\end{aligned}$$

Solution 86.
1.

$$\begin{aligned}
1 &= \int_{-\infty}^{\infty} \mathrm{f}(t)\,\mathrm{d}t \\
&= \int_{0}^{5} ct(5-t)^2\,\mathrm{d}t \\
&= \frac{625c}{12}
\end{aligned}$$

Therefore,

$$c = \frac{12}{625}$$

2.

$$
\begin{aligned}
F_T(t) &= \int_{-\infty}^{t} f(s)\,ds \\
&= \begin{cases} 0 & ; \quad t < 0 \\ \frac{12}{625}\left(\frac{25t^2}{2} - \frac{10t^3}{3} + \frac{t^4}{4}\right) & ; \quad 0 \le t \le 5 \\ 1 & ; \quad 5 < t \end{cases}
\end{aligned}
$$

3.

$$
\begin{aligned}
P(T \ge 2) &= \int_{0}^{2} f(t)\,dt \\
&= \int_{0}^{2} \frac{12}{625} t(5-t)^2\,dt \\
&= \frac{328}{625}
\end{aligned}
$$

Solution 87.

1.

$$
\begin{aligned}
1 &= \int_{-\infty}^{\infty} f(t)\,dt \\
&= \int_{0}^{100} ct(100-t) \\
&= -c\left(-50t^2 + \frac{t^3}{3}\right)\Bigg|_{0}^{100} \\
&= \frac{500000c}{3}
\end{aligned}
$$

Therefore,

$$c = \frac{3}{500000}$$

2.

$$
\begin{aligned}
F_X(x) &= \int\limits_{-\infty}^{x} F_X(x)\,dx \\
&= \begin{cases} 0 & x < 0 \\ -\frac{3}{500000}\left(-50x^2 + \frac{x^3}{3}\right) & ; \quad 0 \leq t \leq 100 \\ 1 & 100 < x \end{cases}
\end{aligned}
$$

3.

$$
\begin{aligned}
P(X < 55) &= \int\limits_{0}^{55} f(t)\,dt \\
&= \int\limits_{0}^{55} \frac{3}{500000} t(100 - t)\,dt \\
&= \frac{2299}{4000}
\end{aligned}
$$

4.

$$
\begin{aligned}
P(X < 55 | X > 54) &= \frac{P(54 < X < 55)}{P(X > 54)} \\
&= \frac{\frac{7439}{500000}}{\frac{6877}{15625}} \\
&= \frac{7439}{220064}
\end{aligned}
$$

$$
\begin{aligned}
P(X < 55 | X > 40) &= \frac{P(40 < X < 55)}{P(X > 54)} \\
&= \frac{\frac{891}{4000}}{\frac{81}{125}} \\
&= \frac{11}{32}
\end{aligned}
$$

Solution 88.

1.

$$
\begin{aligned}
P(T > 15) &= \int\limits_{15}^{\infty} f(T)\,dt \\
&= 0.875
\end{aligned}
$$

2.

$$\begin{aligned}P(20 < T < 35) &= \int_{20}^{35} f(T)\,dt \\ &= 0.625\end{aligned}$$

3.

$$\begin{aligned}P(T < 30) &= \int_{30}^{\infty} f(T)\,dt \\ &= 0.75\end{aligned}$$

4.

$$\begin{aligned}P(T > 36) &= \int_{36}^{\infty} f(T)\,dt \\ &= 0.1\end{aligned}$$

Solution 92.

1.

$$\begin{aligned}P(T > 15) &= \int_{15}^{\infty} f(T)\,dt \\ &= 0.875\end{aligned}$$

2.

$$\begin{aligned}P(20 < T < 35) &= \int_{20}^{35} f(T)\,dt \\ &= 0.625\end{aligned}$$

3.

$$\begin{aligned}P(T < 30) &= \int_{30}^{\infty} f(T)\,dt \\ &= 0.75\end{aligned}$$

4.

$$\begin{aligned} P(T > 36) &= \int_{36}^{\infty} f(T)\,dt \\ &= 0.1 \end{aligned}$$

Solution 93.
As $f(x)$ is a probability density function,

$$\begin{aligned} 1 &= \int_{-\infty}^{\infty} f(x)\,dx \\ &= \int_{0}^{2} cx^4 \\ &= c\left(\frac{32}{5}\right) \end{aligned}$$

Therefore,

$$c = \frac{5}{32}$$

Therefore,

$$\begin{aligned} E[X] &= \int_{-\infty}^{\infty} x\,f(x)\,dx \\ &= \frac{5}{32}\int_{0}^{2} x^5\,dx \\ &= \frac{5}{32}\frac{64}{6} \\ &= \frac{5}{3} \end{aligned}$$

Therefore,

$$\begin{aligned}
\mathrm{E}\left[g(X)\right] &= \int\limits_{-\infty}^{\infty} g(x)\,\mathrm{f}(x)\,\mathrm{d}x \\
\therefore \mathrm{E}\left[X^2\right] &= \frac{5}{32}\int\limits_{0}^{2} x^2x^4 \\
&= \frac{5}{32}\int\limits_{0}^{2} x^6 \\
&= \frac{5}{32}\frac{128}{7} \\
&= \frac{20}{7}
\end{aligned}$$

Therefore,

$$\begin{aligned}
\mathrm{V}(X) &= \mathrm{E}\left[X^2\right] - \mathrm{E}[X]^2 \\
&= \frac{20}{7} - \frac{25}{9} \\
&= \frac{5}{63}
\end{aligned}$$

Solution 97.

1. Let X be the obtained score.
   Let

$$\begin{aligned}
Z &= \frac{X-\mu}{\sigma} \\
&= \frac{X-100}{15}
\end{aligned}$$

   Therefore, Z is a standard normal random variable. Therefore,

$$\begin{aligned}
\mathrm{P}(X > 125) &= \mathrm{P}\left(Z > \frac{125-100}{15}\right) \\
&= \mathrm{P}\left(\frac{5}{3}\right) \\
&= 1 - \Phi\left(\frac{5}{3}\right)
\end{aligned}$$

2. Let X be the obtained score.
   Let

$$\begin{aligned}
Z &= \frac{X-\mu}{\sigma} \\
&= \frac{X-100}{15}
\end{aligned}$$

Therefore, Z is a standard normal random variable. Therefore,

$$\begin{aligned}
P(90 < X < 100) &= P\left(\frac{90-100}{15} < Z < \frac{110-100}{15}\right) \\
&= P\left(\frac{-2}{3} < Z < \frac{2}{3}\right) \\
&= \Phi\left(\frac{2}{3}\right) - \Phi\left(-\frac{2}{3}\right) \\
&= \Phi\left(\frac{2}{3}\right) - \left(1 - \Phi\left(\frac{2}{3}\right)\right) \\
&= 2\Phi\left(\frac{2}{3}\right) - 1
\end{aligned}$$

Solution 98.

1. Let X be the life of the tyre.
   Let

$$\begin{aligned}
Z &= \frac{X-\mu}{\sigma} \\
&= \frac{X-34000}{4000}
\end{aligned}$$

   Therefore, Z is a standard normal random variable. Therefore,

$$\begin{aligned}
P(X > 40000) &= P\left(Z > \frac{40000-34000}{4000}\right) \\
&= P\left(Z > \frac{3}{2}\right) \\
&= 1 - \Phi\left(\frac{3}{2}\right)
\end{aligned}$$

2. Let X be the life of the tyre.
   Let

$$\begin{aligned}
Z &= \frac{X-\mu}{\sigma} \\
&= \frac{X-34000}{4000}
\end{aligned}$$

Therefore, Z is a standard normal random variable. Therefore,

$$\begin{aligned}
P(30000 < X < 35000) &= P\left(\frac{30000-34000}{4000} < Z < \frac{35000-34000}{4000}\right)\\
&= P\left(-1 < Z < \frac{1}{4}\right)\\
&= \Phi\left(\frac{1}{4}\right) - \Phi(-1)\\
&= \Phi\left(\frac{1}{4}\right) - (1-\varphi(1))\\
&= \Phi\left(\frac{1}{4}\right) + \Phi(1) - 1
\end{aligned}$$

3. Let X be the life of the tyre.
   Let

$$\begin{aligned}
Z &= \frac{X-\mu}{\sigma}\\
&= \frac{X-34000}{4000}
\end{aligned}$$

Therefore, Z is a standard normal random variable. Therefore,

$$\begin{aligned}
P(X > 30000) &= P\left(Z > \frac{30000-34000}{4000}\right)\\
&= P(Z > -1)\\
&= 1 - \Phi(1)\\
P(X > 40000) &= P\left(Z > \frac{40000-34000}{4000}\right)\\
&= P\left(Z > \frac{3}{2}\right)\\
&= 1 - \Phi\left(\frac{3}{2}\right)
\end{aligned}$$

Therefore,

$$\begin{aligned}
P(X > 40000 | X > 30000) &= \frac{P(X > 40000 \cap X > 30000}{P(X > 30000)}\\
&= \frac{P(X > 40000)}{P(X > 30000)}\\
&= \frac{1-\Phi\left(\frac{3}{2}\right)}{1-\Phi(1)}
\end{aligned}$$

Solution 99.

1. Let X be the rainfall in a year.
   Let

$$\begin{aligned} Z &= \frac{X-\mu}{\sigma} \\ &= \frac{X-30.2}{8.4} \end{aligned}$$

   Therefore, Z is a standard normal random variable. Therefore,

$$\begin{aligned} \mathrm{P}(X>44) &= \mathrm{P}\left(Z>\frac{44-30.2}{8.4}\right) \\ &= \mathrm{P}\left(Z>\frac{13.8}{8.4}\right) \\ &= \mathrm{P}\left(Z>\frac{23}{14}\right) \\ &= 1-\Phi\left(\frac{23}{14}\right) \end{aligned}$$

2. For any year, as the rainfall in every year is independent of the previous years, the probability that the rainfall will exceed 44 inches is

$$\mathrm{P}(X>44) = 1-\Phi\left(\frac{23}{14}\right)$$

   Therefore, the probability that the rainfall will exceed 44 inches in exactly 3 of the next 7 years is $\frac{7}{3}\left(1-\Phi\left(\frac{23}{14}\right)\right)^3\left(\Phi\left(\frac{23}{14}\right)\right)^4$.

Appendix B

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