A-LEVEL NOTES

CALCULUS
(for Emily)

May 2021
version 0.2

CALCULUS notes
for A-LEVEL Mathematics
and Further Mathematics
(May 2021)

This document is a self contained set of lecture notes for A level Mathematics and Further Mathematics. These notes are maintained by Hugh Murrell and available for free as a *pdf* document from:
https://hughmurrell.github.io/.

This document is based on a set of *free undergraduate notes* from the University of Wisconsin that were constructed by **Sigurd Angenent** with input from **Joel Robbin** and **A. Miller**, The original versions including LaTex source code can be found here:
http://www.math.wisc.edu/~angenent/Free-Lecture-Notes/

This collection also includes worked problems from past A-level and STEP (Sixth Term Examination Paper) papers. Further worked problems will be added in due course depending on reader engagement with the collection.

The intention is that this set of Calculus notes will provide preparation material for the following A-LEVEL papers:

Mathematics	Further Mathematics	STEP
Paper 1 (Pure)	Paper 1 (Pure)	Section A (Pure)
Paper 3 (Pure)	Paper 2 (Pure)	

Prospective A-level students and students planning to write the *Sixth Term Examination Paper* (STEP), are encouraged to make use of this text to supplement their A-level materials. Answers and hints to selected exercises can be found in the appendix of this text.

Further assistance with challenging problems can be obtained via email. To obtain help, point out errors or propose new problems or YouTube videos please feel free to email hugh.murrell@gmail.com.

Contents

Chapter 1

Numbers and Functions

The subject of this course is "functions of one real variable" so we begin by wondering what a real number "really" is, and then, in the next section, what a function is.

1.1 What is a number?

1.1.1 Different kinds of numbers.

The simplest numbers are the ***positive integers***

$$1, 2, 3, 4, \cdots$$

the number ***zero***

$$0,$$

and the ***negative integers***

$$\cdots, -4, -3, -2, -1.$$

Together these form the integers or "whole numbers."

Next, there are the numbers you get by dividing one whole number by another (nonzero) whole number. These are the so called fractions or ***rational numbers*** such as

$$\frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, \frac{4}{3}, \cdots$$

or

$$-\frac{1}{2}, -\frac{1}{3}, -\frac{2}{3}, -\frac{1}{4}, -\frac{2}{4}, -\frac{3}{4}, -\frac{4}{3}, \cdots$$

By definition, any whole number is a rational number (in particular zero is a rational number.)

You can add, subtract, multiply and divide any pair of rational numbers and the result will again be a rational number (provided you don't try to divide by zero).

One day in middle school you were told that there are other numbers besides the rational numbers, and the first example of such a number is the square root of two. It has been known ever since the time of the greeks that no rational number exists whose square is exactly 2, i.e. you can't find a fraction $\frac{m}{n}$ such that

$$\left(\frac{m}{n}\right)^2 = 2, \text{ i.e. } m^2 = 2n^2.$$

x	x^2
1.2	1.44
1.3	1.69
1.4	$1.96 < 2$
1.5	$2.25 > 2$
1.6	2.56

Nevertheless, if you compute x^2 for some values of x between 1 and 2, and check if you get more or less than 2, then it looks like there should be some number x between 1.4 and 1.5 whose square is exactly 2. So, we *assume* that there is such a number, and we call it the square root of 2, written as $\sqrt{2}$. This raises several questions. How do we know there really is a number between 1.4 and 1.5 for which $x^2 = 2$? How many other such numbers are we going to assume into existence? Do these new numbers obey the same algebra rules (like $a + b = b + a$) as the rational numbers? If we knew precisely what these numbers (like $\sqrt{2}$) were then we could perhaps answer such questions. It turns out to be rather difficult to give a precise description of what a number is, and in this course we won't try to get anywhere near the bottom of this issue. Instead we will think of numbers as "infinite decimal expansions" as follows.

One can represent certain fractions as decimal fractions, e.g.

$$\frac{279}{25} = \frac{1116}{100} = 11.16.$$

Not all fractions can be represented as decimal fractions. For instance, expanding $\frac{1}{3}$ into a decimal fraction leads to an unending decimal fraction

$$\frac{1}{3} = 0.333\,333\,333\,333\,333\ \cdots$$

It is impossible to write the complete decimal expansion of $\frac{1}{3}$ because it contains infinitely many digits. But we can describe the expansion: each digit is a three. An electronic calculator, which always represents numbers as ***finite*** decimal numbers, can never hold the number $\frac{1}{3}$ exactly.

Every fraction can be written as a decimal fraction which may or may not be finite. If the decimal expansion doesn't end, then it must repeat. For instance,

$$\frac{1}{7} = 0.142857\,142857\,142857\,142857\ \ldots$$

Conversely, any infinite repeating decimal expansion represents a rational number.

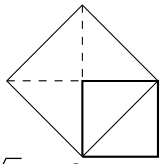
A ***real number*** is specified by a possibly unending decimal expansion. For instance,

$$\sqrt{2} = 1.414\,213\,562\,373\,095\,048\,801\,688\,724\,209\,698\,078\,569\,671\,875\,376\,9\ldots$$

Of course you can never write *all* the digits in the decimal expansion, so you only write the first few digits and hide the others behind dots. To give a precise description of a real number (such as $\sqrt{2}$) you have to explain how you could *in principle* compute as many digits in the expansion as you would like. In this text we will not be going into the details of how this should be done.

1.1.2 A reason to believe in $\sqrt{2}$.

The Pythagorean theorem says that the hypotenuse of a right triangle with sides 1 and 1 must be a line segment of length $\sqrt{2}$. In middle or high school you learned something similar to the following geometric construction of a line segment whose length is $\sqrt{2}$.



Take a square with side of length 1, and construct a new square one of whose sides is the diagonal of the first square. The figure you get consists of 5 triangles of equal area and by counting triangles you see that the larger square has exactly twice the area of the smaller square. Therefore the diagonal of the smaller square, being the side of the larger square, is $\sqrt{2}$ as long as the side of the smaller square. To find out more about the properties of the number $\sqrt{2}$ watch this YouTube by Numberphile .

1.1.3 Why are real numbers called real?

All the numbers we will use in this first part of this text are "real numbers." At some point (later in the text) it becomes useful to assume that there is a number whose square is -1. No real number has this property since the square of any real number is positive, so it was decided to call this new imagined number "imaginary" and to refer to the numbers we already have (rationals, $\sqrt{2}$-like things) as "real."

1.1.4 The real number line and intervals.

It is customary to visualize the real numbers as points on a straight line. We imagine a line, and choose one point on this line, which we call the ***origin***. We also decide which direction we call "left" and hence which we call "right." Some draw the number line vertically and use the words "up" and "down."

To plot any real number x one marks off a distance x from the origin, to the right (up) if $x > 0$, to the left (down) if $x < 0$.

The ***distance along the number line*** between two numbers x and y is $|x - y|$. In particular, the distance is never a negative number.

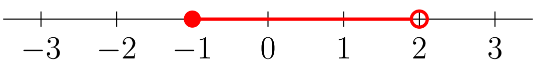


Figure 1.1: To draw the half open interval $[-1, 2)$ use a filled dot to mark the endpoint which is included and an open dot for an excluded endpoint.

Almost every equation involving variables x, y, etc. we write down in this course will be true for some values of x but not for others. In modern abstract mathematics a collection of real numbers (or any other kind of mathematical objects) is called a ***set***. Below are some examples of sets of real numbers. We will use the notation from these examples throughout this course.

The collection of all real numbers between two given real numbers form an interval. The following notation is used

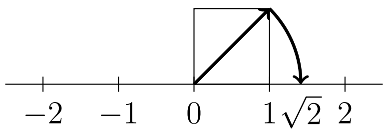


Figure 1.2: To find $\sqrt{2}$ on the real line you draw a square of sides 1 and drop the diagonal onto the real line.

- (a, b) is the set of all real numbers x which satisfy $a < x < b$.
- $[a, b)$ is the set of all real numbers x which satisfy $a \le x < b$.
- $(a, b]$ is the set of all real numbers x which satisfy $a < x \le b$.
- $[a, b]$ is the set of all real numbers x which satisfy $a \le x \le b$.

If the endpoint is not included then it may be ∞ or $-\infty$. E.g. $(-\infty, 2]$ is the interval of all real numbers (both positive and negative) which are ≤ 2.

1.1.5 Set notation.

A common way of describing a set is to say it is the collection of all real numbers which satisfy a certain condition. One uses this notation

$$\mathcal{A} = \big\{x \mid x \text{ satisfies this or that condition}\big\}$$

Most of the time we will use upper case letters in a calligraphic font to denote sets. ($\mathcal{A}$,$\mathcal{B}$,$\mathcal{C}$,$\mathcal{D}$, ...)
For instance, the interval (a, b) can be described as

$$(a, b) = \big\{x \mid a < x < b\big\}$$

The set

$$\mathcal{B} = \big\{x \mid x^2 - 1 > 0\big\}$$

consists of all real numbers x for which $x^2 - 1 > 0$, i.e. it consists of all real numbers x for which either $x > 1$ or $x < -1$ holds. This set consists of two parts: the interval $(-\infty, -1)$ and the interval $(1, \infty)$.
You can try to draw a set of real numbers by drawing the number line and coloring the points belonging to that set red, or by marking them in some other way.
Some sets can be very difficult to draw. For instance,

$$\mathcal{C} = \big\{x \mid x \text{ is a rational number}\big\}$$

can't be accurately drawn. In this course we will try to avoid such sets.
Sets can also contain just a few numbers, like

$$\mathcal{D} = \{1, 2, 3\}$$

which is the set containing the numbers one, two and three. Or the set

$$\mathcal{E} = \{x \mid x^3 - 4x^2 + 1 = 0\}$$

which consists of the solutions of the equation $x^3 - 4x^2 + 1 = 0$. (There are three of them, but it is not easy to give a formula for the solutions.)

If $\mathcal{A}$ and $\mathcal{B}$ are two sets then ***the union of*** $\mathcal{A}$ ***and*** $\mathcal{B}$ is the set which contains all numbers that belong either to $\mathcal{A}$ or to $\mathcal{B}$. The following notation is used

$$\mathcal{A} \cup \mathcal{B} = \{x \mid x \text{ belongs to } \mathcal{A} \text{ or to } \mathcal{B} \text{ or both.}\}$$

Similarly, the ***intersection of two sets*** $\mathcal{A}$ ***and*** $\mathcal{B}$ is the set of numbers which belong to both sets. This notation is used:

$$\mathcal{A} \cap \mathcal{B} = \{x \mid x \text{ belongs to both } \mathcal{A} \text{ and } \mathcal{B}.\}$$

1.2 Functions

Wherein we meet the main characters of this text

1.2.1 Definition.

To specify a ***function*** f you must

1. give a ***rule*** which tells you how to compute the value $f(x)$ of the function for a given real number x, and:

2. say for which real numbers x the rule may be applied.

The set of numbers for which a function is defined is called its ***domain***. The set of all possible numbers $f(x)$ as x runs over the domain is called the ***range*** of the function. The rule must be ***unambiguous:*** the same $x$must always lead to the same $f(x)$.

For instance, one can define a function f by putting $f(x) = \sqrt{x}$ for all $x \geq 0$. Here the rule defining f is "take the square root of whatever number you're given", and the function f will accept all nonnegative real numbers.

The rule which specifies a function can come in many different forms. Most often it is a formula, as in the square root example of the previous paragraph. Sometimes you need a few formulas, as in

$$g(x) = \begin{cases} 2x & \text{for } x < 0 \\ x^2 & \text{for } x \geq 0 \end{cases} \qquad \text{domain of } g = \text{all real numbers.}$$

Functions which are defined by different formulas on different intervals are sometimes called ***piecewise defined functions.***

1.2.2 Graphing a function.

You get the ***graph of a function*** f by drawing all points whose coordinates are (x, y) where x must be in the domain of f and $y = f(x)$.

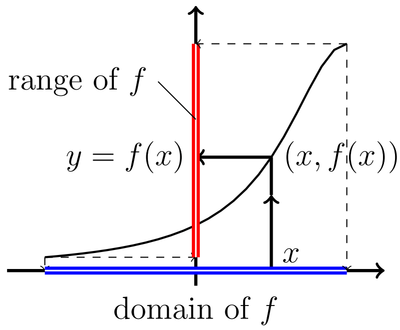


Figure 1.3: The graph of a function f. The domain of f consists of all x values at which the function is defined, and the range consists of all possible values f can have.

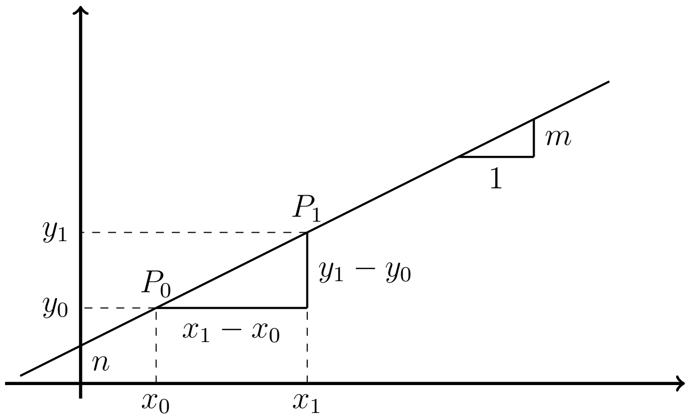


Figure 1.4: A straight line and its slope. The line is the graph of $f(x) = mx + n$. It intersects the y-axis at height n, and the ratio between the amounts by which y and x increase as you move from one point to another on the line is $\frac{y_1 - y_0}{x_1 - x_0} = m$.

1.2.3 Linear functions.

A function which is given by the formula

$$f(x) = mx + n$$

where m and n are constants is called a ***linear function***. Its graph is a straight line. The constants m and n are the ***slope*** and ***y-intercept*** of the line. Conversely, any straight line which is not vertical (i.e. not parallel to the y-axis) is the graph of a linear function. If you know two points (x_0, y_0) and (x_1, y_1) on the line, then then one can compute the slope m from the "rise-over-run" formula

$$m = \frac{y_1 - y_0}{x_1 - x_0}.$$

This formula actually contains a theorem from Euclidean geometry, namely it says that the ratio $(y_1 - y_0) : (x_1 - x_0)$ is the same for every pair of points (x_0, y_0) and (x_1, y_1) that you could pick on the line.

1.2.4 Domain and "biggest possible domain."

In this course we will usually not be careful about specifying the domain of the function. When this happens the domain is understood to be the set of all x for which the rule which tells you how to compute $f(x)$ is meaningful. For instance, if we say that h is the function

$$h(x) = \sqrt{x}$$

then the domain of h is understood to be the set of all nonnegative real numbers

$$\text{domain of } h = [0, \infty)$$

since $\sqrt{x}$ is well-defined for all $x \geq 0$ and undefined for $x < 0$.

A systematic way of finding the domain and range of a function for which you are only given a formula is as follows:

- The domain of f consists of all x for which $f(x)$ is well-defined ("makes sense")
- The range of f consists of all y for which you can solve the equation $f(x) = y$.

1.2.5 Example – find the domain and range of $f(x) = 1/x^2$.

The expression $1/x^2$ can be computed for all real numbers x except $x = 0$ since this leads to division by zero. Hence the domain of the function $f(x) = 1/x^2$ is

$$\text{"all real numbers except 0"} = \big\{x \mid x \neq 0\big\} = (-\infty, 0) \cup (0, \infty).$$

To find the range we ask "for which y can we solve the equation $y = f(x)$ for x," i.e. we for which y can you solve $y = 1/x^2$ for x?

If $y = 1/x^2$ then we must have $x^2 = 1/y$, so first of all, since we have to divide by y, y can't be zero. Furthermore, $1/y = x^2$ says that y must be positive. On the other hand, if $y > 0$ then $y = 1/x^2$ has a solution (in fact two solutions), namely $x = \pm 1/\sqrt{y}$. This shows that the range of f is

$$\text{"all positive real numbers"} = \{x \mid x > 0\} = (0, \infty).$$

1.2.6 Functions in "real life."

One can describe the motion of an object using a function. If some object is moving along a straight line, then you can define the following function: Let $x(t)$ be the distance from the object to a fixed marker on the line, at the time t. Here the domain of the function is the set of all times t for which we know the position of the object, and the rule is

Given t, measure the distance between the object and the marker at time t.

There are many examples of this kind. For instance, a biologist could describe the growth of a cell by defining $m(t)$ to be the mass of the cell at time t (measured since the birth of the cell). Here the domain is the interval $[0, T]$, where T is the life time of the cell, and the rule that describes the function is

Given t, weigh the cell at time t.

1.2.7 The Vertical Line Property.

Generally speaking graphs of functions are curves in the plane but they distinguish themselves from arbitrary curves by the way they intersect vertical lines: ***The graph of a function cannot intersect a vertical line "$x =$ constant" in more than one point.*** The reason why this is true is very simple: if two points lie on a vertical line, then they have the same x coordinate, so if they also lie on the graph of a function f, then their y-coordinates must also be equal, namely $f(x)$.

1.2.8 Examples.

The graph of $f(x) = x^3 - x$ "goes up and down," and, even though it intersects several horizontal lines in more than one point, it intersects ***every*** vertical line in exactly one point.

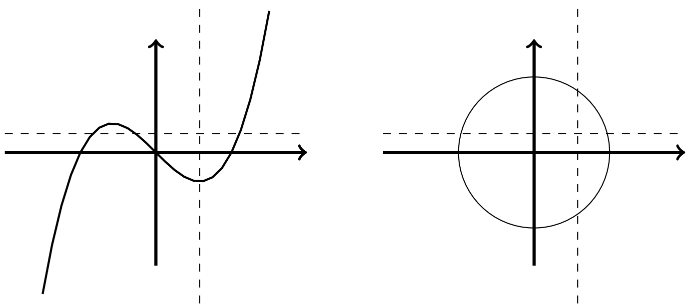


Figure 1.5: The graph of $y = x^3 - x$ fails the "horizontal line test," but it passes the "vertical line test." The circle fails both tests.

The collection of points determined by the equation $x^2 + y^2 = 1$ is a circle. It is not the graph of a function since the vertical line $x = 0$ (the y-axis) intersects the graph in two points $P_1(0, 1)$ and $P_2(0, -1)$. See Figure 1.6.

1.3 Implicit functions

For many functions the rule which tells you how to compute it is not an explicit formula, but instead an equation which you still must solve. A function which is defined in this way is called an "implicit function."

1.3.1 Example.

One can define a function f by saying that for each x the value of $f(x)$ is the solution y of the equation

$$x^2 + 2y - 3 = 0.$$

In this example you can solve the equation for y,

$$y = \frac{3 - x^2}{2}.$$

Thus we see that the function we have defined is $f(x) = (3 - x^2)/2$.
Here we have two definitions of the same function, namely

(i) "$y = f(x)$ is defined by $x^2 + 2y - 3 = 0$," and

(ii) "f is defined by $f(x) = (3 - x^2)/2$."

The first definition is the implicit definition, the second is explicit. You see that with an "implicit function" it isn't the function itself, but rather the way it was defined that's implicit.

1.3.2 Another example: domain of an implicitly defined function.

Define g by saying that for any x the value $y = g(x)$ is the solution of

$$x^2 + xy - 3 = 0.$$

Just as in the previous example one can then solve for y, and one finds that

$$g(x) = y = \frac{3 - x^2}{x}.$$

Unlike the previous example this formula does not make sense when $x = 0$, and indeed, for $x = 0$ our rule for g says that $g(0) = y$ is the solution of

$$0^2 + 0 \cdot y - 3 = 0, \text{ i.e. } y \text{ is the solution of } 3 = 0.$$

That equation has no solution and hence $x = 0$ does not belong to the domain of our function g.

1.3.3 Example: the equation alone does not determine the function.

Define $y = h(x)$ to be the solution of

$$x^2 + y^2 = 1.$$

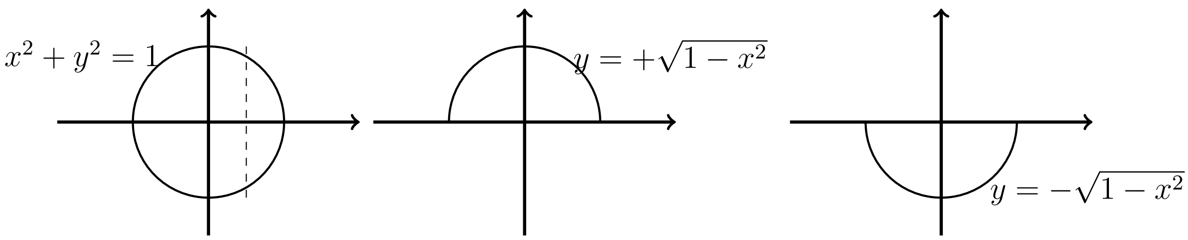


Figure 1.6: The circle determined by $x^2+y^2=1$ is not the graph of a function, but it contains the graphs of the two functions $h_1(x)=\sqrt{1-x^2}$ and $h_2(x)=-\sqrt{1-x^2}$.

If $x>1$ or $x<-1$ then $x^2>1$ and there is no solution, so $h(x)$ is at most defined when $-1\le x\le 1$. But when $-1<x<1$ there is another problem: not only does the equation have a solution, but it even has two solutions:

$$x^2+y^2=1 \iff y=\sqrt{1-x^2} \text{ or } y=-\sqrt{1-x^2}.$$

The rule which defines a function must be unambiguous, and since we have not specified which of these two solutions is $h(x)$ the function is not defined for $-1<x<1$.

One can fix this by making a choice, but there are many possible choices. Here are three possibilities:

$$h_1(x) = \text{the nonnegative solution } y \text{ of } x^2+y^2=1$$
$$h_2(x) = \text{the nonpositive solution } y \text{ of } x^2+y^2=1$$
$$h_3(x) = \begin{cases} h_1(x) & \text{when } x<0 \\ h_2(x) & \text{when } x\ge 0 \end{cases}$$

1.3.4 Why use implicit functions?

In all the examples we have done so far we could replace the implicit description of the function with an explicit formula. This is not always possible or if it is possible the implicit description is much simpler than the explicit formula. For instance, you can define a function f by saying that $y=f(x)$ if and only if

$$y^3+3y+2x=0. \tag{1.1}$$

This means that the recipe for computing $f(x)$ for any given x is "solve the equation $y^3+3y+2x=0$." E.g. to compute $f(0)$ you set $x=0$ and solve $y^3+3y=0$. The only solution is $y=0$, so $f(0)=0$. To compute $f(1)$ you have to solve $y^3+3y+2\cdot 1=0$, and if you're lucky you see that $y=-1$ is the solution, and $f(1)=-1$.

In general, no matter what x is, the equation (1.1) turns out to have exactly one solution y (which depends on x, this is how you get the function f). Solving (1.1) is not easy. In the early 1500s Cardano and Tartaglia discovered a formula[1] for the solution. Here it is:

$$y=f(x)=\sqrt[3]{-x+\sqrt{1+x^2}}-\sqrt[3]{x+\sqrt{1+x^2}}.$$

[1]To see the solution and its history visit

http://www.gap-system.org/~history/HistTopics/Quadratic_etc_equations.html

The implicit description looks a lot simpler, and when we try to differentiate this function later on, it will be much easier to use "implicit differentiation" than to use the Cardano-Tartaglia formula directly.

1.4 Inverse functions.

If you have a function f, then you can try to define a new function f^{-1}, the so-called ***inverse function of*** f**,** by the following prescription:

$$\text{For any given } x \text{ we say that } y = f^{-1}(x) \text{ if } y \text{ is the solution to the equation } f(y) = x. \tag{1.2}$$

So to find $y = f^{-1}(x)$ you solve the equation $x = f(y)$. If this is to define a function then the prescription (1.2) must be unambiguous and the equation $f(y) = x$ has to have a solution and cannot have more than one solution.

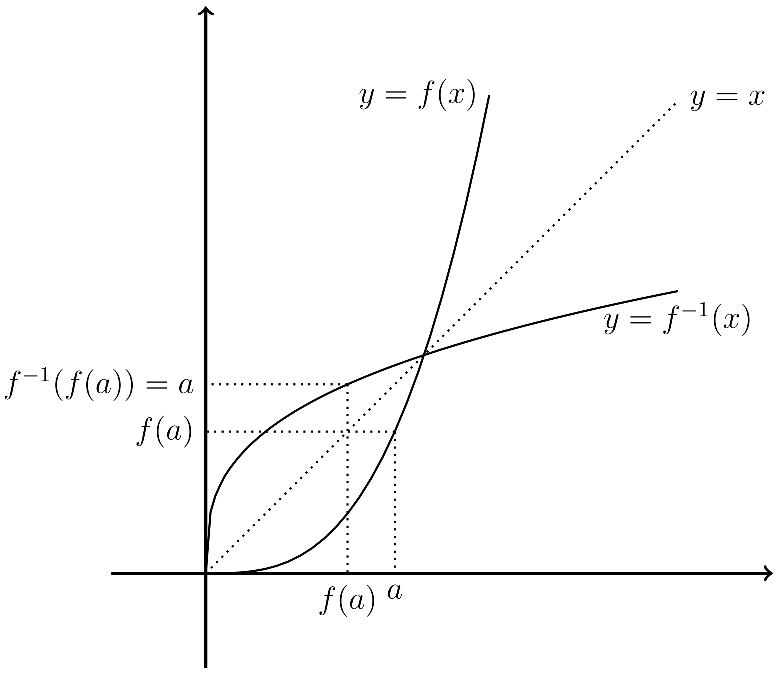


Figure 1.7: The graph of a function and its inverse are mirror images of each other.

1.4.1 Examples.

Consider the function f with $f(x) = 2x + 3$. Then the equation $f(y) = x$ works out to be

$$2y + 3 = x$$

and this has the solution

$$y = \frac{x-3}{2}.$$

So $f^{-1}(x)$ is defined for all x, and it is given by $f^{-1}(x) = (x - 3)/2$.

Next we consider the function $g(x) = x^2$ with domain all positive real numbers. To see for which x the inverse $g^{-1}(x)$ is defined we try to solve the equation $g(y) = x$, i.e. we try to solve $y^2 = x$. If $x < 0$ then this equation has no solutions since $y^{\geq}0$ for all y. But if $x \geq 0$ then $y^{=}x$ does have a solution, namely $y = \sqrt{x}$.

So we see that $g^{-1}(x)$ is defined for all nonnegative real numbers x, and that it is given by $g^{-1}(x) = \sqrt{x}$.

1.4.2 Inverse trigonometric functions.

The familiar trigonometric functions Sine, Cosine and Tangent have inverses which are called arcsine, arccosine and arctangent.

$y = f(x)$		$x = f^{-1}(y)$	
$y = \sin x$	$(-\pi/2 \leq x \leq \pi/2)$	$x = \arcsin(y)$	$(-1 \leq y \leq 1)$
$y = \cos x$	$(0 \leq x \leq \pi)$	$x = \arccos(y)$	$(-1 \leq y \leq 1)$
$y = \tan x$	$(-\pi/2 < x < \pi/2)$	$x = \arctan(y)$	

The notations $\arcsin y = \sin^{-1} y$, $\arccos x = \cos^{-1} x$, and $\arctan u = \tan^{-1} u$ are also commonly used for the inverse trigonometric functions. We will avoid the $\sin^{-1} y$ notation because it is ambiguous. Namely, everybody writes the square of $\sin y$ as

$$\left(\sin y\right)^2 = \sin^2 y.$$

Replacing the 2's by -1's would lead to

$$\arcsin y = \sin^{-1} y \overset{?!?}{=} \left(\sin y\right)^{-1} = \frac{1}{\sin y}, \quad \textbf{which is not true!}$$

To reinforce your understanding of what an inverse function is consider watching this YouTube by Mario

1.5 PROBLEMS

NUMBERS

1. What is the 2007th digit after the period in the expansion of $\frac{1}{7}$? †378

2. Which of the following fractions have finite decimal expansions?

$$a = \frac{2}{3}, \quad b = \frac{3}{25}, \quad c = \frac{276937}{15625}.$$

3. Draw the following sets of real numbers. Each of these sets is the union of one or more intervals. Find those intervals. Which of thee sets are finite?

$\mathcal{A} = \left\{x \mid x^2 - 3x + 2 \leq 0\right\}$
$\mathcal{B} = \left\{x \mid x^2 - 3x + 2 \geq 0\right\}$

$\mathcal{C} = \{x \mid x^2 - 3x > 3\}$
$\mathcal{D} = \{x \mid x^2 - 5 > 2x\}$
$\mathcal{E} = \{t \mid t^2 - 3t + 2 \le 0\}$
$\mathcal{F} = \{\alpha \mid \alpha^2 - 3\alpha + 2 \ge 0\}$
$\mathcal{G} = (0, 1) \cup (5, 7]$
$\mathcal{H} = \big(\{1\} \cup \{2, 3\}\big) \cap (0, 2\sqrt{2})$
$\mathcal{Q} = \{\theta \mid \sin\theta = \frac{1}{2}\}$
$\mathcal{R} = \{\varphi \mid \cos\varphi > 0\}$

4. Suppose $\mathcal{A}$ and $\mathcal{B}$ are intervals. Is it always true that $\mathcal{A} \cap \mathcal{B}$ is an interval? How about $\mathcal{A} \cup \mathcal{B}$?

5. Consider the sets

$$\mathcal{M} = \{x \mid x > 0\} \text{ and } \mathcal{N} = \{y \mid y > 0\}.$$

Are these sets the same?

6. Write the numbers

$$x = 0.3131313131\ldots, \quad y = 0.273273273273\ldots$$
$$\text{and } z = 0.21541541541541541\ldots$$

as fractions (i.e. write them as $\frac{m}{n}$, specifying m and n.)
(Hint: show that $100x = x + 31$. A similar trick works for y, but z is a little harder.)
†378

7. Is the number whose decimal expansion after the period consists only of nines, i.e.

$$x = 0.99999999999999999\ldots$$

an integer?

FUNCTIONS

8. The functions f and g are defined by

$$f(x) = x^2 \text{ and } g(s) = s^2.$$

Are f and g the same functions or are they different? †378

9. Find a formula for the function f which is defined by

$$y = f(x) \iff x^2y + y = 7.$$

What is the domain of f?

10. Find a formula for the function f which is defined by

$$y = f(x) \iff x^2y - y = 6.$$

What is the domain of f?

11. Let f be the function defined by $y = f(x) \iff y$ is the largest solution of

$$y^2 = 3x^2 - 2xy.$$

Find a formula for f. What are the domain and range of f?

12. Find a formula for the function f which is defined by

$$y = f(x) \iff 2x + 2xy + y^2 = 5 \text{ and } y > -x.$$

Find the domain of f.

13. Use a calculator to compute $f(1.2)$ in three decimals where f is the implicitly defined function from §1.3.4. (There are (at least) two different ways of finding $f(1.2)$)

14. **(a)** True or false:
for all x one has $\sin\big(\arcsin x\big) = x$?
(b) True or false:
for all x one has $\arcsin\big(\sin x\big) = x$? †378

15. On a graphing calculator plot the graphs of the following functions, and explain the results. (Hint: first do the previous exercise.)

$$\begin{aligned}
f(x) &= \arcsin(\sin x), \quad -2\pi \le x \le 2\pi \\
g(x) &= \arcsin(x) + \arccos(x), \quad 0 \le x \le 1 \\
h(x) &= \arctan \frac{\sin x}{\cos x}, \quad |x| < \pi/2 \\
k(x) &= \arctan \frac{\cos x}{\sin x}, \quad |x| < \pi/2 \\
l(x) &= \arcsin(\cos x), \quad -\pi \le x \le \pi \\
m(x) &= \cos(\arcsin x), \quad -1 \le x \le 1
\end{aligned}$$

16. Find the inverse of the function f which is given by $f(x) = \sin x$ and ***whose domain is*** $\pi \le x \le 2\pi$. Sketch the graphs of both f and f^{-1}.

17. Find a number a such that the function $f(x) = \sin(x+\pi/4)$ with domain $a \le x \le a+\pi$ has an inverse. Give a formula for $f^{-1}(x)$ using the arcsine function.

18. Draw the graph of the function h_3 from §1.3.3.

19. A function f is given which satisfies $f(2x+3) = x^2$ for all real numbers x.
Compute

(a) $f(0)$	(b) $f(3)$	(c) $f(x)$
(d) $f(y)$	(e) $f(f(2))$	

where x and y are arbitrary real numbers.
What are the range and domain of f?

20. A function f is given which satisfies $f\big(\frac{1}{x+1}\big) = 2x - 12$ for all real numbers x.

Compute

(a) $f(1)$ (b) $f(0)$ (c) $f(x)$

(d) $f(t)$ (e) $f(f(2))$

where x and t are arbitrary real numbers.

What are the range and domain of f?

21. Does there exist a function f which satisfies $f(x^2) = x + 1$ for ***all*** real numbers x?

22. Explain how you "complete the square" in a quadratic expression like $ax^2 + bx$.

23. Find the range of the following functions:

$$f(x) = 2x^2 + 3$$
$$g(x) = -2x^2 + 4x$$
$$h(x) = 4x + x^2$$
$$k(x) = 4\sin x + \sin^2 x$$
$$\ell(x) = 1/(1 + x^2)$$
$$m(x) = 1/(3 + 2x + x^2).$$

24. For each real number a we define a line ℓ_a with equation $y = ax + a^2$.

(a) Draw the lines corresponding to $a = -2, -1, -\frac{1}{2}, 0, \frac{1}{2}, 1, 2$.

(b) Does the point with coordinates $(3, 2)$ lie on one or more of the lines ℓ_a (where a can be any number, not just the five values from part (a))? If so, for which values of a does $(3, 2)$ lie on ℓ_a?

(c) Which points in the plane lie on at least one of the lines ℓ_a?.

25. For which values of m and n does the graph of $f(x) = mx + n$ intersect the graph of $g(x) = 1/x$ in exactly one point and also contain the point $(-1, 1)$?

26. For which values of m and n does the graph of $f(x) = mx + n$ ***not*** intersect the graph of $g(x) = 1/x$?

Chapter 2

Derivatives

To work with derivatives you have to know what a limit is, but to motivate why we are going to study limits let's first look at the two classical problems that gave rise to the notion of a derivative: the tangent to a curve, and the instantaneous velocity of a moving object.

2.1 The tangent to a curve

Suppose you have a function $y = f(x)$ and you draw its graph. If you want to find the tangent to the graph of f at some given point on the graph of f, how would you do that?

Let P be the point on the graph at which want to draw the tangent. If you are making a real paper and ink drawing you would take a ruler, make sure it goes through P and then turn it until it doesn't cross the graph anywhere else.

If you are using equations to describe the curve and lines, then you could pick a point Q on the graph and construct the line through P and Q ("construct" means "find an equation for"). This line is called a "secant," and it is of course not the tangent that you're looking for. But if you choose Q to be very close to P then the secant will be close to the tangent.

So this is our recipe for constructing the tangent through P: pick another point Q on the graph, find the line through P and Q, and see what happens to this line as you take Q closer and closer to P. The resulting secants will then get closer and closer to some line, and that line is the tangent.

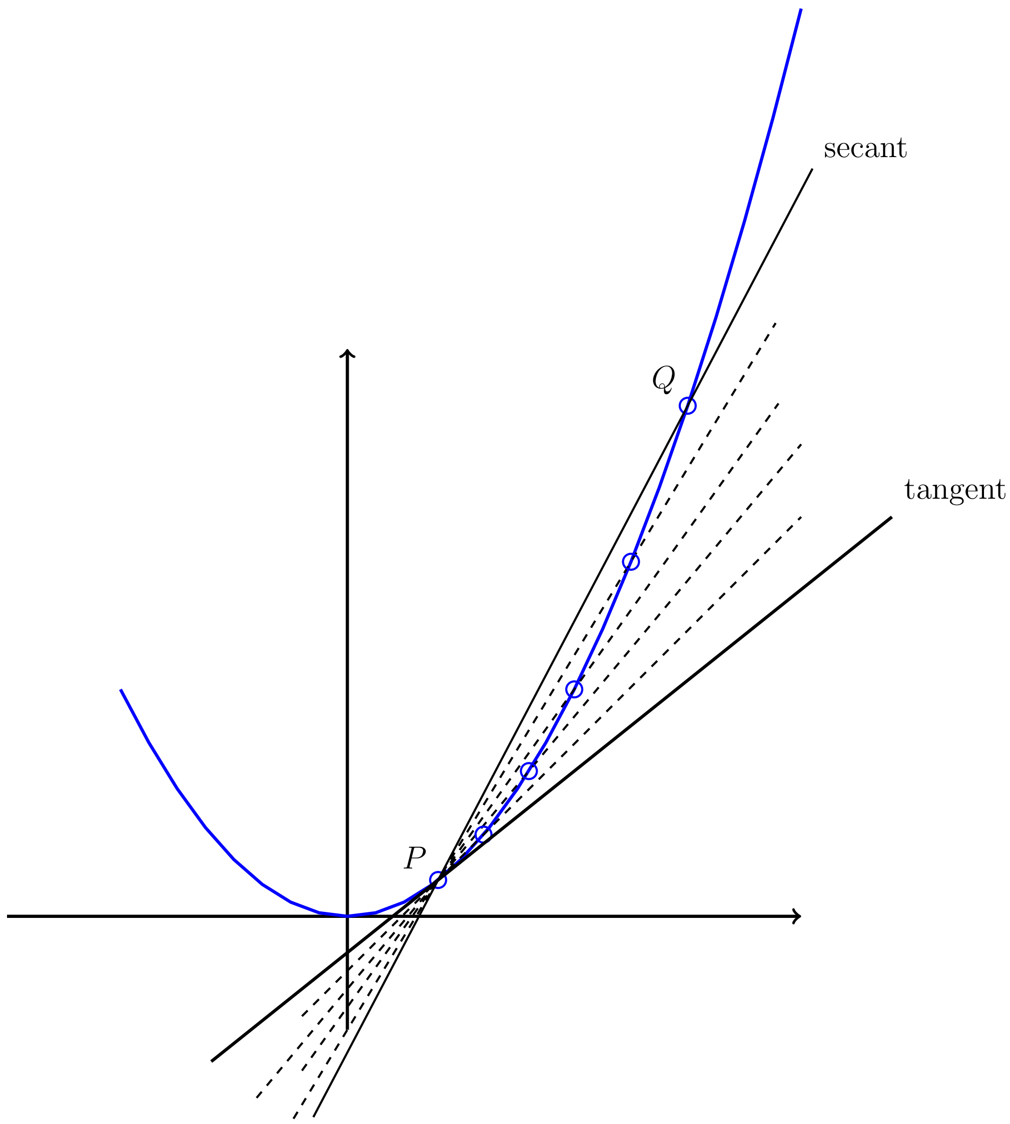


Figure 2.1: Constructing the tangent by letting $Q \to P$

We'll write this in formulas in a moment, but first let's worry about how close Q should be to P. We can't set Q equal to P, because then P and Q don't determine a line (you need ***two*** points to determine a line). If you choose Q different from P then you don't get the tangent, but at best something that is "close" to it. Some people have suggested that one should take Q "infinitely close" to P, but it isn't clear what that would mean. The concept of a limit is meant to solve this confusing problem.

2.2 An example – tangent to a parabola

To make things more concrete, suppose that the function we had was $f(x) = x^2$, and that the point was $(1, 1)$. The graph of f is of course a parabola.

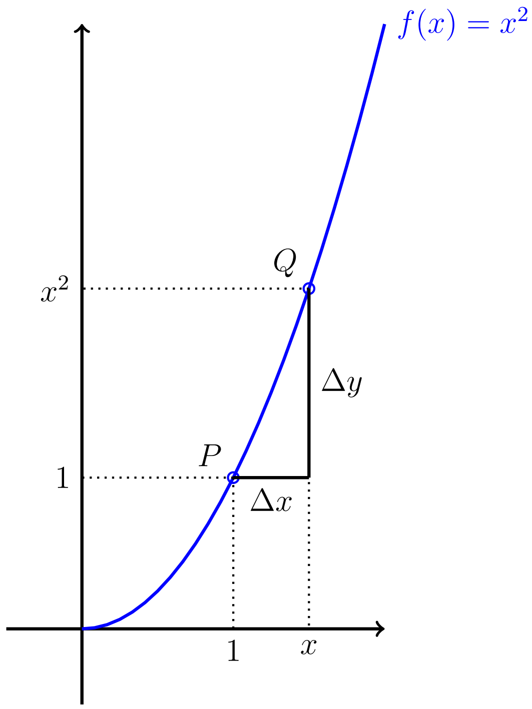


Figure 2.2: finding the slope of a tangent to a parabola by letting $Q \to P$

Any line through the point $P(1,1)$ has equation

$$y - 1 = m(x - 1)$$

where m is the slope of the line. So instead of finding the equation of the secant and tangent lines we will find their slopes.

Let Q be the other point on the parabola, with coordinates (x, x^2). We can "move Q around on the graph" by changing x. Whatever x we choose, it must be different from 1, for otherwise P and Q would be the same point. What we want to find out is how the line through P and Q changes if x is changed (and in particular, if x is chosen very close to a). Now, as one changes x one thing stays the same, namely, the secant still goes through P. So to describe the secant we only need to know its slope. By the "rise over run" formula, the slope of the secant line joining P and Q is

$$m_{PQ} = \frac{\Delta y}{\Delta x} \quad \text{where} \quad \Delta y = x^2 - 1 \quad \text{and} \quad \Delta x = x - 1.$$

By factoring $x^2 - 1$ we can rewrite the formula for the slope as follows

$$m_{PQ} = \frac{\Delta y}{\Delta x} = \frac{x^2 - 1}{x - 1} = \frac{(x-1)(x+1)}{x - 1} = x + 1. \tag{2.1}$$

As x gets closer to 1, the slope m_{PQ}, being $x + 1$, gets closer to the value $1 + 1 = 2$. We say that

the limit of the slope m_{PQ} as Q approaches P is 2.

In symbols,

$$\lim_{Q \to P} m_{PQ} = 2,$$

or, since Q approaching P is the same as x approaching 1,

$$\lim_{x \to 1} m_{PQ} = 2. \tag{2.2}$$

So we find that the tangent line to the parabola $y = x^2$ at the point $(1, 1)$ has equation

$$y - 1 = 2(x - 1), \text{ i.e. } y = 2x - 1.$$

A warning: you cannot substitute $x = 1$ in equation (2.1) to get (2.2) even though it looks like that's what we did. The reason why you can't do that is that when $x = 1$ the point Q coincides with the point P so "the line through P and Q" is not defined; also, if $x = 1$ then $\Delta x = \Delta y = 0$ so that the rise-over-run formula for the slope gives

$$m_{PQ} = \frac{\Delta x}{\Delta y} = \frac{0}{0} = \text{ undefined.}$$

It is only after the algebra trick in (2.1) that setting $x = 1$ gives something that is well defined. But if the intermediate steps leading to $m_{PQ} = x + 1$ aren't valid for $x = 1$ why should the final result mean anything for $x = 1$?

Something more complicated has happened. We did a calculation which is valid for all $x \neq 1$, and later looked at what happens if x gets "very close to 1." This is the concept of a limit and we'll study it in more detail later in this section, but first another example.

2.3 Instantaneous velocity

If you try to define "instantaneous velocity" you will again end up trying to divide zero by zero. Here is how it goes: When you are driving in your car the speedometer tells you how fast your are going, i.e. what your velocity is. What is this velocity? What does it mean if the speedometer says "50mph"?

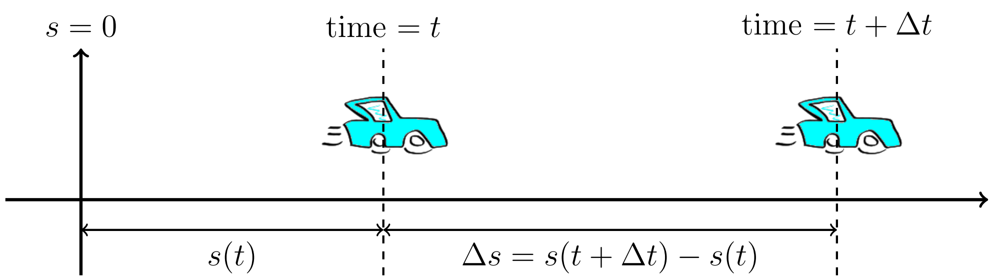


Figure 2.3: calculating the instantaneous velocity of a car

We all know what ***average velocity*** is. Namely, if it takes you two hours to cover 100 miles, then your average velocity was

$$\frac{\text{distance traveled}}{\text{time it took}} = 50 \text{ miles per hour.}$$

This is not the number the speedometer provides you – it doesn't wait two hours, measure how far you went and compute distance/time. If the speedometer in your car tells you that you are driving 50mph, then that should be your velocity ***at the moment*** that you look at your speedometer, i.e. "distance traveled over time it took" at the moment you look at the speedometer. But during the moment you look at your speedometer no time goes by (because a moment has no length) and you didn't cover any distance, so your velocity at that moment is $\frac{0}{0}$, i.e. undefined. Your velocity at ***any*** moment is undefined. But then what is the speedometer telling you?

To put all this into formulas we need to introduce some notation. Let t be the time (in hours) that has passed since we got onto the road, and let $s(t)$ be the distance we have covered since then.

Instead of trying to find the velocity exactly at time t, we find a formula for the average velocity during some (short) time interval beginning at time t. We'll write Δt for the length of the time interval.

At time t we have traveled $s(t)$ miles. A little later, at time $t + \Delta t$ we have traveled $s(t+\Delta t)$. Therefore during the time interval from t to $t+\Delta t$ we have moved $s(t+\Delta t)-s(t)$ miles. Our average velocity in that time interval is therefore

$$\frac{s(t + \Delta t) - s(t)}{\Delta t} \quad \text{miles per hour.}$$

The shorter you make the time interval, i.e. the smaller you choose Δt, the closer this number should be to the instantaneous velocity at time t.

So we have the following formula (definition, really) for the velocity at time t

$$v(t) = \lim_{\Delta t \to 0} \frac{s(t + \Delta t) - s(t)}{\Delta t}. \tag{2.3}$$

The viewing of YouTube by 3Blue1Brown is recommended to reinforce these ideas.

2.4 Rates of change

The two previous examples have much in common. If we ignore all the details about geometry, graphs, highways and motion, the following happened in both examples:

We had a function $y = f(x)$, and we wanted to know how much $f(x)$ changes if x changes. If you change x to $x + \Delta x$, then y will change from $f(x)$ to $f(x + \Delta x)$. The change in y is therefore

$$\Delta y = f(x + \Delta x) - f(x),$$

and the average rate of change is

$$\frac{\Delta y}{\Delta x} = \frac{f(x + \Delta x) - f(x)}{\Delta x}. \tag{2.4}$$

This is the average rate of change of f over the interval from x to $x + \Delta x$. To define ***the rate of change of the function*** f ***at*** x we let the length Δx of the interval become smaller and smaller, in the hope that the average rate of change over the shorter and shorter time intervals will get closer and closer to some number. If that happens then that "limiting number" is called the rate of change of f at x, or, the ***derivative*** of f at x. It is written as

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}. \tag{2.5}$$

Derivatives and what you can do with them are what the calculus is all about. To see further how *tangent lines* and *derivatives* are linked consider watching this YouTube by Socratica .

The description we just went through shows that to understand what a derivative is you need to know what a limit is. In the next chapter we'll study limits so that we get a less vague understanding of formulas like (2.5).

2.5 Examples of rates of change

2.5.1 Acceleration as the rate at which velocity changes.

As you are driving in your car your velocity does not stay constant, it changes with time. Suppose $v(t)$ is your velocity at time t (measured in miles per hour). You could try to figure out how fast your velocity is changing by measuring it at one moment in time (you get $v(t)$), then measuring it a little later (you get $v(\Delta t)$)). You conclude that your velocity increased by $\Delta v = v(t+\Delta t) - v(t)$ during a time interval of length Δt, and hence

$$\left\{ \begin{array}{c} \text{average rate at which} \\ \text{your velocity changed} \end{array} \right\} = \frac{\Delta v}{\Delta \tau} = \frac{v(t + \Delta t) - v(t)}{\Delta t}.$$

This rate of change is called your *average acceleration* (over the time interval from t to $t+\Delta t$). Your *instantaneous acceleration* at time t is the limit of your average acceleration as you make the time interval shorter and shorter:

$$\{\text{acceleration at time } t\} = a = \lim_{\Delta t \to 0} \frac{v(t + \Delta t) - v(t)}{\Delta t}.$$

th the average and instantaneous accelerations are measured in "miles per hour per hour," i.e. in

$$(\text{mi/h})/\text{h} = \text{mi/h}^2.$$

Or, if you had measured distances in meters and time in seconds then velocities would be measured in meters per second, and acceleration in meters per second per second, which is the same as meters per second^2, i.e. "meters per squared second."

2.5.2 Reaction rates.

Think of a chemical reaction in which two substances A and B react to form AB_2 according to the reaction

$$A + 2B \longrightarrow AB_2.$$

If the reaction is taking place in a closed reactor, then the "amounts" of A and B will be decreasing, while the amount of AB_2 will increase. Chemists write [A] for the amount of “A” in the chemical reactor (measured in moles). Clearly [A] changes with time so it defines a *function*. We're mathematicians so we will write “[A](t)” for the number of moles of A present at time t.

To describe how fast the amount of A is changing we consider the derivative of [A] with respect to time, i.e.

$$[A]'(t) = \lim_{\Delta t \to 0} \frac{[\mathrm{A}](t+\Delta t) - [\mathrm{A}](t)}{\Delta t}.$$

This quantity is the rate of change of [A]. The notation “[A]$'(t)$” is really only used by calculus professors. If you open a paper on chemistry you will find that the derivative is written in LEIBNIZ notation:

$$\frac{d[\mathrm{A}]}{dt}$$

More on this in §4.1.2

How fast does the reaction take place? If you add more A or more B to the reactor then you would expect that the reaction would go faster, i.e. that more AB_2 is being produced per second. The law of *mass-action kinetics* from chemistry states this more precisely. For our particular reaction it would say that the rate at which A is consumed is given by

$$\frac{d[\mathrm{A}]}{dt} = k\,[\mathrm{A}]\,[\mathrm{B}]^2,$$

in which the constant k is called the *reaction constant.* It's a constant that you could try to measure by timing how fast the reaction goes.

Before attempting this sections problems the reader should consider viewing YouTube by AFmath .

2.6 PROBLEMS

RATES OF CHANGE

27. Repeat the reasoning in §2.2 to find the slope at the point $(\frac{1}{2}, \frac{1}{4})$, or more generally at any point (a, a^2) on the parabola with equation $y = x^2$.

28. Repeat the reasoning in §2.2 to find the slope at the point $(\frac{1}{2}, \frac{1}{8})$, or more generally at any point (a, a^3) on the curve with equation $y = x^3$.

29.

Should you trust your calculator?

Find the slope of the tangent to the parabola $y = x^2$ at the point $(\frac{1}{3}, \frac{1}{9})$ (You have already done this: see exercise 27).

Instead of doing the algebra you could try to compute the slope by using a calculator. This exercise is about how you do that and what happens if you try (too hard).

Compute $\frac{\Delta y}{\Delta x}$ for various values of Δx:

$$\Delta x = 0.1, 0.01, 0.001, 10^{-6}, 10^{-12}.$$

As you choose Δx smaller your computed $\frac{\Delta y}{\Delta x}$ ought to get closer to the actual slope. Use at least 10 decimals and organize your results in a table like this:

Δx	$f(a)$	$f(a+\Delta x)$	Δy	$\Delta y/\Delta x$
0.1	...	...	...	...
0.01	...	...	...	...
0.001	...	...	...	...
10^{-6}	...	...	...	...
10^{-12}	...	...	...	...

Look carefully at the ratios $\Delta y/\Delta x$. Do they look like they are converging to some number? Compare the values of $\frac{\Delta y}{\Delta x}$ with the true value you got in the beginning of this problem.

30. Simplify the algebraic expressions you get when you compute Δy and $\Delta y/\Delta x$ for the following functions

$$(a)\ y = x^2 - 2x + 1$$
$$(b)\ y = \frac{1}{x}$$
$$(c)\ y = 2^x$$

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31. Suppose that some quantity y is a function of some other quantity x, and suppose that y is a mass, i.e. y is measured in pounds, and x is a length, measured in feet. What units do the increments Δy and Δx, and the derivative dy/dx have? †379

32. A tank is filling with water. The volume (in gallons) of water in the tank at time t (seconds) is $V(t)$. What units does the derivative $V'(t)$ have? †379

33. Let $A(x)$ be the area of an equilateral triangle whose sides measure x inches.

(a) Show that $\frac{dA}{dx}$ has the units of a length.

(b) Which length does $\frac{dA}{dx}$ represent geometrically? [Hint: draw two equilateral triangles, one with side x and another with side $x + \Delta x$. Arrange the triangles so that they both have the origin as their lower left hand corner, and so there base is on the x-axis.] †379

34. Let $A(x)$ be the area of a square with side x, and let $L(x)$ be the perimeter of the square (sum of the lengths of all its sides). Using the familiar formulas for $A(x)$ and $L(x)$ show that $A'(x) = \frac{1}{2}L(x)$.

Give a geometric interpretation that explains why $\Delta A \approx \frac{1}{2}L(x)\Delta x$ for small Δx.

35. Let $A(r)$ be the area enclosed by a circle of radius r, and let $L(r)$ be the length of the circle. Show that $A'(r) = L(r)$. (Use the familiar formulas from geometry for the area and perimeter of a circle.)

36. Let $V(r)$ be the volume enclosed by a sphere of radius r, and let $S(r)$ be the its surface area. Show that $V'(r) = S(r)$. (Use the formulas $V(r) = \frac{4}{3}\pi r^3$ and $S(r) = 4\pi r^2$.) (To understand where $S(r) = 4\pi r^2$ comes from consider watching YouTube by 3Blue1Brown .)

Chapter 3

Limits and Continuous Functions

3.1 Informal definition of limits

While it is easy to define precisely in a few words what a square root is ($\sqrt{a}$ is the positive number whose square is a) the definition of the limit of a function runs over several terse lines, and most people don't find it very enlightening when they first see it. (See §3.2.) So we postpone this for a while and fine tune our intuition for another page.

3.1.1 Definition of the limit (1st attempt).

If f is some function then

$$\lim_{x\to a} f(x) = L$$

is read "the limit of $f(x)$ as x approaches a is L." It means that if you choose values of x which are close ***but not equal*** to a, then $f(x)$ will be close to the value L; moreover, $f(x)$ gets closer and closer to L as x gets closer and closer to a.

The following alternative notation is sometimes used

$$f(x) \to L \quad \text{as} \quad x \to a;$$

(read "$f(x)$ approaches L as x approaches a" or "$f(x)$ goes to L is x goes to a".)

3.1.2 Example.

If $f(x) = x + 3$ then

$$\lim_{x\to 4} f(x) = 7,$$

is true, because if you substitute numbers x close to 4 in $f(x) = x + 3$ the result will be close to 7.

3.1.3 Example: substituting numbers to guess a limit.

What (if anything) is

$$\lim_{x \to 2} \frac{x^2 - 2x}{x^2 - 4}?$$

Here $f(x) = (x^2 - 2x)/(x^2 - 4)$ and $a = 2$.
We first try to substitute $x = 2$, but this leads to

$$f(2) = \frac{2^2 - 2 \cdot 2}{2^2 - 4} = \frac{0}{0}$$

which does not exist. Next we try to substitute values of x close but not equal to 2. Table 3.1 suggests that $f(x)$ approaches 0.5.

x	$f(x)$	x	$g(x)$
3.000000	0.600000	1.000000	1.009990
2.500000	0.555556	0.500000	1.009980
2.100000	0.512195	0.100000	1.009899
2.010000	0.501247	0.010000	1.008991
2.001000	0.500125	0.001000	1.000000

Table 3.1: Finding limits by substituting values of x "close to a." (Values of $f(x)$ and $g(x)$ rounded to six decimals.)

3.1.4 Example: Substituting numbers can suggest the wrong answer.

The previous example shows that our first definition of "limit" is not very precise, because it says "x close to a," but how close is close enough? Suppose we had taken the function

$$g(x) = \frac{101\,000x}{100\,000x + 1}$$

and we had asked for the limit $\lim_{x \to 0} g(x)$.
Then substitution of some "small values of x" could lead us to believe that the limit is 1.000 .... Only when you substitute even smaller values do you find that the limit is 0 (zero)!
See also problem 29.

3.2 The formal, authoritative, definition of limit

The informal description of the limit uses phrases like "closer and closer" and "really very small." In the end we don't really know what they mean, although they are suggestive. "Fortunately" there is a good definition, i.e. one which is unambiguous and can be used to settle any dispute about the question of whether $\lim_{x\to a} f(x)$ equals some number L or not. Here is the definition. It takes a while to digest, so read it once, look at the examples, do a few exercises, read the definition again. Go on to the next sections. Throughout this text the student is urged to come back to this section and read it again.

Definition 3.2.1. Definition of $\lim_{x\to a} f(x) = L$.
We say that L is the limit of $f(x)$ as $x \to a$, if

1. $f(x)$ need not be defined at $x = a$, but it must be defined for all other x in some interval which contains a.

2. for every $\varepsilon > 0$ one can find a $\delta > 0$ such that for all x in the domain of f one has

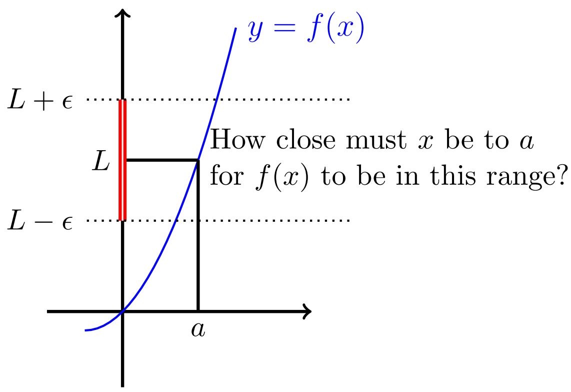
$$|x - a| < \delta \text{ implies } |f(x) - L| < \varepsilon. \tag{3.1}$$

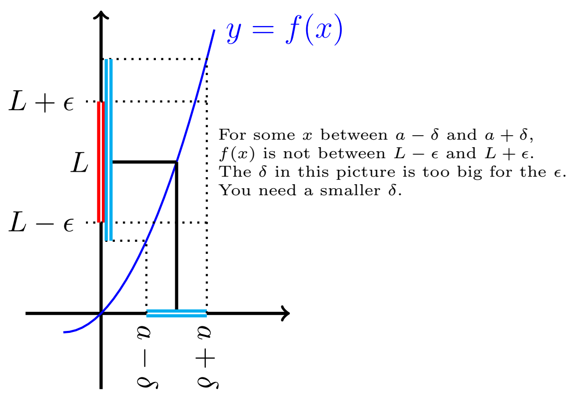
3.2.1 Why the absolute values?

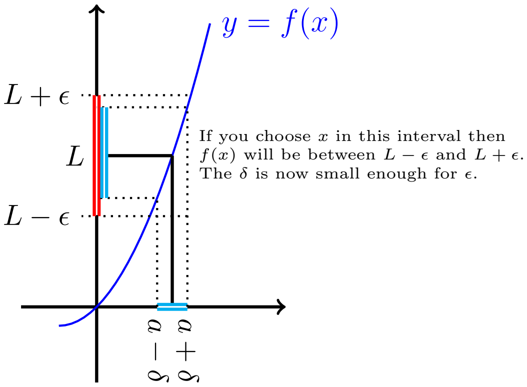
The quantity $|x - y|$ is the distance between the points x and y on the number line, and one can measure how close x is to y by calculating $|x - y|$. The inequality $|x - y| < \delta$ says that "the distance between x and y is less than δ," or that "x and y are closer than δ."

3.2.2 What are ε and δ?

The quantity ε is how close you would like $f(x)$ to be to its limit L; the quantity δ is how close you have to choose x to a to achieve this. To prove that $\lim_{x\to a} f(x) = L$ you must assume that someone has given you an unknown $\varepsilon > 0$, and then find a postive δ for which (3.1) holds. The δ you find will depend on ε.







3.2.3 examples

3.2.3.1 Show that $\lim_{x\to 5} 2x + 1 = 11$.

We have $f(x) = 2x+1$, $a = 5$ and $L = 11$, and the question we must answer is "how close should x be to 5 if want to be sure that $f(x) = 2x + 1$ differs less than ε from $L = 11$?"
To figure this out we try to get an idea of how big $|f(x) - L|$ is:

$$|f(x) - L| = \big|(2x + 1) - 11\big| = |2x - 10| = 2 \cdot |x - 5| = 2 \cdot |x - a|.$$

So, if $2|x - a| < \varepsilon$ then we have $|f(x) - L| < \varepsilon$, i.e.

$$\text{if } |x - a| < \tfrac{1}{2}\varepsilon \text{ then } |f(x) - L| < \varepsilon.$$

We can therefore choose $\delta = \frac{1}{2}\varepsilon$. No matter what $\varepsilon > 0$ we are given our δ will also be positive, and if $|x - 5| < \delta$ then we can guarantee $|(2x + 1) - 11| < \varepsilon$. That shows that $\lim_{x\to 5} 2x + 1 = 11$.

3.2.3.2 Show that $\lim_{x\to 1} x^2 = 1$

We have $f(x) = x^2$, $a = 1$, $L = 1$, and again the question is, "how small should $|x - 1|$ be to guarantee $|x^2 - 1| < \varepsilon$?"
We begin by estimating the difference $|x^2 - 1|$

$$|x^2 - 1| = |(x - 1)(x + 1)| = |x + 1| \cdot |x - 1|.$$

As x approaches 1 the factor $|x - 1|$ becomes small, and if the other factor $|x + 1|$ were a constant (e.g. 2 as in the previous example) then we could find δ as before, by dividing ε by that constant.
Here is a trick that allows you to replace the factor $|x + 1|$ with a constant. We hereby agree *that we always choose our δ so that $\delta \le 1$.* If we do that, then we will always have

$$|x - 1| < \delta \le 1, \text{i.e. } |x - 1| < 1,$$

and x will always be beween 0 and 2. Therefore

$$|x^2 - 1| = |x + 1| \cdot |x - 1| < 3|x - 1|.$$

If we now want to be sure that $|x^2 - 1| < \varepsilon$, then this calculation shows that we should require $3|x - 1| < \varepsilon$, i.e. $|x - 1| < \frac{1}{3}\varepsilon$. So we should choose $\delta \le \frac{1}{3}\varepsilon$. We must also live up to our promise never to choose $\delta > 1$, so if we are handed an ε for which $\frac{1}{3}\varepsilon > 1$, then we choose $\delta = 1$ instead of $\delta = \frac{1}{3}\varepsilon$. To summarize, we are going to choose

$$\delta = \text{the smaller of 1 and } \frac{1}{3}\varepsilon.$$

We have shown that if you choose δ this way, then $|x - 1| < \delta$ implies $|x^2 - 1| < \varepsilon$, no matter what $\varepsilon > 0$ is.

The expression "the smaller of a and b" shows up often, and is abbreviated to $\min(a, b)$. We could therefore say that in this problem we will choose δ to be

$$\delta = \min\big(1, \tfrac{1}{3}\varepsilon\big).$$

To get a better understanding of this process consider viewing this YouTube by rootmath .

3.2.3.3 Show that $\lim_{x\to 4} 1/x = 1/4$.

Solution: We apply the definition with $a = 4$, $L = 1/4$ and $f(x) = 1/x$. Thus, for any $\varepsilon > 0$ we try to show that if $|x - 4|$ is small enough then one has $|f(x) - 1/4| < \varepsilon$.

We begin by estimating $|f(x) - \frac{1}{4}|$ in terms of $|x - 4|$:

$$|f(x) - 1/4| = \left|\frac{1}{x} - \frac{1}{4}\right| = \left|\frac{4 - x}{4x}\right| = \frac{|x - 4|}{|4x|} = \frac{1}{|4x|}\,|x - 4|.$$

As before, things would be easier if $1/|4x|$ were a constant. To achieve that we again agree not to take $\delta > 1$. If we always have $\delta \le 1$, then we will always have $|x-4| < 1$, and hence $3 < x < 5$. How large can $1/|4x|$ be in this situation? Answer: the quantity $1/|4x|$ increases as you decrease x, so if $3 < x < 5$ then it will never be larger than $1/|4\cdot 3| = \frac{1}{12}$. We see that if we never choose $\delta > 1$, we will always have

$$|f(x) - \tfrac{1}{4}| \le \tfrac{1}{12}|x - 4| \quad \text{for} \quad |x - 4| < \delta.$$

To guarantee that $|f(x) - \frac{1}{4}| < \varepsilon$ we could threfore require

$$\tfrac{1}{12}|x - 4| < \varepsilon, \quad \text{i.e.} \quad |x - 4| < 12\varepsilon.$$

Hence if we choose $\delta = 12\varepsilon$ or any smaller number, then $|x-4| < \delta$ implies $|f(x)-4| < \varepsilon$. Of course we have to honor our agreement never to choose $\delta > 1$, so our choice of δ is

$$\delta = \text{the smaller of 1 and } 12\varepsilon = \min\big(1, 12\varepsilon\big).$$

3.3 Variations on the limit theme

Not all limits are "for $x \to a$." here we describe some possible variations on the concept of limit.

3.3.1 Left and right limits.

When we let "x approach a" we allow x to be both larger or smaller than a, as long as x gets close to a. If we explicitly want to study the behaviour of $f(x)$ as x approaches a through values larger than a, then we write

$$\lim_{x\searrow a} f(x) \text{ or } \lim_{x\to a+} f(x) \text{ or } \lim_{x\to a+0} f(x) \text{ or } \lim_{x\to a, x>a} f(x).$$

All four notations are in use. Similarly, to designate the value which $f(x)$ approaches as x approaches a through values below a one writes

$$\lim_{x\nearrow a} f(x) \text{ or } \lim_{x\to a-} f(x) \text{ or } \lim_{x\to a-0} f(x) \text{ or } \lim_{x\to a, x<a} f(x).$$

The precise definition of right limits goes like this:

Definition 3.3.1. right and left limits
Let f be a function. Then the *right-limit*

$$\lim_{x\searrow a} f(x) = L. \tag{3.2}$$

means that for every $\varepsilon > 0$ one can find a $\delta > 0$ such that

$$a < x < a + \delta \implies |f(x) - L| < \varepsilon$$

holds for all x in the domain of f.
The left-limit, i.e. the one-sided limit in which x approaches a through values less than a is defined in a similar way.
The following theorem tells you how to use one-sided limits to decide if a function $f(x)$ has a limit at $x = a$.

Theorem 3.3.1. If both one-sided limits

$$\lim_{x\searrow a} f(x) = L_+, \text{ and } \lim_{x\nearrow a} f(x) = L_-$$

exist, then

$$\lim_{x\to a} f(x) \text{ exists} \iff L_+ = L_-.$$

In other words, if a function has both left- and right-limits at some $x = a$, then that function has a limit at $x = a$ if the left- and right-limits are equal.

3.3.2 Limits at infinity.

Instead of letting x approach some finite number, one can let x become "larger and larger" and ask what happens to $f(x)$. If there is a number L such that $f(x)$ gets arbitrarily close to L if one chooses x sufficiently large, then we write

$$\lim_{x\to\infty} f(x) = L, \quad \text{or} \quad \lim_{x\uparrow\infty} f(x) = L, \quad \text{or} \quad \lim_{x\nearrow\infty} f(x) = L.$$

("The limit for x going to infinity is L.")

3.3.3 Example – Limit of $1/x$.

The larger you choose x, the smaller its reciprocal $1/x$ becomes. Therefore, it seems reasonable to say

$$\lim_{x\to\infty} \frac{1}{x} = 0.$$

Here is the precise definition:

3.3.4 Definition of limit at ∞.

Let f be some function which is defined on some interval $x_0 < x < \infty$. If there is a number L such that for every $\varepsilon > 0$ one can find an A such that

$$x > A \implies |f(x) - L| < \varepsilon$$

for all x, then we say that the limit of $f(x)$ for $x \to \infty$ is L.

The definition is very similar to the original definition of the limit. Instead of δ which specifies how close x should be to a, we now have a number A which says how large x should be, which is a way of saying "how close x should be to infinity."

3.3.5 Example – Limit of $1/x$ (again) .

To ***prove*** that $\lim_{x\to\infty} 1/x = 0$ we apply the definition to $f(x) = 1/x$, $L = 0$.

For given $\varepsilon > 0$ we need to show that

$$\left| \frac{1}{x} - L \right| < \varepsilon \text{ for all } x > A \tag{3.3}$$

provided we choose the right A.

How do we choose A? A is not allowed to depend on x, but it may depend on ε.

If we assume for now that we will only consider positive values of x, then (3.3) simplifies to

$$\frac{1}{x} < \varepsilon$$

which is equivalent to

$$x > \frac{1}{\varepsilon}.$$

This tells us how to choose A. Given any positive ε, we will simply choose

$$A = \frac{1}{\varepsilon}$$

Then one has $|\frac{1}{x} - 0| = \frac{1}{x} < \varepsilon$ for all $x > A$. Hence we have proved that $\lim_{x\to\infty} 1/x = 0$.

3.4 Properties of the Limit

The precise definition of the limit is not easy to use, and fortunately we won't use it very often in this class. Instead, there are a number of properties that limits have which allow you to compute them without having to resort to "epsiloncy."

The following properties also apply to the variations on the limit from 3.3. I.e. the following statements remain true if one replaces each limit by a one-sided limit, or a limit for $x \to \infty$.

Limits of constants and of x. If a and c are constants, then

$$\lim_{x\to a} c = c \tag{P_1}$$

and

$$\lim_{x\to a} x = a. \tag{P_2}$$

Limits of sums, products and quotients. Let F_1 and F_2 be two given functions whose limits for $x \to a$ we know,

$$\lim_{x\to a} F_1(x) = L_1, \qquad \lim_{x\to a} F_2(x) = L_2.$$

Then

$$\lim_{x\to a}\big(F_1(x) + F_2(x)\big) = L_1 + L_2, \tag{P_3}$$

$$\lim_{x\to a}\big(F_1(x) - F_2(x)\big) = L_1 - L_2, \tag{P_4}$$

$$\lim_{x\to a}\big(F_1(x) \cdot F_2(x)\big) = L_1 \cdot L_2 \tag{P_5}$$

Finally, if $\lim_{x\to a} F_2(x) \neq 0$,

$$\lim_{x\to a} \frac{F_1(x)}{F_2(x)} = \frac{L_1}{L_2}. \tag{P_6}$$

In other words the limit of the sum is the sum of the limits, etc. One can prove these laws using the definition of limit in §3.2 but we will not do this here. However, I hope these laws seem like common sense: if, for x close to a, the quantity $F_1(x)$ is close to L_1 and $F_2(x)$ is close to L_2, then certainly $F_1(x) + F_2(x)$ should be close to $L_1 + L_2$.

There are two more properties of limits which we will add to this list later on. They are the "Sandwich Theorem" (§3.8) and the substitution theorem (§3.9).

3.5 Examples of limit computations

3.5.1 Find $\lim_{x\to 2} x^2$.

One has

$$\begin{aligned}
\lim_{x\to 2} x^2 &= \lim_{x\to 2} x \cdot x \\
&= \big(\lim_{x\to 2} x\big) \cdot \big(\lim_{x\to 2} x\big) && \text{by } (P_5) \\
&= 2 \cdot 2 = 4.
\end{aligned}$$

Similarly,

$$\begin{aligned}\lim_{x\to 2} x^3 &= \lim_{x\to 2} x \cdot x^2 \\ &= \left(\lim_{x\to 2} x\right) \cdot \left(\lim_{x\to 2} x^2\right) && (P_5) \text{ again} \\ &= 2 \cdot 4 = 8,\end{aligned}$$

and, by (P_4)

$$\lim_{x\to 2} x^2 - 1 = \lim_{x\to 2} x^2 - \lim_{x\to 2} 1 = 4 - 1 = 3,$$

and, by (P_4) again,

$$\lim_{x\to 2} x^3 - 1 = \lim_{x\to 2} x^3 - \lim_{x\to 2} 1 = 8 - 1 = 7,$$

Putting all this together, one gets

$$\lim_{x\to 2} \frac{x^3 - 1}{x^2 - 1} = \frac{2^3 - 1}{2^2 - 1} = \frac{8 - 1}{4 - 1} = \frac{7}{3}$$

because of (P_6). To apply (P_6) we must check that the denominator ("L_2") is not zero. Since the denominator is 3 everything is OK, and we were allowed to use (P_6).

3.5.2 Try the examples 3.1.3 and 3.1.4 using the limit properties.

To compute $\lim_{x\to 2}(x^2 - 2x)/(x^2 - 4)$ we first use the limit properties to find

$$\lim_{x\to 2} x^2 - 2x = 0 \text{ and } \lim_{x\to 2} x^2 - 4 = 0.$$

to complete the computation we would like to apply the last property (P_6) about quotients, but this would give us

$$\lim_{x\to 2} f(x) = \frac{0}{0}.$$

The denominator is zero, so we were not allowed to use (P_6) (and the result doesn't mean anything anyway). We have to do something else.

The function we are dealing with is a ***rational function***, which means that it is the quotient of two polynomials. For such functions there is an algebra trick which always allows you to compute the limit even if you first get $\frac{0}{0}$. The thing to do is to divide numerator and denominator by $x - 2$. In our case we have

$$x^2 - 2x = (x - 2) \cdot x, \qquad x^2 - 4 = (x - 2) \cdot (x + 2)$$

so that

$$\lim_{x\to 2} f(x) = \lim_{x\to 2} \frac{(x - 2) \cdot x}{(x - 2) \cdot (x + 2)} = \lim_{x\to 2} \frac{x}{x + 2}.$$

After this simplification we ***can*** use the properties $(P_{\ldots})$ to compute

$$\lim_{x\to 2} f(x) = \frac{2}{2 + 2} = \frac{1}{2}.$$

3.5.3 Example – Find $\lim_{x\to 2}\sqrt{x}$.

Of course, you would think that $\lim_{x\to 2}\sqrt{x} = \sqrt{2}$ and you can indeed prove this using δ & ε (See problem 43.) But is there an easier way? There is nothing in the limit properties which tells us how to deal with a square root, and using them we can't even prove that there is a limit. However, if you ***assume*** that the limit exists then the limit properties allow us to find this limit.

The argument goes like this: suppose that there is a number L with

$$\lim_{x\to 2}\sqrt{x} = L.$$

Then property (P_5) implies that

$$L^2 = \big(\lim_{x\to 2}\sqrt{x}\big)\cdot\big(\lim_{x\to 2}\sqrt{x}\big) = \lim_{x\to 2}\sqrt{x}\cdot\sqrt{x} = \lim_{x\to 2} x = 2.$$

In other words, $L^2 = 2$, and hence L must be either $\sqrt{2}$ or $-\sqrt{2}$. We can reject the latter because whatever x does, its squareroot is always a positive number, and hence it can never "get close to" a negative number like $-\sqrt{2}$.

Our conclusion: if the limit exists, then

$$\lim_{x\to 2}\sqrt{x} = \sqrt{2}.$$

The result is not surprising: if x gets close to 2 then $\sqrt{x}$ gets close to $\sqrt{2}$.

3.5.4 Example – The derivative of $\sqrt{x}$ at $x = 2$.

Find

$$\lim_{x\to 2}\frac{\sqrt{x}-\sqrt{2}}{x-2}$$

assuming the result from the previous example.

Solution: The function is a fraction whose numerator and denominator vanish when $x = 2$, i.e. the limit is of the form $\frac{0}{0}$. We use the same algebra trick as before, namely we factor numerator and denominator:

$$\frac{\sqrt{x}-\sqrt{2}}{x-2} = \frac{\sqrt{x}-\sqrt{2}}{(\sqrt{x}-\sqrt{2})(\sqrt{x}+\sqrt{2})} = \frac{1}{\sqrt{x}+\sqrt{2}}.$$

Now one can use the limit properties to compute

$$\lim_{x\to 2}\frac{\sqrt{x}-\sqrt{2}}{x-2} = \lim_{x\to 2}\frac{1}{\sqrt{x}+\sqrt{2}} = \frac{1}{2\sqrt{2}} = \frac{\sqrt{2}}{4}.$$

3.5.5 Limit as $x \to \infty$ of rational functions.

A rational function is the quotient of two polynomials, so

$$R(x) = \frac{a_n x^n + \cdots + a_1 x + a_0}{b_m x^m + \cdots + b_1 x + b_0}. \tag{3.4}$$

We have seen that

$$\lim_{x\to\infty} \frac{1}{x} = 0$$

We even proved this in example 3.3.5. Using this you can find the limit at ∞ for any rational function $R(x)$ as in (3.4). One could turn the outcome of the calculation of $\lim_{x\to\infty} R(x)$ into a recipe/formula involving the degrees n and m of the numerator and denominator, and also their coefficients a_i, b_j, which students would then memorize, but it is better to remember "the trick."

To find $\lim_{x\to\infty} R(x)$ divide numerator and denominator by x^m (the highest power of x occurring in the denominator).

For example, let's compute

$$\lim_{x\to\infty} \frac{3x^2+3}{5x^2+7x-39}.$$

Remember the trick and divide top and bottom by x^2, and you get

$$\begin{aligned}
\lim_{x\to\infty} \frac{3x^2+3}{5x^2+7x-39} &= \lim_{x\to\infty} \frac{3+3/x^2}{5+7/x-39/x^2} \\
&= \frac{\lim_{x\to\infty} 3+3/x^2}{\lim_{x\to\infty} 5+7/x-39/x^2} \\
&= \frac{3}{5}
\end{aligned}$$

Here we have used the limit properties (P_*) to break the limit down into little pieces like $\lim_{x\to\infty} 39/x^2$ which we can compute as follows

$$\lim_{x\to\infty} 39/x^2 = \lim_{x\to\infty} 39 \cdot \left(\frac{1}{x}\right)^2 = \left(\lim_{x\to\infty} 39\right) \cdot \left(\lim_{x\to\infty} \frac{1}{x}\right)^2 = 39 \cdot 0^2 = 0.$$

3.5.6 Another example with a rational function .

Compute

$$\lim_{x\to\infty} \frac{x}{x^3+5}.$$

We apply "the trick" again and divide numerator and denominator by x^3. This leads to

$$\lim_{x\to\infty} \frac{x}{x^3+5} = \lim_{x\to\infty} \frac{1/x^2}{1+5/x^3} = \frac{\lim_{x\to\infty} 1/x^2}{\lim_{x\to\infty} 1+5/x^3} = \frac{0}{1} = 0.$$

To show all possible ways a limit of a rational function can turn out we should do yet another example, but that one belongs in the next section (see example 3.6.6.)

3.6 When limits fail to exist

In the last couple of examples we worried about the possibility that a limit $\lim_{x\to a} g(x)$ actually might not exist. This can actually happen, and in this section we'll see a few examples of what failed limits look like. First let's agree on what we will call a "failed limit."

3.6.1 Definition.

If there is no number L such that $\lim_{x\to a} f(x) = L$, then we say that the limit $\lim_{x\to a} f(x)$ does not exist.

3.6.2 The sign function near $x = 0$.

The "sign function[1]" is defined by

$$\operatorname{sign}(x) = \begin{cases} -1 & \text{for } x < 0 \\ 0 & \text{for } x = 0 \\ 1 & \text{for } x > 0 \end{cases}$$

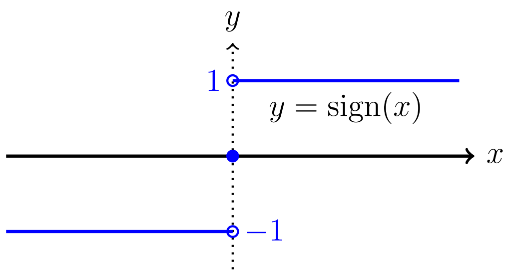


Figure 3.1: The sign function.

Note that "the sign of zero" is defined to be zero. But does the sign function have a limit at $x = 0$, i.e. does $\lim_{x\to 0} \operatorname{sign}(x)$ exist? And is it also zero? The answers are ***no*** and ***no***, and here is why: suppose that for some number L one had

$$\lim_{x\to 0} \operatorname{sign}(x) = L,$$

then since for arbitrary small positive values of x one has $\operatorname{sign}(x) = +1$ one would think that $L = +1$. But for arbitrarily small negative values of x one has $\operatorname{sign}(x) = -1$, so one

[1]Some people don't like the notation $\operatorname{sign}(x)$, and prefer to write

$$g(x) = \frac{x}{|x|}$$

instead of $g(x) = \operatorname{sign}(x)$. If you think about this formula for a moment you'll see that $\operatorname{sign}(x) = x/|x|$ for all $x \neq 0$. When $x = 0$ the quotient $x/|x|$ is of course not defined.

would conclude that $L = -1$. But one number L can't be both $+1$ and -1 at the same time, so there is no such L, i.e. there is no limit.

$$\lim_{x\to 0} \operatorname{sign}(x) \text{ does not exist.}$$

In this example the one-sided limits do exist, namely,

$$\lim_{x\searrow 0} \operatorname{sign}(x) = 1 \text{ and } \lim_{x\nearrow 0} \operatorname{sign}(x) = -1.$$

All this says is that when x approaches 0 through positive values, its sign approaches $+1$, while if x goes to 0 through negative values, then its sign approaches -1.

3.6.3 The example of the backward sine.

Contemplate the limit as $x \to 0$ of the "backward sine," i.e.

$$\lim_{x\to 0} \sin\left(\frac{\pi}{x}\right).$$

When $x = 0$ the function $f(x) = \sin(\pi/x)$ is not defined, because its definition involves division by x. What happens to $f(x)$ as $x \to 0$? First, π/x becomes larger and larger ("goes to infinity") as $x \to 0$. Then, taking the sine, we see that $\sin(\pi/x)$ oscillates between $+1$ and -1 infinitely often as $x \to 0$. This means that $f(x)$ gets close to any number between -1 and $+1$ as $x \to 0$, but that the function $f(x)$ ***never stays close*** to any particular value because it keeps oscillating up and down.

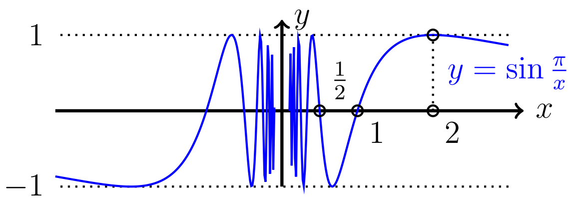


Figure 3.2: Graph of $y = \sin\frac{\pi}{x}$ for $-3 < x < 3$, $x \neq 0$.

Here again, the limit $\lim_{x\to 0} f(x)$ does not exist. We have arrived at this conclusion by only considering what $f(x)$ does for small positive values of x. So the limit fails to exist in a stronger way than in the example of the sign-function. There, even though the limit didn't exist, the one-sided limits existed. In the present example we see that even the one-sided limit

$$\lim_{x\searrow 0} \sin\frac{\pi}{x}$$

does not exist.

3.6.4 Trying to divide by zero using a limit.

The expression $1/0$ is not defined, but what about

$$\lim_{x\to 0} \frac{1}{x}?$$

This limit also does not exist. Here are two reasons:

It is common wisdom that if you divide by a small number you get a large number, so as $x \searrow 0$ the quotient $1/x$ will not be able to stay close to any particular finite number, and the limit can't exist.

"Common wisdom" is not always a reliable tool in mathematical proofs, so here is a better argument. The limit can't exist, because that would contradict the limit properties $(P_1) \cdots (P_6)$. Namely, suppose that there were an number L such that

$$\lim_{x\to 0} \frac{1}{x} = L.$$

Then the limit property (P_5) would imply that

$$\lim_{x\to 0} \bigl(\frac{1}{x} \cdot x\bigr) = \bigl(\lim_{x\to 0} \frac{1}{x}\bigr) \cdot \bigl(\lim_{x\to 0} x\bigr) = L \cdot 0 = 0.$$

On the other hand $\frac{1}{x} \cdot x = 1$ so the above limit should be 1! A number can't be both 0 and 1 at the same time, so we have a contradiction. The assumption that $\lim_{x\to 0} 1/x$ exists is to blame, so it must go.

3.6.5 Using limit properties to show a limit does *not* exist.

The limit properties tell us how to prove that certain limits exist (and how to compute them). Although it is perhaps not so obvious at first sight, they also allow you to prove that certain limits do not exist. The previous example shows one instance of such use. Here is another.

Property (P_3) says that if both $\lim_{x\to a} g(x)$ and $\lim_{x\to a} h(x)$ exist then $\lim_{x\to a} g(x)+h(x)$ also must exist. You can turn this around and say that if $\lim_{x\to a} g(x) + h(x)$ does not exist then either $\lim_{x\to a} g(x)$ or $\lim_{x\to a} h(x)$ does not exist (or both limits fail to exist).

For instance, the limit

$$\lim_{x\to 0} \frac{1}{x} - x$$

can't exist, for if it did, then the limit

$$\lim_{x\to 0} \frac{1}{x} = \lim_{x\to 0} \bigl(\frac{1}{x} - x + x\bigr) = \lim_{x\to 0} \bigl(\frac{1}{x} - x\bigr) + \lim_{x\to 0} x$$

would also have to exist, and we know $\lim_{x\to 0} \frac{1}{x}$ doesn't exist.

3.6.6 Limits at ∞ which don't exist.

If you let x go to ∞, then x will not get "closer and closer" to any particular number L, so it seems reasonable to guess that

$$\lim_{x\to\infty} x \text{ does not exist.}$$

One can prove this from the limit definition (and see exercise 70).

Let's consider

$$L = \lim_{x\to\infty} \frac{x^2 + 2x - 1}{x + 2}.$$

Once again we divide numerator and denominator by the highest power in the denominator (i.e. x)

$$L = \lim_{x\to\infty} \frac{x + 2 - \frac{1}{x}}{1 + 2/x}$$

Here the denominator has a limit ('tis 1), but the numerator does not, for if $\lim_{x\to\infty} x + 2 - \frac{1}{x}$ existed then, since $\lim_{x\to\infty}(2 - 1/x) = 2$ exists,

$$\lim_{x\to\infty} x = \lim_{x\to\infty} \left[\big(x + 2 - \frac{1}{x}\big) - \big(2 - \frac{1}{x}\big) \right]$$

would also have to exist, and $\lim_{x\to\infty} x$ doesn't exist.

So we see that L is the limit of a fraction in which the denominator has a limit, but the numerator does not. In this situation the limit L itself can never exist. If it did, then

$$\lim_{x\to\infty} \big(x + 2 - \frac{1}{x}\big) = \lim_{x\to\infty} \frac{x + 2 - \frac{1}{x}}{1 + 2/x} \cdot (1 + 2/x)$$

would also have to have a limit.

3.7 What's in a name?

There is a big difference between the variables x and a in the formula

$$\lim_{x\to a} 2x + 1,$$

namely a is a ***free variable***, while x is a ***dummy variable*** (or "placeholder" or a "bound variable.")

The difference between these two kinds of variables is this:

- if you replace a dummy variable in some formula consistently by some other variable then the value of the formula does not change. On the other hand, it never makes sense to substitute a number for a dummy variable.

- the value of the formula may depend on the value of the free variable.

To understand what this means consider the example $\lim_{x\to a} 2x + 1$ again. The limit is easy to compute:

$$\lim_{x\to a} 2x + 1 = 2a + 1.$$

If we replace x by, say u (systematically) then we get

$$\lim_{u\to a} 2u + 1$$

which is again equal to $2a + 1$. This computation says that *if some number gets close to a then two times that number plus one gets close to* $2a + 1$. This is a very wordy way of expressing the formula, and you can shorten things by giving a name (like x or u) to the number which approaches a. But the result of our computation shouldn't depend on the name we choose, i.e. it doesn't matter if we call it x or u.

Since the name of the variable x doesn't matter it is called a dummy variable. Some prefer to call x a bound variable, meaning that in

$$\lim_{x\to a} 2x + 1$$

the x in the expression $2x+1$ is bound to the x written underneath the limit – you can't change one without changing the other.

Substituting a number for a dummy variable usually leads to complete nonsense. For instance, let's try setting $x = 3$ in our limit, i.e. what is

$$\lim_{3\to a} 2 \cdot 3 + 1\,?$$

Of course $2 \cdot 3 + 1 = 7$, but what does 7 do when 3 gets closer and closer to the number a? That's a silly question, because 3 is a constant and it doesn't "get closer" to some other number like a! If you ever see 3 get closer to another number then it's time to take a vacation.

On the other hand the variable a is free: you can assign it particular values, and its value will affect the value of the limit. For instance, if we set $a = 3$ (but leave x alone) then we get

$$\lim_{x\to 3} 2x + 1$$

and there's nothing strange about that (the limit is $2\cdot 3+1 = 7$, no problem.) You could substitute other values of a and you would get a different answer. In general you get $2a + 1$.

3.8 Limits and Inequalities

This section has two theorems which let you compare limits of different functions. The properties in these theorems are not formulas that allow you to compute limits like the properties $(P_1)\ldots(P_6)$ from §3.4. Instead, they allow you to ***reason*** about limits, i.e. they let you say that this or that limit is positive, or that it must be the same as some other limit which you find easier to think about.

The first theorem should not surprise you – all it says is that bigger functions have bigger limits.

Theorem 3.8.1. Let f and g be functions whose limits for $x \to a$ exist, and assume that $f(x) \le g(x)$ holds for all x. Then

$$\lim_{x\to a} f(x) \le \lim_{x\to a} g(x).$$

A useful special case arises when you set $f(x) = 0$. The theorem then says that if a function g never has negative values, then its limit will also never be negative.

The statement may seem obvious, but it still needs a proof, starting from the ε-δ definition of limit. This will be done in lecture.

Here is the second theorem about limits and inequalities.

Theorem 3.8.2. The Sandwich Theorem.

Suppose that

$$f(x) \le g(x) \le h(x)$$

(for all x) and that

$$\lim_{x\to a} f(x) = \lim_{x\to a} h(x).$$

Then

$$\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = \lim_{x\to a} h(x).$$

The theorem is useful when you want to know the limit of g, and when you can ***sandwich*** it between two functions f and h whose limits are easier to compute. The Sandwich Theorem looks like the first theorem of this section, but there is an important difference: in the Sandwich Theorem you don't have to assume that the limit of g exists. The inequalities $f \le g \le h$ combined with the circumstance that f and h have the same limit are enough to guarantee that the limit of g exists.

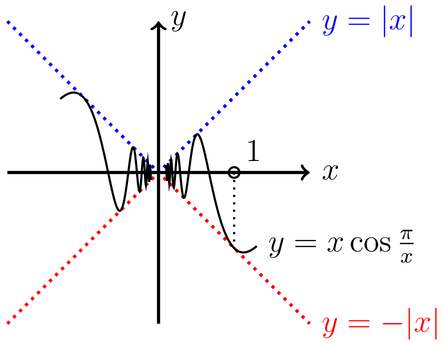


Figure 3.3: Graphs of $|x|$, $-|x|$ and $x\cos\frac{\pi}{x}$ for $-1.2 < x < 1.2$

3.8.1 Example: a Backward Cosine Sandwich.

The Sandwich Theorem says that if the function $g(x)$ is sandwiched between two functions $f(x)$ and $h(x)$ and the limits of the outside functions f and h exist and are equal, then the limit of the inside function g exists and equals this common value. For example

$$-|x| \le x\cos\frac{1}{x} \le |x|$$

since the cosine is always between -1 and 1. Since

$$\lim_{x\to 0} -|x| = \lim_{x\to 0} |x| = 0$$

the sandwich theorem tells us that

$$\lim_{x\to 0} x \cos \frac{1}{x} = 0.$$

Note that the limit $\lim_{x\to 0} \cos(1/x)$ does ***not*** exist, for the same reason that the "backward sine" did not have a limit for $x \to 0$ (see example 3.6.3). Multiplying with x changed that.

3.9 Continuity

3.9.1 Definition.

*A function g is **continuous** at a if*

$$\lim_{x\to a} g(x) = g(a) \tag{3.5}$$

A function is continuous if it is continuous at every a in its domain.

Note that when we say that a function is continuous on some interval it is understood that the domain of the function includes that interval. For example, the function $f(x) = 1/x^2$ is continuous on the interval $1 < x < 5$ but is ***not*** continuous on the interval $-1 < x < 1$.

3.9.2 Polynomials are continuous.

For instance, let us show that $P(x) = x^2 + 3x$ is continuous at $x = 2$. To show that you have to prove that

$$\lim_{x\to 2} P(x) = P(2),$$

i.e.

$$\lim_{x\to 2} x^2 + 3x = 2^2 + 3 \cdot 2.$$

You can do this two ways: using the definition with ε and δ (i.e. the hard way), or using the limit properties $(P_1)\ldots(P_6)$ from §3.4 (just as good, and easier, even though it still takes a few lines to write it out – do both!)

3.9.3 Rational functions are continuous.

Let $R(x) = \frac{P(x)}{Q(x)}$ be a rational function, and let a be any number in the domain of R, i.e. any number for which $Q(a) \neq 0$. Then one has

$$\begin{aligned}
\lim_{x\to a} R(x) &= \lim_{x\to a} \frac{P(x)}{Q(x)} \\
&= \frac{\lim_{x\to a} P(x)}{\lim_{x\to a} Q(x)} && \text{property } (P_6) \\
&= \frac{P(a)}{Q(a)} && P \text{ and } Q \text{ are continuous} \\
&= R(a).
\end{aligned}$$

This shows that R is indeed continuous at a.

3.9.4 Some discontinuous functions.

If $\lim_{x\to a} g(x)$ does not exist, then it certainly cannot be equal to $g(a)$, and therefore any failed limit provides an example of a discontinuous function.

For instance, the sign function $g(x) = \text{sign}(x)$ from example 3.1 is not continuous at $x = 0$.

Is the backward sine function $g(x) = \sin(1/x)$ from example 3.2 also discontinuous at $x = 0$? No, it is not, for two reasons: first, the limit $\lim_{x\to 0} \sin(1/x)$ does not exist, and second, we haven't even defined the function $g(x)$ at $x = 0$, so even if the limit existed, we would have no value $g(0)$ to compare it with.

3.9.5 How to make functions discontinuous.

Here is a discontinuous function:

$$f(x) = \begin{cases} x^2 & \text{if } x \neq 3, \\ 47 & \text{if } x = 3. \end{cases}$$

In other words, we take a continuous function like $g(x) = x^2$, and change its value somewhere, e.g. at $x = 3$. Then

$$\lim_{x\to 3} f(x) = 9 \neq 47 = f(3).$$

The reason that the limit is 9 is that our new function $f(x)$ coincides with our old continuous function $g(x)$ for all x except $x = 3$. Therefore the limit of $f(x)$ as $x \to 3$ is the same as the limit of $g(x)$ as $x \to 3$, and since g is continuous this is $g(3) = 9$.

3.9.6 Sandwich in a bow tie.

We return to the function from example 3.3. Consider

$$f(x) = \begin{cases} x\cos\left(\frac{1}{x}\right) & \text{for } x \neq 0, \\ 0 & \text{for } x = 0 \end{cases}$$

Then f is continuous at $x = 0$ by the Sandwich Theorem (see Example 3.3).
If we change the definition of f by picking a different value at $x = 0$ the new function will not be continuous, since changing f at $x = 0$ does not change the limit $\lim_{x\to 0} f(x)$. Since this limit is zero, $f(0) = 0$ is the only possible choice of $f(0)$ which makes f continuous at $x = 0$.

3.10 Substitution in Limits

Given two functions f and g one can consider their composition $h(x) = f(g(x))$. To compute the limit

$$\lim_{x\to a} f\big(g(x)\big)$$

we write $u = g(x)$, so that we want to know

$$\lim_{x\to a} f(u) \text{ where } u = g(x).$$

Suppose that you can find the limits

$$L = \lim_{x\to a} g(x) \text{ and } \lim_{u\to L} f(u) = M.$$

Then it seems reasonable that as x approaches a, $u = g(x)$ will approach L, and $f(g(x))$ approaches M.
This is in fact a theorem:

Theorem 3.10.1. If $\lim_{x\to a} g(x) = L$, and if the function f is continuous at $u = L$, then

$$\lim_{x\to a} f\big(g(x)\big) = \lim_{u\to L} f(u) = f(L).$$

Another way to write this is

$$\lim_{x\to a} f\big(g(x)\big) = f\big(\lim_{x\to a} g(x)\big).$$

3.10.1 Example: compute $\lim_{x\to 3} \sqrt{x^3 - 3x^2 + 2}$.

The given function is the composition of two functions, namely

$$\sqrt{x^3 - 3x^2 + 2} = \sqrt{u}, \text{ with } u = x^3 - 3x^2 + 2,$$

or, in function notation, we want to find $\lim_{x\to 3} h(x)$ where

$$h(x) = f(g(x)), \text{ with } g(x) = x^3 - 3x^2 + 2 \text{ and } g(x) = \sqrt{x}.$$

Either way, we have

$$\lim_{x\to 3} x^3 - 3x^2 + 2 = 2 \quad \text{and} \quad \lim_{u\to 2} \sqrt{u} = \sqrt{2}.$$

You get the first limit from the limit properties $(P_1)\ldots(P_5)$. The second limit says that taking the square root is a continuous function, which it is. We have not proved that (yet), but this particular limit is the one from example 3.5.3. Putting these two limits together we conclude that the limit is $\sqrt{2}$.

Normally, you write this whole argument as follows:

$$\lim_{x\to 3} \sqrt{x^3 - 3x^2 + 2} = \sqrt{\lim_{x\to 3} x^3 - 3x^2 + 2} = \sqrt{2},$$

where you must point out that $f(x) = \sqrt{x}$ is a continuous function to justify the first step.

Another possible way of writing this is

$$\lim_{x\to 3} \sqrt{x^3 - 3x^2 + 2} = \lim_{u\to 2} \sqrt{u} = \sqrt{2},$$

where you must say that you have substituted $u = x^3 - 3x^2 + 2$.

3.11 Two Limits in Trigonometry

In this section we'll derive a few limits involving the trigonometric functions. You can think of them as saying that for small angles θ one has

$$\sin\theta \approx \theta \quad \text{and} \quad \cos\theta \approx 1 - \tfrac{1}{2}\theta^2.$$

We will use these limits when we compute the derivatives of Sine, Cosine and Tangent.

Theorem 3.11.1. $\displaystyle\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1.$

Proof. The proof requires a few sandwiches and some geometry on the unit circle.

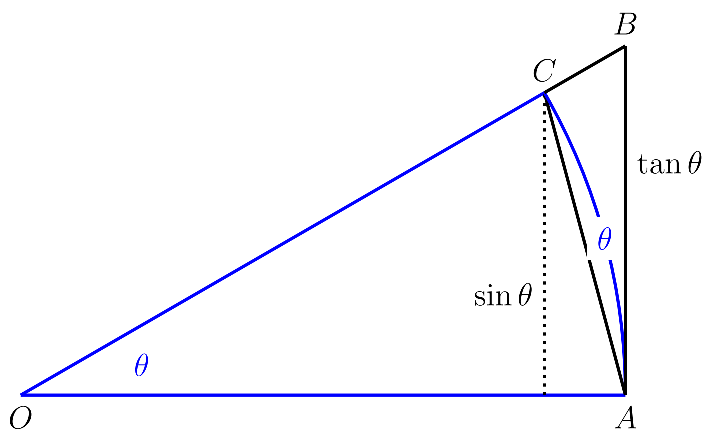


Figure 3.4: The sandwitch: $\sin\theta < \theta < \tan\theta$

We begin by considering figure 3.5 with $OA = OC = 1$ and $0 < \theta < \pi/2$.

Since the wedge OAC contains the triangle OAC its area must be larger. The area of the wedge is $\frac{1}{2}\theta$ and the area of the triangle is $\frac{1}{2}\sin\theta$, so we find that

$$0 < \sin\theta < \theta \text{ for } 0 < \theta < \frac{\pi}{2}. \tag{3.6}$$

The Sandwich Theorem implies that

$$\lim_{\theta\searrow 0} \sin\theta = 0. \tag{3.7}$$

Moreover, we also have

$$\lim_{\theta\searrow 0} \cos\theta = \lim_{\theta\searrow 0} \sqrt{1-\sin^2\theta} = 1. \tag{3.8}$$

Next we compare the areas of the wedge OAC and the larger triangle OAB. Since OAB has area $\frac{1}{2}\tan\theta$ we find that

$$\theta < \tan\theta$$

for $0 < \theta < \frac{\pi}{2}$. Since $\tan\theta = \frac{\sin\theta}{\cos\theta}$ we can multiply with $\cos\theta$ and divide by θ to get

$$\cos\theta < \frac{\sin\theta}{\theta} \text{ for } 0 < \theta < \frac{\pi}{2}$$

If we go back to (3.6) and divide by θ, then we get

$$\cos\theta < \frac{\sin\theta}{\theta} < 1$$

The Sandwich Theorem can be used once again, and now it gives

$$\lim_{\theta\searrow 0} \frac{\sin\theta}{\theta} = 1.$$

This is a one-sided limit. To get the limit in which $\theta \nearrow 0$, you use that $\sin\theta$ is an odd function. □

3.11.1 An example.

We will show that

$$\lim_{\theta\to 0} \frac{1-\cos\theta}{\theta^2} = \frac{1}{2}. \tag{3.9}$$

This follows from $\sin^2\theta + \cos^2\theta = 1$. Namely,

$$\begin{aligned}
\frac{1-\cos\theta}{\theta^2} &= \frac{1}{1+\cos\theta}\,\frac{1-\cos^2\theta}{\theta^2} \\
&= \frac{1}{1+\cos\theta}\,\frac{\sin^2\theta}{\theta^2} \\
&= \frac{1}{1+\cos\theta}\left\{\frac{\sin\theta}{\theta}\right\}^2.
\end{aligned}$$

We have just shown that $\cos\theta \to 1$ and $\frac{\sin\theta}{\theta} \to 1$ as $\theta \to 0$, so (3.9) follows.

After watching YouTube by Mathologer on indeterminate forms, the reader should try using L'Hopital's rule to recompute the previous two limits.

3.12 PROBLEMS

LIMITS FROM FIRST PRINCIPALS

37. Emily offers to make square sheets of paper for Kate. Given $x > 0$ Emily plans to mark off a length x and cut out a square of side x. Kate asks Emily for a square with area 4 square foot. Emily tells Kate that she can't measure ***exactly*** 2 foot and the area of the square she produces will only be approximately 4 square foot. Kate doesn't mind as long as the area of the square doesn't differ more than 0.01 square foot from what he really asked for (namely, 4 square foot).

(a) What is the biggest error Emily can afford to make when she marks off the length x?

(b) Bronwyn also wants square sheets, with area 4 square feet. However, she needs the error in the area to be less than 0.00001 square foot. (She's paying).

How accurate must Emily measure the side of the squares she's going to cut for Bronwyn?

Use the ε–δ definition to prove the following limits

38. $\lim_{x\to 1} 2x - 4 = 6$ †379

39. $\lim_{x\to 2} x^2 = 4.$ †379

40. $\lim_{x\to 2} x^2 - 7x + 3 = -7$ †379

41. $\lim_{x\to 3} x^3 = 27$ †380

42. $\lim_{x\to 2} x^3 + 6x^2 = 32.$

43. $\lim_{x\to 4} \sqrt{x} = 2.$ †380

44. $\lim_{x\to 3} \sqrt{x+6} = 9.$ †380

45. $\lim_{x\to 2} \dfrac{1+x}{4+x} = \frac{1}{2}.$ †381

46. $\lim_{x\to 1} \dfrac{2-x}{4-x} = \frac{1}{3}.$

47. $\lim_{x\to 3} \dfrac{x}{6-x} = 1.$

48. $\lim_{x\to 0} \sqrt{|x|} = 0$

49. *(Emily goes cubic.)* Emily is offering to build cubes of side x. Airline regulations allow you take a cube on board provided its volume and surface area add up to less than 33 (everything measured in feet). For instance, a cube with 2 foot sides has volume+area equal to $2^3 + 6 \times 2^2 = 32$.

If you ask Emily to build a cube whose volume plus total surface area is 32 cubic feet with an error of at most ε, then what error can she afford to make when she measures the side of the cube he's making?

50. Our definition of a derivative in (2.5) contains a limit. What is the function "f" there, and what is the variable? †381

SUBSTITUTION

Find the following limits.

51. $\lim_{x\to -7}(2x+5)$

52. $\lim_{x\to 7-}(2x+5)$

53. $\lim_{x\to -\infty}(2x+5)$

54. $\lim_{x\to -4}(x+3)^{2006}$

55. $\lim_{x\to -4}(x+3)^{2007}$

56. $\lim_{x\to -\infty}(x+3)^{2007}$

57. $\lim_{t\to 1}\dfrac{t^2+t-2}{t^2-1}$

58. $\lim_{t\nearrow 1}\dfrac{t^2+t-2}{t^2-1}$

59. $\lim_{t\to -1}\dfrac{t^2+t-2}{t^2-1}$

60. $\lim_{x\to\infty}\dfrac{x^2+3}{x^2+4}$

61. $\lim_{x\to\infty}\dfrac{x^5+3}{x^2+4}$

62. $\lim_{x\to\infty}\dfrac{x^2+1}{x^5+2}$

63. $\lim_{x\to\infty}\dfrac{(2x+1)^4}{(3x^2+1)^2}$

64. $\lim_{u\to\infty}\dfrac{(2u+1)^4}{(3u^2+1)^2}$

65. $\lim_{t\to 0}\dfrac{(2t+1)^4}{(3t^2+1)^2}$

66. If $\lim_{x\to a} f(x)$ exists then f is continuous at $x=a$. *True or false?* †381

67. Give two examples of functions for which $\lim_{x\searrow 0} f(x)$ does not exist. †381

68. If $\lim_{x\to 0} f(x)$ and $\lim_{x\to 0} g(x)$ both do not exist, then $\lim_{x\to 0}\big(f(x)+g(x)\big)$ also does not exist. *True or false?* †381

69. If $\lim_{x\to 0} f(x)$ and $\lim_{x\to 0} g(x)$ both do not exist, then $\lim_{x\to 0}(f(x)/g(x))$ also does not exist. *True or false?* †381

70. In the text we proved that $\lim_{x\to\infty}\frac{1}{x}=0$. Show that this implies that $\lim_{x\to\infty} x$ does not exist. Hint: Suppose $\lim_{x\to\infty} x = L$ for some number L. Apply the limit properties to $\lim_{x\to\infty} x\cdot\dfrac{1}{x}$.

71. Evaluate $\lim_{x\to 9}\dfrac{\sqrt{x}-3}{x-9}$. Hint: Multiply top and bottom by $\sqrt{x}+3$.

72. Evaluate $\lim_{x\to 2}\dfrac{\frac{1}{x}-\frac{1}{2}}{x-2}$.

73. Evaluate $\lim_{x\to 2}\dfrac{\frac{1}{\sqrt{x}}-\frac{1}{\sqrt{2}}}{x-2}$.

74. A function f is defined by

$$f(x)=\begin{cases} x^3 & \text{for } x<-1\\ ax+b & \text{for } -1\le x<1\\ x^2+2 & \text{for } x\ge 1.\end{cases}$$

where a and b are constants. The function f is continuous. What are a and b?

75. Find a constant k such that the function

$$f(x)=\begin{cases} 3x+2 & \text{for } x<2\\ x^2+k & \text{for } x\ge 2.\end{cases}$$

is continuous. Hint: Compute the one-sided limits.

76. Find constants a and c such that the function

$$f(x) = \begin{cases} x^3 + c & \text{for } x < 0 \\ ax + c^2 & \text{for } 0 \le x < 1 \\ \arctan x & \text{for } x \ge 1. \end{cases}$$

is continuous for all x.

LIMITS INVOLVING TRIGONOMETRIC FUNCTIONS

Find each of the following limits or show that it does not exist. Distinguish between limits which are infinite and limits which do not exist.

77. $\lim_{\alpha \to 0} \dfrac{\sin 2\alpha}{\sin \alpha}$ (two ways: with and without the double angle formula!) †381

78. $\lim_{x \to 0} \dfrac{\sin 3x}{2x}$. †381

79. $\lim_{\theta \to 0} \dfrac{\tan \theta}{\theta}$. †381

80. $\lim_{\alpha \to 0} \dfrac{\tan 4\alpha}{\sin 2\alpha}$. †381

81. $\lim_{x \to 0} \dfrac{1 - \cos x}{x \sin x}$. †381

82. $\lim_{\theta \to \pi/2} \dfrac{1 - \sin \theta}{\theta - \pi/2}$ †381

83. $\lim_{x \to \infty} \dfrac{2x^3 + 3x^2 \cos x}{(x+2)^3}$.

84. $\lim_{x \to 0} \dfrac{\sin^2 x}{1 - \cos x}$.

85. $\lim_{x \to 0} \dfrac{\sin(x^2)}{x^2}$.

86. $\lim_{x \to 0} \dfrac{x(1 - \cos x)}{\tan^3 x}$.

87. $\lim_{x \to 0} \dfrac{\sin(x^2)}{1 - \cos x}$.

88. $\lim_{x \to \pi/2} \dfrac{x - \frac{\pi}{2}}{\cos x}$. †381

89. $\lim_{x \to \pi/2} (x - \frac{\pi}{2}) \tan x$. †381

90. $\lim_{x \to 0} \dfrac{\cos x}{x^2 + 9}$.

91. $\lim_{x \to \pi} \dfrac{\sin x}{x - \pi}$. †381

92. $\lim_{x \to 0} \dfrac{\sin x}{x + \sin x}$.

93. $A = \lim_{x \to \infty} \dfrac{\sin x}{x}$. **(!!)** †382

94. $B = \lim_{x \to \infty} \dfrac{\cos x}{x}$. **(!! again)**

95. Is there a constant k such that the function

$$f(x) = \begin{cases} \sin(1/x) & \text{for } x \ne 0 \\ k & \text{for } x = 0. \end{cases}$$

is continuous? If so, find it; if not, say why. †382

96. Find a constant A so that the function

$$f(x) = \begin{cases} \dfrac{\sin x}{2x} & \text{for } x \ne 0 \\ A & \text{when } x = 0 \end{cases}$$

†382

97. Compute $\lim_{x\to\infty} x \sin\frac{\pi}{x}$ and $\lim_{x\to\infty} x \tan\frac{\pi}{x}$. (Hint: substitute something).

98. (*Geometry & Trig review*) Let A_n be the area of the regular n-gon inscribed in the unit circle, and let B_n be the area of the regular n-gon whose inscribed circle has radius 1.

(a) Show that $A_n < \pi < B_n$.

(b) Show that

$$A_n = \frac{n}{2}\sin\frac{2\pi}{n} \text{ and } B_n = n\tan\frac{\pi}{n}$$

(c) Compute $\lim_{n\to\infty} A_n$ and $\lim_{n\to\infty} B_n$.

Here is a picture of A_{12} (in blue), B_6 (in cyan) and π (in red):

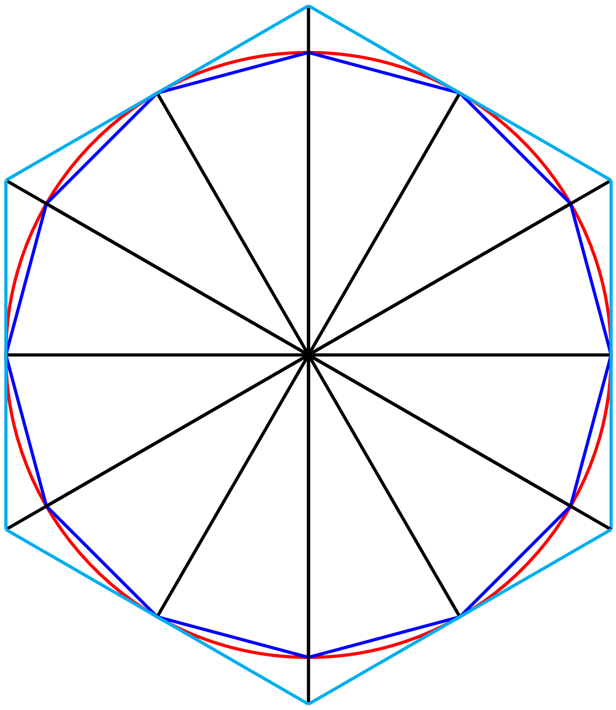


Figure 3.5: Archimedes uses the sandwitch to compute π

On a historical note: Archimedes managed to compute A_{96} and B_{96} and by doing this got the most accurate approximation for π that was known in his time. See also:

http://www-history.mcs.st-andrews.ac.uk/HistTopics/Pi_through_the_ages.html

Chapter 4

Derivatives (continued)

"Leibniz never thought of the derivative as a limit"

http://www.gap-system.org/~history/Biographies/Leibniz.html

In chapter 2 we saw two mathematical problems which led to expressions of the form $\frac{0}{0}$. Now that we know how to handle limits, we can state the definition of the derivative of a function. After computing a few derivatives using the definition we will spend most of this section developing the *differential calculus,* which is a collection of rules that allow you to compute derivatives without always having to use basic definition.

4.1 Derivatives Defined

4.1.1 Definition.

Let f be a function which is defined on some interval (c, d) and let a be some number in this interval.

*The **derivative of the function f at a** is the value of the limit*

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}. \tag{4.1}$$

*f is said to be **differentiable at a** if this limit exists.*

*f is called **differentiable on the interval (c, d)** if it is differentiable at every point a in (c, d).*

4.1.2 Other notations.

One can substitute $x = a + h$ in the limit (4.1) and let $h \to 0$ instead of $x \to a$. This gives the formula

$$f'(a) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}, \tag{4.2}$$

Often you will find this equation written with x instead of a and Δx instead of h, which makes it look like this:

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}.$$

The interpretation is the same as in equation (2.4) from §2.4. The numerator $f(x+\Delta x) - f(x)$ represents the amount by which the function value of f changes if one increases its argument x by a (small) amount Δx. If you write $y = f(x)$ then we can call the increase in f

$$\Delta y = f(x + \Delta x) - f(x),$$

so that the derivative $f'(x)$ is

$$f'(x) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}.$$

Gottfried Wilhelm von Leibniz, one of the inventors of calculus, came up with the idea that one should write this limit as

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x},$$

the idea being that after letting Δx go to zero it didn't vanish, but instead became an infinitely small quantity which Leibniz called "dx." The result of increasing x by this infinitely small quantity dx is that $y = f(x)$ increased by another infinitely small quantity dy. The ratio of these two infinitely small quantities is what we call the derivative of $y = f(x)$.

There are no "infinitely small real numbers," and this makes Leibniz' notation difficult to justify. In the 20th century mathematicians have managed to create a consistent theory of "infinitesimals" which allows you to compute with "dx and dy" as Leibniz and his contemporaries would have done. This theory is called "non standard analysis." We won't mention it any further[1]. Nonetheless, even though we won't use infinitely small numbers, Leibniz' notation is very useful and we will use it.

At this point the reader should watch YouTube by 3Blue1Brown .

4.2 Direct computation of derivatives

4.2.1 Example – The derivative of $f(x) = x^2$ is $f'(x) = 2x$.

We have done this computation before in §2.2. The result was

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} = \lim_{h \to 0} \frac{(x + h)^2 - x^2}{h} = \lim_{h \to 0}(2x + h) = 2x.$$

Leibniz would have written

$$\frac{dx^2}{dx} = 2x.$$

[1]But if you want to read more on this you should see Keisler's calculus text at

http://www.math.wisc.edu/~keisler/calc.html

I would not recommend using Keisler's text and this text at the same time, but if you like math you should remember that it exists, and look at it later.

4.2.2 The derivative of $g(x) = x$ is $g'(x) = 1$.

Indeed, one has

$$g'(x) = \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} = \lim_{h\to 0} \frac{(x+h) - x}{h} = \lim_{h\to 0} \frac{h}{h} = 1.$$

In Leibniz' notation:

$$\frac{dx}{dx} = 1.$$

This is an example where Leibniz' notation is most misleading, because if you divide dx by dx then you should of course get 1. Nonetheless, this is not what is going on. The expression $\frac{dx}{dx}$ is not really a fraction since there are no two "infinitely small" quantities dx which we are dividing.

4.2.3 The derivative of any constant function is zero .

Let $k(x) = c$ be a constant function. Then we have

$$k'(x) = \lim_{h\to 0} \frac{k(x+h) - k(x)}{h} = \lim_{h\to 0} \frac{c - c}{h} = \lim_{h\to 0} 0 = 0.$$

Leibniz would have said that if c is a constant, then

$$\frac{dc}{dx} = 0.$$

4.2.4 Derivative of x^n for $n = 1, 2, 3, \ldots$.

To differentiate $f(x) = x^n$ one proceeds as follows:

$$f'(a) = \lim_{x\to a} \frac{f(x) - f(a)}{x - a} = \lim_{x\to a} \frac{x^n - a^n}{x - a}.$$

We need to simplify the fraction $(x^n - a^n)/(x - a)$. For $n = 2$ we have

$$\frac{x^2 - a^2}{x - a} = x + a.$$

For $n = 1, 2, 3, \ldots$ the geometric sum formula tells us that

$$\frac{x^n - a^n}{x - a} = x^{n-1} + x^{n-2}a + x^{n-3}a^2 + \cdots + xa^{n-2} + a^{n-1}. \tag{4.3}$$

If you don't remember the geometric sum formula, then you could also just verify (4.3) by carefully multiplying both sides with $x - a$. For instance, when $n = 3$ you would get

$$\begin{array}{rcllll} x \times (x^2 + xa + a^2) & = & x^3 & +ax^2 & +a^2x & \\ -a \times (x^2 + xa + a^2) & = & & -ax^2 & -a^2x & -a^3 \\ \hline (x - a) \times (x^2 + ax + a^2) & = & x^3 & & & -a^3 \end{array}$$

With formula (4.3) in hand we can now easily find the derivative of x^n:

$$\begin{aligned} f'(a) &= \lim_{x\to a} \frac{x^n - a^n}{x - a} \\ &= \lim_{x\to a} \bigl\{ x^{n-1} + x^{n-2}a + x^{n-3}a^2 + \cdots + xa^{n-2} + a^{n-1} \bigr\} \\ &= a^{n-1} + a^{n-2}a + a^{n-3}a^2 + \cdots + a\, a^{n-2} + a^{n-1}. \end{aligned}$$

Here there are n terms, and they all are equal to a^{n-1}, so the final result is

$$f'(a) = na^{n-1}.$$

One could also write this as $f'(x) = nx^{n-1}$, or, in Leibniz' notation

$$\frac{dx^n}{dx} = nx^{n-1}.$$

This formula turns out to be true in general, but here we have only proved it for the case in which n is a positive integer.

4.3 Differentiable implies Continuous

Theorem 4.3.1. If a function f is differentiable at some a in its domain, then f is also continuous at a.

Proof. We are given that

$$\lim_{x\to a} \frac{f(x) - f(a)}{x - a}$$

exists, and we must show that

$$\lim_{x\to a} f(x) = f(a).$$

This follows from the following computation

$$\begin{aligned} \lim_{x\to a} f(x) &= \lim_{x\to a} \bigl(f(x) - f(a) + f(a) \bigr) && \text{(algebra)} \\ &= \lim_{x\to a} \frac{f(x) - f(a)}{x - a} \cdot (x - a) + f(a) && \text{(more algebra)} \\ &= \left\{ \lim_{x\to a} \frac{f(x) - f(a)}{x - a} \right\} \cdot \lim_{x\to a} (x - a) + \lim_{x\to a} f(a) && \text{(Limit Properties)} \\ &= f'(a) \cdot 0 + f(a) && (f'(a) \text{ exists}) \\ &= f(a). \end{aligned}$$

□

4.4 Some non-differentiable functions

4.4.1 A graph with a corner.

Consider the function

$$f(x) = |x| = \begin{cases} x & \text{for } x \geq 0, \\ -x & \text{for } x < 0. \end{cases}$$

This function is continuous at all x, but it is not differentiable at $x = 0$.

To see this try to compute the derivative at 0,

$$f'(0) = \lim_{x\to 0} \frac{|x| - |0|}{x - 0} = \lim_{x\to 0} \frac{|x|}{x} = \lim_{x\to 0} \operatorname{sign}(x).$$

We know this limit does not exist (see §3.6.2)

If you look at the graph of $f(x) = |x|$ then you see what is wrong: the graph has a corner at the origin and it is not clear which line, if any, deserves to be called the tangent to the graph at the origin.

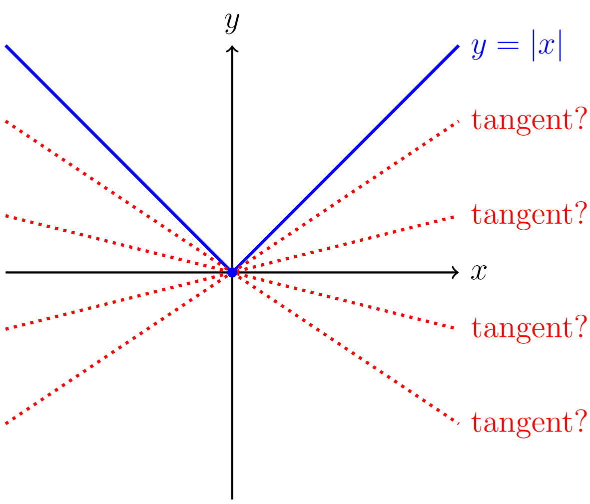


Figure 4.1: The graph of $y = |x|$ has no tangent at the origin.

4.4.2 A graph with a cusp.

Another example of a function without a derivative at $x = 0$ is

$$f(x) = \sqrt{|x|}.$$

When you try to compute the derivative you get this limit

$$f'(0) = \lim_{x\to 0} \frac{\sqrt{|x|}}{x} = ?$$

The limit from the right is

$$\lim_{x\searrow 0} \frac{\sqrt{|x|}}{x} = \lim_{x\searrow 0} \frac{1}{\sqrt{x}},$$

which does not exist (it is "$+\infty$"). Likewise, the limit from the left also does not exist ('tis "$-\infty$). Nonetheless, a drawing for the graph of f suggests an obvious tangent to the graph at $x = 0$, namely, the y-axis. That observation does not give us a derivative, because the y-axis is vertical and hence has no slope.

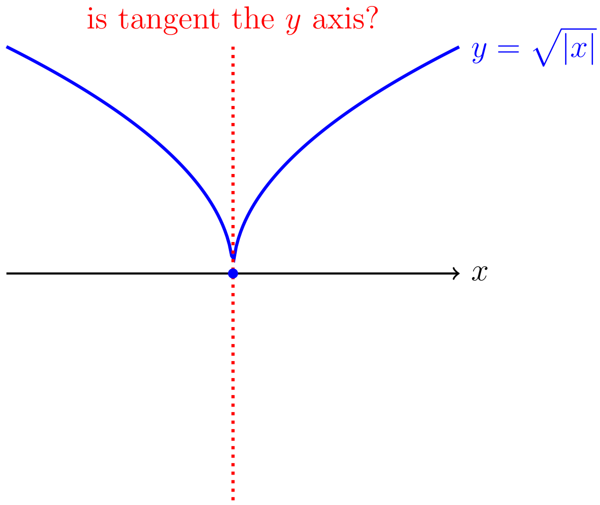


Figure 4.2: Tangent to the graph of $y = |x|^{1/2}$ at the origin.

4.4.3 A graph with absolutely no tangents, *anywhere.*

The previous two examples were about functions which did not have a derivative at $x = 0$. In both examples the point $x = 0$ was the only point where the function failed to have a derivative. It is easy to give examples of functions which are not differentiable at more than one value of x, but here I would like to show you a function f which doesn't have a derivative ***anywhere in its domain.***

To keep things short I won't write a formula for the function, and merely show you a graph. In this graph you see a typical path of a Brownian motion, i.e. t is time, and $x(t)$ is the position of a particle which undergoes a Brownian motion.

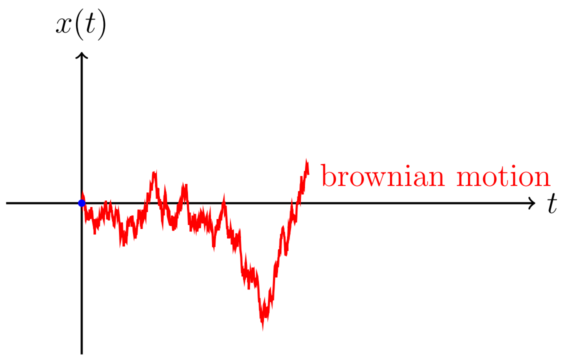


Figure 4.3: A Brownian motion. Note how the graph doesn't have a tangent anywhere at all.

The interested reader can watch this YouTube by Luke Harmon for a further explanation of Brownian motion. To see a similar graph check the Dow Jones or Nasdaq in the upper left hand corner of the web page at http://finance.yahoo.com in the afternoon on any weekday.

4.5 The Differentiation Rules

You could go on and compute more derivatives from the definition. Each time you would have to compute a new limit, and hope that there is some trick that allows you to find that limit. This is fortunately not necessary. It turns out that if you know a few basic derivatives (such as $dx^n/dx = nx^{n-1}$) the you can find derivatives of arbitrarily complicated functions by breaking them into smaller pieces. In this section we'll look at rules which tell you how to differentiate a function which is either the sum, difference, product or quotient of two other functions.

Constant rule:	$c' = 0$	$\dfrac{dc}{dx} = 0$
Sum rule:	$(u \pm v)' = u' \pm v'$	$\dfrac{du \pm v}{dx} = \dfrac{du}{dx} \pm \dfrac{dv}{dx}$
Product rule:	$(u \cdot v)' = u' \cdot v + u \cdot v'$	$\dfrac{duv}{dx} = \dfrac{du}{dx}v + u\dfrac{dv}{dx}$
Quotient rule:	$\left(\dfrac{u}{v}\right)' = \dfrac{u' \cdot v - u \cdot v'}{v^2}$	$\dfrac{d\frac{u}{v}}{dx} = \dfrac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

Table 4.1: The differentiation rules

The situation is analogous to that of the "limit-properties" $(P_1)\ldots(P_6)$ from the previous chapter which allowed us to compute limits without always having to go back to the epsilon-delta definition.

4.5.1 Sum, product and quotient rules.

In the following c and n are constants, u and v are functions of x, and $'$ denotes differentiation. The Differentiation Rules in function notation, and Leibniz notation, are listed in figure 4.1.

Note that we already proved the Constant Rule in example 4.2.2. We will now prove the sum, product and quotient rules.

4.5.2 Proof of the Sum Rule.

Suppose that $f(x) = u(x) + v(x)$ for all x where u and v are differentiable. Then

$$\begin{aligned}
f'(a) &= \lim_{x\to a} \frac{f(x)-f(a)}{x-a} && \text{(definition of } f'\text{)}\\
&= \lim_{x\to a} \frac{\big(u(x)+v(x)\big)-\big(u(a)+v(a)\big)}{x-a} && \text{(use } f = u+v\text{)}\\
&= \lim_{x\to a} \left(\frac{u(x)-u(a)}{x-a} + \frac{v(x)-v(a)}{x-a}\right) && \text{(algebra)}\\
&= \lim_{x\to a} \frac{u(x)-u(a)}{x-a} + \lim_{x\to a} \frac{v(x)-v(a)}{x-a} && \text{(limit property)}\\
&= u'(a) + v'(a) && \text{(definition of } u',\ v'\text{)}
\end{aligned}$$

4.5.3 Proof of the Product Rule.

Let $f(x) = u(x)v(x)$. To find the derivative we must express the change of f in terms of the changes of u and v

$$\begin{aligned}
f(x) - f(a) &= u(x)v(x) - u(a)v(a)\\
&= u(x)v(x) - u(x)v(a) + u(x)v(a) - u(a)v(a)\\
&= u(x)\big(v(x)-v(a)\big) + \big(u(x)-u(a)\big)v(a)
\end{aligned}$$

Now divide by $x - a$ and let $x \to a$:

$$\begin{aligned}
\lim_{x\to a} \frac{f(x)-f(a)}{x-a} &= \lim_{x\to a} u(x)\frac{v(x)-v(a)}{x-a} + \frac{u(x)-u(a)}{x-a}v(a)\\
&\qquad\qquad \text{(use the limit properties)}\\
&= \Big(\lim_{x\to a} u(x)\Big)\Big(\lim_{x\to a}\frac{v(x)-v(a)}{x-a}\Big) + \Big(\lim_{x\to a}\frac{u(x)-u(a)}{x-a}\Big)v(a)\\
&= u(a)v'(a) + u'(a)v(a),
\end{aligned}$$

as claimed. In this last step we have used that

$$\lim_{x\to a}\frac{u(x)-u(a)}{x-a} = u'(a) \quad \text{and} \quad \lim_{x\to a}\frac{v(x)-v(a)}{x-a} = v'(a)$$

and also that

$$\lim_{x\to a} u(x) = u(a)$$

This last limit follows from the fact that u is continuous, which in turn follows from the fact that u is differentiable.

4.5.4 Proof of the Quotient Rule .

We can break the proof into two parts. First we do the special case where $f(x) = 1/v(x)$, and then we use the product rule to differentiate

$$f(x) = \frac{u(x)}{v(x)} = u(x) \cdot \frac{1}{v(x)}.$$

So let $f(x) = 1/v(x)$. We can express the change in f in terms of the change in v

$$f(x) - f(a) = \frac{1}{v(x)} - \frac{1}{v(a)} = \frac{v(x) - v(a)}{v(x)v(a)}.$$

Dividing by $x - a$ we get

$$\frac{f(x) - f(a)}{x - a} = \frac{1}{v(x)v(a)} \frac{v(x) - v(a)}{x - a}.$$

Now we want to take the limit $x \to a$. We are given the v is differentiable, so it must also be continuous and hence

$$\lim_{x \to a} v(x) = v(a).$$

Therefore we find

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{x \to a} \frac{1}{v(x)v(a)} \lim_{x \to a} \frac{v(x) - v(a)}{x - a} = \frac{v'(a)}{v(a)^2}.$$

That completes the first step of the proof. In the second step we use the product rule to differentiate $f = u/v$

$$f' = \left(\frac{u}{v}\right)' = \left(u \cdot \frac{1}{v}\right)' = u' \cdot \frac{1}{v} + u \cdot \left(\frac{1}{v}\right)' = \frac{u'}{v} - u\frac{v'}{v^2} = \frac{u'v - uv'}{v^2}.$$

4.5.5 A shorter, but not quite perfect derivation of the Quotient Rule .

The Quotient Rule can be derived from the Product Rule as follows: if $w = u/v$ then

$$w \cdot v = u \tag{4.4}$$

By the product rule we have

$$w' \cdot v + w \cdot v' = u',$$

so that

$$w' = \frac{u' - w \cdot v'}{v} = \frac{u' - (u/v) \cdot v'}{v} = \frac{u' \cdot v - u \cdot v'}{v^2}.$$

Unlike the proof in §4.5.4 above, this argument does not prove that w is differentiable if u and v are. It only says that ***if the derivative exists*** then it must be what the Quotient Rule says it is.

The trick which is used here, is a special case of a method called "implicit differentiation." We have an equation (4.4) which the quotient w satisfies, and from by differentiating this equation we find w'.

4.5.6 Differentiating a constant multiple of a function .

Note that the rule

$$(cu)' = cu'$$

follows from the Constant Rule and the Product Rule.

4.5.7 Picture of the Product Rule.

If u and v are quantities which depend on x, and if increasing x by Δx causes u and v to change by Δu and Δv, then the product of u and v will change by

$$\Delta(uv) = (u + \Delta u)(v + \Delta v) - uv = u\Delta v + v\Delta u + \Delta u\Delta v. \tag{4.5}$$

If u and v are differentiable functions of x, then the changes Δu and Δv will be of the same order of magnitude as Δx, and thus one expects $\Delta u\Delta v$ to be much smaller. One therefore ignores the last term in (4.5), and thus arrives at

$$\Delta(uv) = u\Delta v + v\Delta u.$$

Leibniz would now divide by Δx and replace Δ's by d's to get the product rule:

$$\frac{\Delta(uv)}{\Delta x} = u\frac{\Delta v}{\Delta x} + v\frac{\Delta u}{\Delta x}.$$

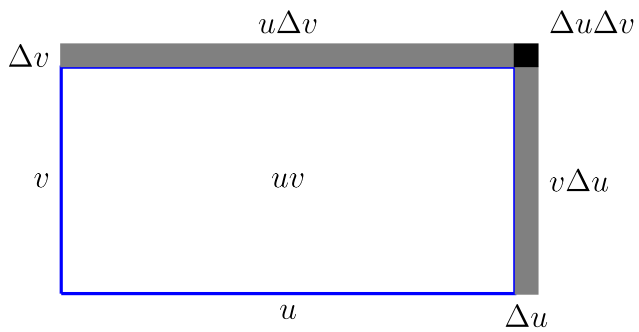


Figure 4.4: The Product Rule. *How much does the area of a rectangle change if its sides u and v are increased by Δu and Δv?* Most of the increase is accounted for by the two thin rectangles whose areas are $u\Delta v$ and $v\Delta u$. So the increase in area is approximately $u\Delta v + v\Delta u$, which explains why the product rule says $(uv)' = uv' + vu'$.

4.6 Differentiating powers of functions

4.6.1 Product rule with more than one factor.

If a function is given as the product of n functions, i.e.

$$f(x) = u_1(x) \times u_2(x) \times \cdots \times u_n(x),$$

then you can differentiate it by applying the product rule $n-1$ times (there are n factors, so there are $n-1$ multiplications.)
After the first step you would get

$$f' = u_1'\big(u_2\cdots u_n\big) + u_1\big(u_2\cdots u_n\big)'.$$

In the second step you apply the product rule to $(u_2u_3\cdots u_n)'$. This yields

$$\begin{aligned} f' &= u_1'u_2\cdots u_n + u_1\big[u_2'u_3\cdots u_n + u_2(u_3\cdots u_n)'\big] \\ &= u_1'u_2\cdots u_n + u_1u_2'u_3\cdots u_n + u_1u_2\big(u_3\cdots u_n\big)'. \end{aligned}$$

Continuing this way one finds after $n-1$ applications of the product rule that

$$\big(u_1\cdots u_n\big)' = u_1'u_2\cdots u_n + u_1u_2'u_3\cdots u_n + \cdots + u_1u_2u_3\cdots u_n'. \tag{4.6}$$

4.6.2 The Power rule .

If all n factors in the previous paragraph are the same, so that the function f is the n^{th} power of some other function,

$$f(x) = \big(u(x)\big)^n,$$

then all terms in the right hand side of (4.6) are the same, and, since there are n of them, one gets

$$f'(x) = nu^{n-1}(x)u'(x),$$

or, in Leibniz' notation,

$$\frac{du^n}{dx} = nu^{n-1}\frac{du}{dx}. \tag{4.7}$$

4.6.3 The Power Rule for Negative Integer Exponents .

We have just proved the power rule (4.7) assuming n is a positive integer. The rule actually holds for all real exponents n, but the proof is harder.
Here we prove the Power Rule for negative exponents using the Quotient Rule. Suppose $n = -m$ where m is a positive integer. Then the Quotient Rule tells us that

$$(u^n)' = \big(u^{-m}\big)' = \left(\frac{1}{u^m}\right)' \overset{\text{Q.R.}}{=} -\frac{(u^m)'}{(u^m)^2}.$$

Since m is a positive integer, we can use (4.7), so $(u^m)' = mu^{m-1}$, and hence

$$(u^n)' = -\frac{mu^{m-1}\cdot u'}{u^{2m}} = -mu^{-m-1}\cdot u' = nu^{n-1}u'.$$

4.6.4 The Power Rule for Rational Exponents .

So far we have proved that the power law holds if the exponent n is an integer.

We will now see how you can show that the power law holds even if the exponent n is any fraction, $n = p/q$. The following derivation contains the trick called ***implicit differentiation*** which we will study in more detail in Section 4.10.

So let $n = p/q$ where p and q are integers and consider the function

$$w(x) = u(x)^{p/q}.$$

Assuming that both u and w are differentiable functions, we will show that

$$w'(x) = \frac{p}{q}u(x)^{\frac{p}{q}-1}u'(x) \tag{4.8}$$

Raising both sides to the qth power gives

$$w(x)^q = u(x)^p.$$

Here the exponents p and q are integers, so we may apply the Power Rule to both sides. We get

$$qw^{q-1} \cdot w' = pu^{p-1} \cdot u'.$$

Dividing both sides by qw^{q-1} and substituting $u^{p/q}$ for w gives

$$w' = \frac{pu^{p-1} \cdot u'}{qw^{q-1}} = \frac{pu^{p-1} \cdot u'}{qu^{p(q-1)/q}} = \frac{pu^{p-1} \cdot u'}{qu^{p-(p/q)}} = \frac{p}{q} \cdot u^{(p/q)-1} \cdot u'$$

which is the Power Rule for $n = p/q$.

This proof is flawed because we did not show that $w(x) = u(x)^{p/q}$ is differentiable: we only showed what the derivative should be, *if it exists.*

4.6.5 Derivative of x^n for integer n .

If you choose the function $u(x)$ in the Power Rule to be $u(x) = x$, then $u'(x) = 1$, and hence the derivative of $f(x) = u(x)^n = x^n$ is

$$f'(x) = nu(x)^{n-1}u'(x) = nx^{n-1} \cdot 1 = nx^{n-1}.$$

We already knew this of course.

4.6.6 Example – differentiate a polynomial .

Using the Differentiation Rules you can easily differentiate any polynomial and hence any rational function. For example, using the Sum Rule, the Power Rule with $u(x) = x$, the rule $(cu)' = cu'$, the derivative of the polynomial

$$f(x) = 2x^4 - x^3 + 7$$

is

$$f'(x) = 8x^3 - 3x^2.$$

4.6.7 Example – differentiate a rational function.

By the Quotient Rule the derivative of the function

$$g(x) = \frac{2x^4 - x^3 + 7}{1 + x^2}$$

is

$$\begin{aligned} g'(x) &= \frac{(8x^3 - 3x^2)(1 + x^2) - (2x^4 - x^3 + 7)2x}{(1 + x^2)^2} \\ &= \frac{6x^5 - x^4 + 8x^3 - 3x^2 - 14x}{(1 + x^2)^2}. \end{aligned}$$

If you compare this example with the previous then you see that polynomials simplify when you differentiate them while rational functions become more complicated.

4.6.8 Derivative of the square root .

The derivative of $f(x) = \sqrt{x} = x^{1/2}$ is

$$f'(x) = \frac{1}{2}x^{1/2-1} = \frac{1}{2}x^{-1/2} = \frac{1}{2x^{1/2}} = \frac{1}{2\sqrt{x}}$$

where we used the power rule with $n = 1/2$ and $u(x) = x$.

4.7 Higher Derivatives

4.7.1 The derivative is a function.

If the derivative $f'(a)$ of some function f exists for all a in the domain of f, then we have a new function: namely, for each number in the domain of f we compute the derivative of f at that number. This function is called the ***derivative function*** of f, and it is denoted by f'. Now that we have agreed that the derivative of a function is a function, we can repeat the process and try to differentiate the derivative. The result, if it exists, is called the ***second derivative of*** f. It is denoted f''. The derivative of the second derivative is called the third derivative, written f''', and so on.
The nth derivative of f is denoted $f^{(n)}$. Thus

$$f^{(0)} = f, \qquad f^{(1)} = f', \qquad f^{(2)} = f'', \qquad f^{(3)} = f''', \ldots.$$

Leibniz' notation for the nth derivative of $y = f(x)$ is

$$\frac{d^n y}{dx^n} = f^{(n)}(x).$$

4.7.2 Example.

If $f(x) = x^2 - 2x + 3$ then

$$\begin{aligned} f(x) &= x^2 - 2x + 3 \\ f'(x) &= 2x - 2 \\ f''(x) &= 2 \\ f^{(3)}(x) &= 0 \\ f^{(4)}(x) &= 0 \\ &\vdots \end{aligned}$$

All further derivatives of f are zero.

4.7.3 Operator notation.

A common variation on Leibniz' notation for derivatives is the so-called ***operator notation***, as in

$$\frac{d(x^3 - x)}{dx} = \frac{d}{dx}(x^3 - x) = 3x^2 - 1.$$

For higher derivatives one can write

$$\frac{d^2y}{dx^2} = \left(\frac{d}{dx}\right)^2 y$$

Be careful to distinguish the second derivative from the square of the first derivative. Usually

$$\frac{d^2y}{dx^2} \neq \left(\frac{dy}{dx}\right)^2 \textbf{!!!!}$$

This YouTube by 3Blue1Brown gives a nice interpretation to higher order derivatives.

4.8 Differentiating Trigonometric functions

The trigonometric functions Sine, Cosine and Tangent are differentiable, and their derivatives are given by the following formulas

$$\frac{d \sin x}{dx} = \cos x, \qquad \frac{d \cos x}{dx} = -\sin x, \qquad \frac{d \tan x}{dx} = \frac{1}{\cos^2 x}. \tag{4.9}$$

Note the minus sign in the derivative of the cosine!

Proof. By definition one has

$$\sin'(x) = \lim_{h \to 0} \frac{\sin(x+h) - \sin(x)}{h}.$$

To simplify the numerator we use the trigonometric addition formula

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta.$$

with $\alpha = x$ and $\beta = h$, which results in

$$\begin{aligned}\frac{\sin(x+h)-\sin(x)}{h} &= \frac{\sin(x)\cos(h)+\cos(x)\sin(h)-\sin(x)}{h} \\ &= \cos(x)\frac{\sin(h)}{h} + \sin(x)\frac{\cos(h)-1}{h}\end{aligned}$$

Hence by the formulas

$$\lim_{h\to 0}\frac{\sin(h)}{h} = 1 \qquad \text{and} \qquad \lim_{h\to 0}\frac{\cos(h)-1}{h} = 0$$

from Section 3.11 we have

$$\begin{aligned}\sin'(x) &= \lim_{h\to 0}\cos(x)\frac{\sin(h)}{h} + \sin(x)\frac{\cos(h)-1}{h} \\ &= \cos(x)\cdot 1 + \sin(x)\cdot 0 \\ &= \cos(x).\end{aligned}$$

A similar computation leads to the stated derivative of $\cos x$.
To find the derivative of $\tan x$ we apply the quotient rule to

$$\tan x = \frac{\sin x}{\cos x} = \frac{f(x)}{g(x)}.$$

We get

$$\tan'(x) = \frac{\cos(x)\sin'(x) - \sin(x)\cos'(x)}{\cos^2(x)} = \frac{\cos^2(x)+\sin^2(x)}{\cos^2(x)} = \frac{1}{\cos^2(x)}$$

as claimed. □

4.9 The Chain Rule

4.9.1 Composition of functions.

Given two functions f and g, one can define a new function called the ***composition of*** f ***and*** g. The notation for the composition is $f\circ g$, and it is defined by the formula

$$f\circ g(x) = f\big(g(x)\big).$$

The domain of the composition is the set of all numbers x for which this formula gives you something well-defined.

For instance, if $f(x) = x^2 + x$ and $g(x) = 2x + 1$ then

$$f \circ g(x) = f\big(2x+1\big) = (2x+1)^2 + (2x+1)$$
$$\text{and } g \circ f(x) = g\big(x^2+x\big) = 2(x^2+x) + 1$$

Note that $f \circ g$ and $g \circ f$ are not the same fucntion in this example (they hardly ever are the same).

If you think of functions as expressing dependence of one quantity on another, then the composition of functions arises as follows. If a quantity z is a function of another quantity y, and if y itself depends on x, then z depends on x via y.

To get $f \circ g$ from the previous example, we could say $z = f(y)$ and $y = g(x)$, so that

$$z = f(y) = y^2 + y \text{ and } y = 2x + 1.$$

Give x one can compute y, and from y one can then compute z. The result will be

$$z = y^2 + y = (2x+1)^2 + (2x+1),$$

in other notation,

$$z = f(y) = f\big(g(x)\big) = f \circ g(x).$$

One says that ***the composition of*** f ***and*** g ***is the result of subsituting*** g ***in*** f.

Theorem 4.9.1 (Chain Rule)**.** If f and g are differentiable, so is the composition $f \circ g$. The derivative of $f \circ g$ is given by

$$(f \circ g)'(x) = f'(g(x))\, g'(x).$$

The chain rule tells you how to find the derivative of the composition $f \circ g$ of two functions f and g provided you now how to differentiate the two functions f and g.

When written in Leibniz' notation the chain rule looks particularly easy. Suppose that $y = g(x)$ and $z = f(y)$, then $z = f \circ g(x)$, and the derivative of z with respect to x is the derivative of the function $f \circ g$. The derivative of z with respect to y is the derivative of the function f, and the derivative of y with respect to x is the derivative of the function g. In short,

$$\frac{dz}{dx} = (f \circ g)'(x), \quad \frac{dz}{dy} = f'(y) \quad \text{and } \frac{dy}{dx} = g'(x)$$

so that the chain rule says

$$\frac{dz}{dx} = \frac{dz}{dy}\,\frac{dy}{dx}. \tag{4.10}$$

First proof of the chain rule (using Leibniz' notation). We first consider difference quotients instead of derivatives, i.e. using the same notation as above, we consider the effect of an increase of x by an amount Δx on the quantity z.

If x increases by Δx, then $y = g(x)$ will increase by

$$\Delta y = g(x + \Delta x) - g(x),$$

A depends on B depends on C depends on...

Someone is pumping water into a balloon. Assuming that the balloon is spherical you can say how large it is by specifying its radius R. For a growing balloon this radius will change with time t.

The volume of the balloon is a function of its radius, since the volume of a sphere of radius r is given by

$$V = \frac{4}{3}\pi r^3.$$

We now have two functions, the first f turns tells you the radius r of the balloon at time t,

$$r = f(t)$$

and the second tells you the volume of the balloon given its radius

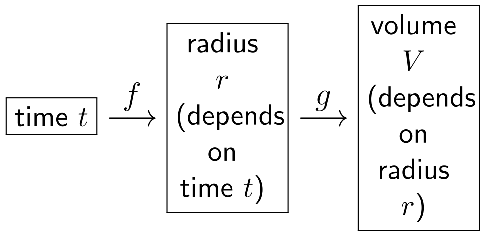
$$V = g(r).$$

The volume of the balloon at time t is then given by

$$V = g\big(f(t)\big) = g \circ f(t),$$

i.e. the function which tells you the volume of the balloon at time t is the composition of first f and then g.

Schematically we can summarize this chain of cause-and-effect relations as follows: you could either say that V depends on r, and r depends on t,



or you could say that V depends directly on t:

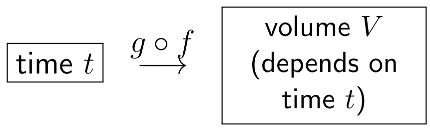


Figure 4.5: A "real world example" of a composition of functions.

and $z = f(y)$ will increase by

$$\Delta z = f(y + \Delta y) - f(y).$$

The ratio of the increase in $z = f(g(x))$ to the increase in x is

$$\frac{\Delta z}{\Delta x} = \frac{\Delta z}{\Delta y} \cdot \frac{\Delta y}{\Delta x}.$$

In contrast to dx, dy and dz in equation (4.10), the Δx, etc. here are finite quantities, so this equation is just algebra: you can cancel the two Δys. If you let the increase Δx go to zero, then the increase Δy will also go to zero, and the difference quotients converge to the derivatives,

$$\frac{\Delta z}{\Delta x} \longrightarrow \frac{dz}{dx}, \quad \frac{\Delta z}{\Delta y} \longrightarrow \frac{dz}{dy}, \quad \frac{\Delta y}{\Delta x} \longrightarrow \frac{dy}{dx}$$

which immediately leads to Leibniz' form of the quotient rule. □

Proof of the chain rule. We verify the formula in Theorem 4.9.1 at some arbitrary value $x = a$, i.e. we will show that

$$(f \circ g)'(a) = f'(g(a))\, g'(a).$$

By definition the left hand side is

$$(f \circ g)'(a) = \lim_{x \to a} \frac{(f \circ g)(x) - (f \circ g)(a)}{x - a} = \lim_{x \to a} \frac{f(g(x)) - f(g(a))}{x - a}.$$

The two derivatives on the right hand side are given by

$$g'(a) = \lim_{x \to a} \frac{g(x) - g(a)}{x - a}$$

and

$$f'(g(a)) = \lim_{y \to a} \frac{f(y) - f(g(a))}{y - g(a)}.$$

Since g is a differentiable function it must also be a continuous function, and hence $\lim_{x \to a} g(x) = g(a)$. So we can substitute $y = g(x)$ in the limit defining $f'(g(a))$

$$f'(g(a)) = \lim_{y \to a} \frac{f(y) - f(g(a))}{y - g(a)} = \lim_{x \to a} \frac{f(g(x)) - f(g(a))}{g(x) - g(a)}. \tag{4.11}$$

Put all this together and you get

$$\begin{aligned}
(f \circ g)'(a) &= \lim_{x \to a} \frac{f(g(x)) - f(g(a))}{x - a} \\
&= \lim_{x \to a} \frac{f(g(x)) - f(g(a))}{g(x) - g(a)} \cdot \frac{g(x) - g(a)}{x - a} \\
&= \lim_{x \to a} \frac{f(g(x)) - f(g(a))}{g(x) - g(a)} \cdot \lim_{x \to a} \frac{g(x) - g(a)}{x - a} \\
&= f'(g(a)) \;\cdot\; g'(a)
\end{aligned}$$

which is what we were supposed to prove – the proof seems complete.

There is one flaw in this proof, namely, we have divided by $g(x) - g(a)$, which is not allowed when $g(x) - g(a) = 0$. This flaw can be fixed but we will not go into the details here.[2] □

Before tackling some examples that require the rules of differentiation the reader is encouraged to view YouTube by 3Blue1Brown .

4.9.2 First example.

We go back to the functions

$$z = f(y) = y^2 + y \text{ and } y = g(x) = 2x + 1$$

[2] Briefly, you have to show that the function

$$h(y) = \begin{cases} \{f(y) - f(g(a))\}/(y - g(a)) & y \neq a \\ f'(g(a)) & y = a \end{cases}$$

is continuous.

from the beginning of this section. The composition of these two functions is

$$z = f(g(x)) = (2x+1)^2 + (2x+1) = 4x^2 + 6x + 2.$$

We can compute the derivative of this composed function, i.e. the derivative of z with respect to x in two ways. First, you simply differentiate the last formula we have:

$$\frac{dz}{dx} = \frac{d(4x^2+6x+2)}{dx} = 8x+6. \tag{4.12}$$

The other approach is to use the chain rule:

$$\frac{dz}{dy} = \frac{d(y^2+y)}{dy} = 2y+1,$$

and

$$\frac{dy}{dx} = \frac{d(2x+1)}{dx} = 2.$$

Hence, by the chain rule one has

$$\frac{dz}{dx} = \frac{dz}{dy}\frac{dy}{dx} = (2y+1)\cdot 2 = 4y+2. \tag{4.13}$$

The two answers (4.12) and (4.13) should be the same. Once you remember that $y = 2x+1$ you see that this is indeed true:

$$y = 2x+1 \implies 4y+2 = 4(2x+1)+2 = 8x+6.$$

The two computations of dz/dx therefore lead to the same answer. In this example there was no clear advantage in using the chain rule. The chain rule becomes useful when the functions f and g become more complicated.

4.9.3 Example where you really need the Chain Rule.

We know what the derivative of $\sin x$ with respect to x is, but none of the rules we have found so far tell us how to differentiate $f(x) = \sin(2x)$.
The function $f(x) = \sin 2x$ is the composition of two simpler functions, namely

$$f(x) = g(h(x)) \text{ where } g(u) = \sin u \text{ and } h(x) = 2x.$$

We know how to differentiate each of the two functions g and h:

$$g'(u) = \cos u, \quad h'(x) = 2.$$

Therefore the chain rule implies that

$$f'(x) = g'(h(x))h'(x) = \cos(2x)\cdot 2 = 2\cos 2x.$$

Leibniz would have decomposed the relation $y = \sin 2x$ between y and x as

$$y = \sin u, \quad u = 2x$$

and then computed the derivative of $\sin 2x$ with respect to x as follows

$$\frac{d\sin 2x}{dx} \overset{u=2x}{=} \frac{d\sin u}{dx} = \frac{d\sin u}{du}\cdot\frac{du}{dx} = \cos u\cdot 2 = 2\cos 2x.$$

4.9.4 The Power Rule and the Chain Rule.

The Power Rule, which says that for any function f and any rational number n one has

$$\frac{d}{dx}\big(f(x)^n\big) = nf(x)^{n-1}f'(x),$$

is a special case of the Chain Rule, for one can regard $y = f(x)^n$ as the composition of two functions

$$y = g(u), \quad u = f(x)$$

where $g(u) = u^n$. Since $g'(u) = nu^{n-1}$ the Chain Rule implies that

$$\frac{du^n}{dx} = \frac{du^n}{du} \cdot \frac{du}{dx} = nu^{n-1}\frac{du}{dx}.$$

Setting $u = f(x)$ and $\frac{du}{dx} = f'(x)$ then gives you the Power Rule.

4.9.5 The volume of an inflating balloon.

Consider the "real world example" from page 73 again. There we considered a growing water balloon of radius

$$r = f(t).$$

The volume of this balloon is

$$V = \frac{4}{3}\pi r^3 = \frac{4}{3}\pi f(t)^3.$$

We can regard this as the composition of two functions, $V = g(r) = \frac{4}{3}\pi r^3$ and $r = f(t)$. According to the chain rule the rate of change of the volume with time is now

$$\frac{dV}{dt} = \frac{dV}{dr}\frac{dr}{dt}$$

i.e. it is the product of the rate of change of the volume with the radius of the balloon and the rate of change of the balloon's radius with time. From

$$\frac{dV}{dr} = \frac{d\frac{4}{3}\pi r^3}{dr} = 4\pi r^2$$

we see that

$$\frac{dV}{dr} = 4\pi r^2\,\frac{dr}{dt}.$$

For instance, if the radius of the balloon is growing at 0.5inch/sec, and if its radius is $r = 3.0$inch, then the volume is growing at a rate of

$$\frac{dV}{dt} = 4\pi(3.0\text{inch})^2 \times 0.5\text{inch/sec} \approx 57\text{inch}^3/\text{sec}.$$

4.9.6 A more complicated example.

Suppose you needed to find the derivative of

$$y = h(x) = \frac{\sqrt{x+1}}{(\sqrt{x+1}+1)^2}$$

We can write this function as a composition of two simpler functions, namely,

$$y = f(u), \quad u = g(x),$$

with

$$f(u) = \frac{u}{(u+1)^2} \text{ and } g(x) = \sqrt{x+1}.$$

The derivatives of f and g are

$$f'(u) = \frac{1\cdot(u+1)^2 - u\cdot 2(u+1)}{(u+1)^4} = \frac{u+1-2}{(u+1)^3} = \frac{u-1}{(u+1)^3},$$

and

$$g'(x) = \frac{1}{2\sqrt{x+1}}.$$

Hence the derivative of the composition is

$$h'(x) = \frac{d}{dx}\left\{\frac{\sqrt{x+1}}{(\sqrt{x+1}+1)^2}\right\} = f'(u)g'(x) = \frac{u-1}{(u+1)^3} \cdot \frac{1}{2\sqrt{x+1}}.$$

The result should be a function of x, and we achieve this by replacing all u's with $u = \sqrt{x+1}$:

$$\frac{d}{dx}\left\{\frac{\sqrt{x+1}}{(\sqrt{x+1}+1)^2}\right\} = \frac{\sqrt{x+1}-1}{(\sqrt{x+1}+1)^3} \cdot \frac{1}{2\sqrt{x+1}}.$$

The last step (where you replace u by its definition in terms of x) is important because the problem was presented to you with only x and y as variables while u was a variable you introduced yourself to do the problem.

Sometimes it is possible to apply the Chain Rule without introducing new letters, and you will simply think "the derivative is the derivative of the outside with respect to the inside times the derivative of the inside." For instance, to compute

$$\frac{d\, 4+\sqrt{7+x^3}}{dx}$$

you could set $u = 7 + x^3$, and compute

$$\frac{d\, 4+\sqrt{7+x^3}}{dx} = \frac{d\, 4+\sqrt{u}}{du} \cdot \frac{du}{dx}.$$

Instead of writing all this explicitly, you could think of $u = 7 + x^3$ as the function "inside the square root," and think of $4 + \sqrt{u}$ as "the outside function." You would then immediately write

$$\frac{d}{dx}(4+\sqrt{7+x^3}) = \frac{1}{2\sqrt{7+x^3}} \cdot 3x^2.$$

4.9.7 The Chain Rule and composing more than two functions.

Often we have to apply the Chain Rule more than once to compute a derivative. Thus if $y = f(u)$, $u = g(v)$, and $v = h(x)$ we have

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dv} \cdot \frac{dv}{dx}.$$

In functional notation this is

$$(f \circ g \circ h)'(x) = f'(g(h(x)) \cdot g'(h(x)) \cdot h'(x).$$

Note that each of the three derivatives on the right is evaluated at a different point. Thus if $b = h(a)$ and $c = g(b)$ the Chain Rule is

$$\left.\frac{dy}{dx}\right|_{x=a} = \left.\frac{dy}{du}\right|_{u=c} \cdot \left.\frac{du}{dv}\right|_{v=b} \cdot \left.\frac{dv}{dx}\right|_{x=a}.$$

For example, if $y = \dfrac{1}{1+\sqrt{9+x^2}}$, then $y = 1/(1+u)$ where $u = 1+\sqrt{v}$ and $v = 9+x^2$ so

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dv} \cdot \frac{dv}{dx} = -\frac{1}{(1+u)^2} \cdot \frac{1}{2\sqrt{v}} \cdot 2x.$$

so

$$\left.\frac{dy}{dx}\right|_{x=4} = \left.\frac{dy}{du}\right|_{u=6} \cdot \left.\frac{du}{dv}\right|_{v=25} \cdot \left.\frac{dv}{dx}\right|_{x=4} = -\frac{1}{7} \cdot \frac{1}{10} \cdot 8.$$

4.10 Implicit differentiation

4.10.1 The recipe.

Recall that an implicitely defined function is a function $y = f(x)$ which is defined by an equation of the form

$$F(x, y) = 0.$$

We call this equation the ***defining equation*** for the function $y = f(x)$. To find $y = f(x)$ for a given value of x you must solve the defining equation $F(x, y) = 0$ for y.

Here is a recipe for computing the derivative of an implicitely defined function.

1. Differentiate the equation $F(x, y) = 0$; you may need the chain rule to deal with the occurences of y in $F(x, y)$;

2. You can rearrange the terms in the result of step 1 so as to get an equation of the form

$$G(x, y)\frac{dy}{dx} + H(x, y) = 0, \tag{4.14}$$

where G and H are expressions containing x and y but not the derivative.

3. Solve the equation in step 2 for $\frac{dy}{dx}$:

$$\frac{dy}{dx} = -\frac{H(x,y)}{G(x,y)} \tag{4.15}$$

4. If you also have an explicit description of the function (i.e. a formula expressing $y = f(x)$ in terms of x) then you can substitute $y = f(x)$ in the expression (4.15) to get a formula for dy/dx in terms of x only.

   Often no explicit formula for y is available and you can't take this last step. In that case (4.15) is as far as you can go.

Observe that by following this procedure you will get a formula for the derivative $\frac{dy}{dx}$ which contains both x and y.

4.10.2 Dealing with equations of the form $F_1(x,y) = F_2(x,y)$.

If the implicit definition of the function is not of the form $F(x,y) = 0$ but rather of the form $F_1(x,y) = F_2(x,y)$ then you move all terms to the left hand side, and proceed as above. E.g. to deal with a function $y = f(x)$ which satisfies

$$y^2 + x = xy$$

you rewrite this equation as

$$y^2 + x - xy = 0$$

and set $F(x,y) = y^2 + x - xy$.

4.10.3 Example – Derivative of $\sqrt[4]{1-x^4}$.

Consider the function

$$f(x) = \sqrt[4]{1-x^4}, \quad -1 \le x \le 1.$$

We will compute its derivative in two ways: first the direct method, and then using the method f implicit differentiation (i.e. the recipe above).

The direct approach goes like this:

$$\begin{aligned} f'(x) &= \frac{d\left(1-x^4\right)^{1/4}}{dx} \\ &= \tfrac{1}{4}\left(1-x^4\right)^{-3/4}\frac{d(1-x^4)}{dx} \\ &= \tfrac{1}{4}\left(1-x^4\right)^{-3/4}\left(-4x^3\right) \\ &= -\frac{x^3}{\left(1-x^4\right)^{3/4}} \end{aligned}$$

To find the derivative using implicit differentiation we must first find a nice implicit description of the function. For instance, we could decide to get rid of all roots or fractional exponents in the function and point out that $y = \sqrt[4]{1-x^4}$ satisfies the equation $y^4 = 1 - x^4$. So our implicit description of the function $y = f(x) = \sqrt[4]{1-x^4}$ is

$$x^4 + y^4 - 1 = 0; \qquad \text{The defining function is therefore } F(x,y) = x^4 + y^4 - 1$$

Differentiate both sides with respect to x (and remember that $y = f(x)$, so y here is a function of x), and you get

$$\frac{dx^4}{dx} + \frac{dy^4}{dx} - \frac{d1}{dx} = 0 \implies 4x^3 + 4y^3\frac{dy}{dx} = 0.$$

The expressions G and H from equation (4.14) in the recipe are $G(x,y) = 4y^3$ and $H(x,y) = 4x^3$.

This last equation can be solved for dy/dx:

$$\frac{dy}{dx} = -\frac{x^3}{y^3}.$$

This is a nice and short form of the derivative, but it contains y as well as x. To express dy/dx in terms of x only, and remove the y dependency we use $y = \sqrt[4]{1-x^4}$. The result is

$$f'(x) = \frac{dy}{dx} = -\frac{x^3}{y^3} = -\frac{x^3}{\left(1-x^4\right)^{3/4}}.$$

4.10.4 Another example.

Let f be a function defined by

$$y = f(x) \iff 2y + \sin y = x, \text{ i.e. } 2y + \sin y - x = 0.$$

For instance, if $x = 2\pi$ then $y = \pi$, i.e. $f(2\pi) = \pi$.

To find the derivative dy/dx we differentiate the defining equation

$$\frac{d(2y + \sin y - x)}{dx} = \frac{d0}{dx} \implies 2\frac{dy}{dx} + \cos y\frac{dy}{dx} - \frac{dx}{dx} = 0 \implies (2 + \cos y)\frac{dy}{dx} - 1 = 0.$$

Solve for $\frac{dy}{dx}$ and you get

$$f'(x) = \frac{1}{2 + \cos y} = \frac{1}{2 + \cos f(x)}.$$

If we were asked to find $f'(2\pi)$ then, since we know $f(2\pi) = \pi$, we could answer

$$f'(2\pi) = \frac{1}{2 + \cos \pi} = \frac{1}{2-1} = 1.$$

If we were asked $f'(\pi/2)$, then all we would be able to say is

$$f'(\pi/2) = \frac{1}{2 + \cos f(\pi/2)}.$$

To say more we would first have to find $y = f(\pi/2)$, which one does by solving

$$2y + \sin y = \frac{\pi}{2}.$$

4.10.5 Derivatives of Arc Sine and Arc Tangent.

Recall that

$$y = \arcsin x \iff x = \sin y \text{ and } -\tfrac{\pi}{2} \le y \le \tfrac{\pi}{2},$$

and

$$y = \arctan x \iff x = \tan y \text{ and } -\tfrac{\pi}{2} < y < \tfrac{\pi}{2}.$$

Theorem 4.10.1.

$$\frac{d \arcsin x}{dx} = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{d \arctan x}{dx} = \frac{1}{1+x^2}$$

Proof. If $y = \arcsin x$ then $x = \sin y$. Differentiate this relation

$$\frac{dx}{dx} = \frac{d \sin y}{dx}$$

and apply the chain rule. You get

$$1 = \big(\cos y\big)\, \frac{dy}{dx},$$

and hence

$$\frac{dy}{dx} = \frac{1}{\cos y}.$$

How do we get rid of the y on the right hand side? We know $x = \sin y$, and also $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$. Therefore

$$\sin^2 y + \cos^2 y = 1 \implies \cos y = \pm\sqrt{1-\sin^2 y} = \pm\sqrt{1-x^2}.$$

Since $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$ we know that $\cos y \ge 0$, so we must choose the positive square root. This leaves us with $\cos y = \sqrt{1-x^2}$, and hence

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}.$$

The derivative of $\arctan x$ is found in the same way, and you should really do this yourself. □

4.11 PROBLEMS

DIFFERENTIATION

99. Compute the derivative of the following functions

$$f(x) = x^2 - 2x \qquad g(x) = \frac{1}{x}$$

$$k(x) = x^3 - 17x \qquad u(x) = \frac{2}{1+x}$$

$$v(x) = \sqrt{x} \qquad w(x) = \frac{1}{\sqrt{x}}$$

using either (4.1) or (4.2).

100. Which of the following functions is differentiable at $x = 0$?

$$f(x) = x|x|, \quad g(x) = x\sqrt{|x|},$$
$$h(x) = x + |x|, \quad k(x) = x^2 \sin\frac{\pi}{x},$$
$$\ell(x) = x \sin\frac{\pi}{x}.$$

These formulas do not define k and ℓ at $x = 0$. We define $k(0) = \ell(0) = 0$.

101. For which value(s) is the function defined by

$$f(x) = \begin{cases} ax + b & \text{for } x < 0 \\ x - x^2 & \text{for } x \geq 0 \end{cases}$$

differentiable at $x = 0$? Sketch the graph of the function f for the values a and b you found.

102. For which value(s) is the function defined by

$$f(x) = \begin{cases} ax^2 + b & \text{for } x < 1 \\ x - x^2 & \text{for } x \geq 1 \end{cases}$$

differentiable at $x = 0$? Sketch the graph of the function f for the values a and b you found.

103. For which value(s) is the function defined by

$$f(x) = \begin{cases} ax^2 & \text{for } x < 2 \\ x + b & \text{for } x \geq 2 \end{cases}$$

differentiable at $x = 0$? Sketch the graph of the function f for the values a and b you found.

104.

True or false: If a function f is continuous at some $x = a$ then it must also be differentiable at $x = a$?

105.

True or false: If a function f is differentiable at some $x = a$ then it must also be continuous at $x = a$?

RULES OF DIFFERENTIATION

106. Let $f(x) = (x^2 + 1)(x^3 + 3)$. Find $f'(x)$ in two ways:
(a) by multiplying and then differentiating,
(b) by using the product rule.
Are your answers the same?

107. Let $f(x) = (1+x^2)^4$. Find $f'(x)$ in two ways, first by expanding to get an expression for $f(x)$ as a polynomial in x and then differentiating, and then by using the power rule. Are the answers the same?

108. Prove the statement in §4.5.6, i.e. show that $(cu)' = c(u')$ follows from the product rule.

Compute the derivatives of the following functions. (try to simplify your answers)

109. $f(x) = x + 1 + (x+1)^2$

110. $f(x) = \dfrac{x-2}{x^4+1}$

111. $f(x) = (\dfrac{1}{1+x})^{-1}$

112. $f(x) = \sqrt{1-x^2}$

113. $f(x) = \dfrac{ax+b}{cx+d}$

114. $f(x) = \dfrac{1}{(1+x^2)^2}$

115. $f(x) = \dfrac{x}{1+\sqrt{x}}$

116. $f(x) = \sqrt{\dfrac{1-x}{1+x}}$

117. $f(x) = \sqrt[3]{x+\sqrt{x}}$

118. $\varphi(t) = \dfrac{t}{1+\sqrt{t}}$

119. $g(s) = \sqrt{\dfrac{1-s}{1+s}}$

120. $h(\rho) = \sqrt[3]{\rho+\sqrt{\rho}}$

121. *Using derivatives to approximate numbers.*
(a) Find the derivative of $f(x) = x^{4/3}$.
(b) Use **(a)** to estimate the number

$$\frac{127^{4/3} - 125^{4/3}}{2}$$

approximately without a calculator. Your answer should have the form p/q where p and q are integers. [*Hint: Note that* $5^3 = 125$ *and take a good look at equation* (4.1).]
(c) Approximate in the same way the numbers $\sqrt{143}$ and $\sqrt{145}$ (Hint: $12 \times 12 = 144$).

122. *Making the product and quotient rules look nicer.*
Instead of looking at the derivative of a function you can look at the ratio of its derivative to the function itself, i.e. you can compute f'/f. This quantity is called the ***logarithmic derivative of the function*** f for reasons that will become clear later.

(a) Compute the logarithmic derivative of these functions (i.e. find $f'(x)/f(x)$)

$$F(x) = x, \quad g(x) = 3x, \quad h(x) = x^2$$
$$k(x) = -x^2, \quad \ell(x) = 2007x^2, \quad m(x) = x^{2007}$$

(b) Show that for any pair of functions u and v one has

$$\frac{(uv)'}{uv} = \frac{u'}{u} + \frac{v'}{v}$$
$$\frac{(u/v)'}{u/v} = \frac{u'}{u} - \frac{v'}{v}$$
$$\frac{(u^n)'}{u^n} = n\,\frac{u'}{u}$$

123. **(a)** Find $f'(x)$ and $g'(x)$ if

$$f(x) = \frac{1+x^2}{2x^4+7}, \qquad g(x) = \frac{2x^4+7}{1+x^2}.$$

Note that $f(x) = 1/g(x)$.
(b) Is it true that $f'(x) = 1/g'(x)$?
(c) Is it true that $f(x) = g^{-1}(x)$?
(d) Is it true that $f(x) = g(x)^{-1}$?

124. **(a)** Let $x(t) = (1-t^2)/(1+t^2)$, $y(t) = 2t/(1+t^2)$ and $u(t) = y(t)/x(t)$. Find dx/dt, dy/dt.
(b) Now that you've done **(a)** there are two different ways of finding du/dt. What are they, and use one of both to find du/dt.

HIGHER DERIVATIVES

125. The equation

$$\frac{2x}{x^2-1} = \frac{1}{x+1} + \frac{1}{x-1} \tag{†}$$

holds for all values of x (except $x = \pm 1$), so you should get the same answer if you differentiate both sides. Check this.

Compute the third derivative of $f(x) = 2x/(x^2-1)$ by using either the left or right hand side (your choice) of (†).

126. Compute the first, second and third derivatives of the following functions

$$f(x) = (x+1)^4 \qquad g(x) = \left(x^2+1\right)^4$$

$$h(x) = \sqrt{x-2} \qquad k(x) = \sqrt[3]{x - \frac{1}{x}}$$

127. Find the derivatives of 10^{th} order of the functions

$$f(x) = x^{12} + x^8 \qquad g(x) = \frac{1}{x}$$

$$h(x) = \frac{12}{1-x} \qquad k(x) = \frac{x^2}{1-x}$$

128. Find $f'(x)$, $f''(x)$ and $f^{(3)}(x)$ if

$$f(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \frac{x^6}{720}.$$

129. Find the 12^{th} derivative of the function $f(x) = \dfrac{1}{x+2}$.

130. Find the n^{th} order derivative of $f(x) = \dfrac{1}{x+2}$(i.e. find a formula for $f^{(n)}(x)$ which is valid for all $n = 0, 1, 2, 3 \ldots$).

131. Find the n^{th} order derivative of $g(x) = \dfrac{x}{x+2}$.

132. Find dy/dx and d^2y/dx^2 if $y = x/(x+2)$.

133. Find du/dt and d^2u/dt^2 if $u = t/(t+2)$. Hint: See previous problem.

134. Find $\dfrac{d}{dx}\left(\dfrac{x}{x+2}\right)$ and $\dfrac{d^2}{dx^2}\left(\dfrac{x}{x+2}\right)$. Hint: See previous problem.

135. Find $\dfrac{d}{dx}\left(\dfrac{x}{x+2}\right)\Bigg|_{x=1}$ and $\dfrac{d}{dx}\left(\dfrac{1}{1+2}\right)$.

136. Find d^2y/dx^2 and $(dy/dx)^2$ if $y = x^3$.

DIFFERENTIATING TRIGONOMETRIC FUNCTIONS

Find the derivatives of the following functions (try to simplify your answers)

137. $f(x) = \sin(x) + \cos(x)$

138. $f(x) = 2\sin(x) - 3\cos(x)$

139. $f(x) = 3\sin(x) + 2\cos(x)$

140. $f(x) = x\sin(x) + \cos(x)$

141. $f(x) = x\cos(x) - \sin x$

142. $f(x) = \dfrac{\sin x}{x}$

143. $f(x) = \cos^2(x)$

144. $f(x) = \sqrt{1 - \sin^2 x}$

145. $f(x) = \sqrt{\dfrac{1 - \sin x}{1 + \sin x}}$

146. $\cot(x) = \dfrac{\cos x}{\sin x}$.

147. Can you find a and b so that the function

$$f(x) = \begin{cases} \cos x & \text{for } x \le \frac{\pi}{4} \\ a + bx & \text{for } x > \frac{\pi}{4} \end{cases}$$

is differentiable at $x = \pi/4$?

148. Can you find a and b so that the function

$$f(x) = \begin{cases} \tan x & \text{for } x < \frac{\pi}{6} \\ a + bx & \text{for } x \ge \frac{\pi}{6} \end{cases}$$

is differentiable at $x = \pi/4$?

149. If f is a given function, and you have another function g which satisfies $g(x) = f(x)+12$ for all x, then f and g have the same derivatives. Prove this. [*Hint: it's a short proof – use the differentiation rules.*]

150.

Show that the functions

$$f(x) = \sin^2 x \text{ and } g(x) = -\cos^2 x$$

have the same derivative by computing $f'(x)$ and $g'(x)$.
With hindsight this was to be expected – why?

151. Find the first and second derivatives of the functions

$$f(x) = \tan^2 x \text{ and } g(x) = \frac{1}{\cos^2 x}.$$

Hint: remember your trig to reduce work! †382

THE CHAIN RULE

152. Let $y = \sqrt{1+x^3}$ and find dy/dx using the Chain Rule. Say what plays the role of $y = f(u)$ and $u = g(x)$.

153. Repeat the previous problem with

$$y = (1 + \sqrt{1+x})^3.$$

154. Emily and Kate differentiated $y = \sqrt{1+x^3}$ with respect to x differently. Emily wrote $y = \sqrt{u}$ and $u = 1 + x^3$ while Kate wrote $y = \sqrt{1+v}$ and $v = x^3$. Assuming neither one made a mistake, did they get the same answer?

155. Let $y = u^3 + 1$ and $u = 3x + 7$. Find $\dfrac{dy}{dx}$ and $\dfrac{dy}{du}$. Express the former in terms of x and the latter in terms of u.

156. Suppose that $f(x) = \sqrt{x}$, $g(x) = 1 + x^2$, $v(x) = f \circ g(x)$, $w(x) = g \circ f(x)$. Find formulas for $v(x)$, $w(x)$, $v'(x)$, and $w'(x)$.

Compute the following derivatives

157. $f(x) = \sin 2x - \cos 3x$ †382

158. $f(x) = \sin \dfrac{\pi}{x}$ †382

159. $f(x) = \sin(\cos 3x)$ †382

160. $f(x) = \dfrac{\sin x^2}{x^2}$ †382

161. $f(x) = \tan\sqrt{1+x^2}$ †382

162. $f(x) = \cos^2 x - \cos x^2$ †382

163. Emily is pouring water into a glass. At time t (seconds) the height of the water in the glass is $h(t)$ (inch). The glass company, which made the glass, says that the volume in the glass to height h is $V = 1.2\ h^2$ (fluid ounces).

(a) The water height in the glass is rising at 2 inch per second at the moment that the height is 2 inch. How fast is Emily pouring water into the glass?

(b) If Emily pours water at a rate of 1 ounce per second, then how fast is the water level in the glass going up when it is 3 inches?

(c) If Emily pours water at 1 ounce per second, and at some moment the water level is going up at 0.5 inch per second. What is the water level at that moment?

164. Find the derivative of $f(x) = x \cos \frac{\pi}{x}$ at the point $x = -\frac{2}{3}$. †382

165. Suppose that $f(x) = x^2 + 1$, $g(x) = x + 5$, and

$$v = f \circ g, \qquad w = g \circ f, \qquad p = f \cdot g, \qquad q = g \cdot f.$$

Find $v(x)$, $w(x)$, $p(x)$, and $q(x)$. †382

166. Suppose that the functions f and g and their derivatives with respect to x have the following values at $x = 0$ and $x = 1$.

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
0	1	1	5	1/3
1	3	-4	-1/3	-8/3

Define

$$v(x) = f(g(x)), \qquad w(x) = g(f(x)),$$
$$p(x) = f(x)g(x), \qquad q(x) = g(x)f(x).$$

Evaluate $v(0)$, $w(0)$, $p(0)$, $q(0)$, $v'(0)$ and $w'(0)$, $p'(0)$, $q'(0)$. If there is insufficient information to answer the question, so indicate.

167. A differentiable function f satisfies $f(3) = 5$, $f(9) = 7$, $f'(3) = 11$ and $f'(9) = 13$. Find an equation for the tangent line to the curve $y = f(x^2)$ at the point $(x, y) = (3, 7)$.

168. There is a function f whose second derivative satisfies

$$f''(x) = -64f(x). \tag{†}$$

(a) One such function is $f(x) = \sin ax$, provided you choose the right constant a: Which value should a have?

(b) For which choices of the constants A, a and b does the function $f(x) = A\sin(ax + b)$ satisfy (†)? †382

169. A cubical sponge, hereafter refered to as 'Bob', is absorbing water, which causes him to expand. His side at time t is $S(t)$. His volume is $V(t)$.

(a) What is the relation between $S(t)$ and $V(t)$, i.e. can you find a function f so that $V(t) = f(S(t))$?

(b) Describe the meaning of the derivatives $S'(t)$ and $V'(t)$ in one plain english sentence each. If we measure lengths in inches and time in minutes, then what units do $t, S(t), V(t), S'(t)$ and $V'(t)$ have?

(c) What is the relation between $S'(t)$ and $V'(t)$?

(d) At the moment that Bob's volume is 8 cubic inches, he is absorbing water at a rate of 2 cubic inch per minute. How fast is his side $S(t)$ growing? †382

IMPLICIT DIFFERENTIATION

For each of the following problems find the derivative $f'(x)$ if $y = f(x)$ satisfies the given equation. State what the expressions $F(x,y)$, $G(x,y)$ and $H(x,y)$ from the recipe in this chapter are.

If you can find an explicit description of the function $y = f(x)$, say what it is.

170. $xy = \dfrac{\pi}{6}$

171. $\sin(xy) = \frac{1}{2}$

172. $\dfrac{xy}{x+y} = 1$

173. $x + y = xy$

174. $(y-1)^2 + x = 0$

175. $(y+1)^2 + y - x = 0$

176. $(y-x)^2 + x = 0$

177. $(y+x)^2 + 2y - x = 0$

178. $\big(y^2 - 1\big)^2 + x = 0$

179. $\big(y^2 + 1\big)^2 - x = 0$

180. $x^3 + xy + y^3 = 3$

181. $\sin x + \sin y = 1$

182. $\sin x + xy + y^5 = \pi$

183. $\tan x + \tan y = 1$

For each of the following explicitly defined functions find an implicit definition which does not involve taking roots. Then use this description to find the derivative dy/dx.

184. $y = f(x) = \sqrt{1-x}$

185. $y = f(x) = \sqrt[4]{x + x^2}$

186. $y = f(x) = \sqrt{1 - \sqrt{x}}$

187. $y = f(x) = \sqrt[4]{x - \sqrt{x}}$

188. $y = f(x) = \sqrt[3]{\sqrt{2x+1} - x^2}$

189. $y = f(x) = \sqrt[4]{x + x^2}$

190. $y = f(x) = \sqrt[3]{x - \sqrt{2x+1}}$

191. $y = f(x) = \sqrt[4]{\sqrt[3]{x}}$

192.

(Inverse trig review) Simplify the following expressions, and indicate for which values of x (or θ, or $\ldots$) your simplification is valid. In case of doubt, try plotting the function on a graphing calculator.

(a) $\sin \arcsin x$

(b) $\cos \arcsin x$

(c) $\arctan(\tan\theta)$

(d) $\cot \arctan x$

(e) $\tan \arctan z$

(f) $\tan \arcsin \theta$

(g) $\arcsin(\sin\theta)$

(h) $\cot \arcsin x$

Now that you know the derivatives of arcsin and arctan, you can find the derivatives of the following functions. What are they?

193. $f(x) = \arcsin(2x)$

194. $f(x) = \arcsin \sqrt{x}$

195. $f(x) = \arctan(\sin x)$

196. $f(x) = \sin \arctan x$

197. $f(x) = \big(\arcsin x\big)^2$

198. $f(x) = \dfrac{1}{1 + (\arctan x)^2}$

199. $f(x) = \sqrt{1 - (\arcsin x)^2}$

200. $f(x) = \dfrac{\arctan x}{\arcsin x}$

RELATED RATES OF CHANGE

201. A 10 foot long pole has one end (B) on the floor and another (A) against a wall. If the bottom of the pole is 8 feet away from the wall, and if it is sliding away from the wall at 7 feet per second, then with what speed is the top (A) going down?

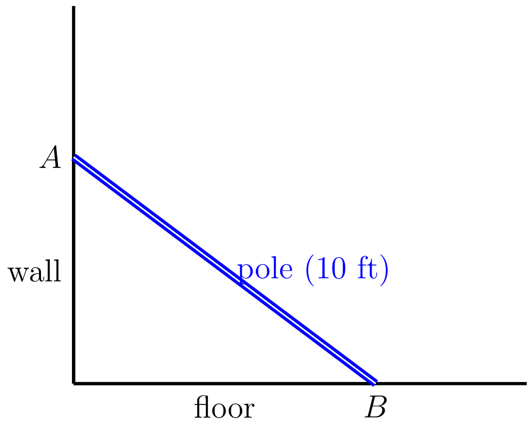


Figure 4.6: A pole leaning against a wall

202. A pole 10 feet long rests against a vertical wall. If the bottom of the pole slides away from the wall at a speed of 2 ft/s, how fast is the angle between the top of the pole and the wall changing when the angle is $\pi/4$ radians?

203. A pole 13 meters long is leaning against a wall. The bottom of the pole is pulled along the ground away from the wall at the rate of 2 m/s. How fast is its height on the wall decreasing when the foot of the pole is 5 m away from the wall?

204. A television camera is positioned 4000 ft from the base of a rocket launching pad. A rocket rises vertically and its speed is 600 ft/s when it has risen 3000 feet.

(a) How fast is the distance from the television camera to the rocket changing at that moment?

(b) How fast is the camera's angle of elevation changing at that same moment? (Assume that the television camera points toward the rocket.)

205. A 2-foot tall dog is walking away from a streetlight which is on a 10-foot pole. At a certain moment, the tip of the dogs shadow is moving away from the streetlight at 5 feet per second. How fast is the dog walking at that moment?

206. An isosceles triangle is changing its shape: the lengths of the two equal sides remain fixed at 2 inch, but the angle $\theta(t)$ between them changes.

Let $A(t)$ be the area of the triangle at time t. If the area increases at a constant rate of $0.5\text{inch}^2/\text{sec}$, then how fast is the angle increasing or decreasing when $\theta = 60°$?

207. A point P is moving in the first quadrant of the plane. Its motion is parallel to the x-axis; its distance to the x-axis is always 10 (feet). Its velocity is 3 feet per second to the left. We write θ for the angle between the positive x-axis and the line segment from the origin to P.

(a) Make a drawing of the point P.

(b) Where is the point when $\theta = \pi/3$?

(c) Compute the rate of change of the angle θ at the moment that $\theta = \frac{\pi}{3}$.

208. The point Q is moving on the line $y = x$ with velocity 3 m/sec. Find the rate of change of the following quantities at the moment in which Q is at the point $(1, 1)$:

(a) the distance from Q to the origin,

(b) the distance from Q to the point $R(2, 0)$,

(c) the angle $\angle ORQ$ where R is again the point $R(2, 0)$.

209. A point P is sliding on the parabola with equation $y = x^2$. Its x-coordinate is increasing at a constant rate of 2 feet/minute.

Find the rate of change of the following quantities at the moment that P is at $(3, 9)$:

(a) the distance from P to the origin,

(b) the area of the rectangle whose lower left corner is the origin and whose upper right corner is P,

(c) the slope of the tangent to the parabola at P,

(d) the angle $\angle OPQ$ where Q is the point $(0, 3)$.

210. A certain amount of gas is trapped in a cylinder with a piston. The ***ideal gas law*** from thermodynamics says that if the cylinder is not heated, and if the piston moves slowly, then one has

$$pV = CT$$

where p is the pressure in the gas, V is its volume, T its temperature (in degrees Kelvin) and C is a constant depending on the amount of gas trapped in the cylinder.

(a) If the pressure is 10psi (pounds per square inch), if the volume is 25inch^3, and if the piston is moving so that the gas volume is expanding at a rate of 2inch^3 per minute, then what is the rate of change of the pressure?

(b) The ideal gas law turns out to be only approximately true. A more accurate description of gases is given by ***van der Waals' equation of state***, which says that

$$\left(p + \frac{a}{V^2}\right)(V - b) = C$$

where a, b, C are constants depending on the temperature and the amount and type of gas in the cylinder.

Suppose that the cylinder contains fictitious gas for which one has $a = 12$ and $b = 3$. Suppose that at some moment the volume of gas is 12in^3, the pressure is 25psi and suppose the gas is expanding at 2 inch^3 per minute. Then how fast is the pressure changing?

Chapter 5

Graph Sketching and Max-Min Problems

The signs of the first and second derivatives of a function tell us something about the shape of its graph. In this chapter we learn how to find that information.

5.1 Tangent and Normal lines to a graph

The slope of the tangent the tangent to the graph of f at the point $(a, f(a))$ is

$$m = f'(a) \tag{5.1}$$

and hence the equation for the tangent is

$$y = f(a) + f'(a)(x - a). \tag{5.2}$$

The slope of the normal line to the graph is $-1/m$ and thus one could write the equation for the normal as

$$y = f(a) - \frac{x - a}{f'(a)}. \tag{5.3}$$

When $f'(a) = 0$ the tangent is horizontal, and hence the normal is vertical. In this case the equation for the normal cannot be written as in (5.3), but instead one gets the simpler equation

$$y = f(a).$$

Both cases are covered by this form of the equation for the normal

$$x = a + f'(a)(f(a) - y). \tag{5.4}$$

Both (5.4) and (5.3) are formulas that you shouldn't try to remember. It is easier to remember that if the slope of the tangent is $m = f'(a)$, then the slope of the normal is $-1/m$.

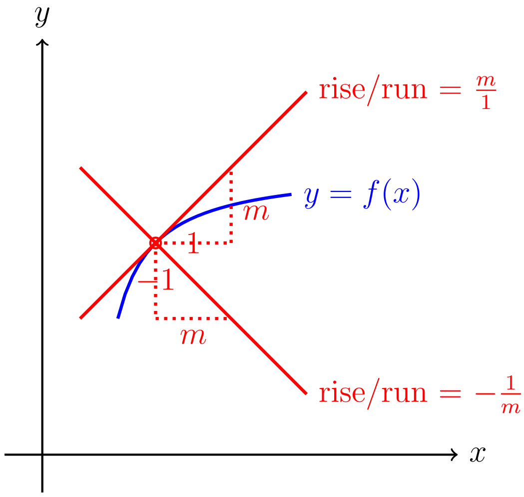


Figure 5.1: Why "slope of normal $= \dfrac{-1}{\text{slope of tangent}}$."

To refresh your knowledge of *tangents* and *normals* consider watching this YouTube by atomi

5.2 The Intermediate Value Theorem

It is said that a function is continuous if you can draw its graph without taking your pencil off the paper. A more precise version of this statement is as follows:

Theorem 5.2.1 (The Intermediate Value Theorem.)**.** If f is a continuous function on an interval $a \leq x \leq b$, and if y is some number between $f(a)$ and $f(b)$, then there is a number c with $a \leq c \leq b$ such that $f(c) = y$.

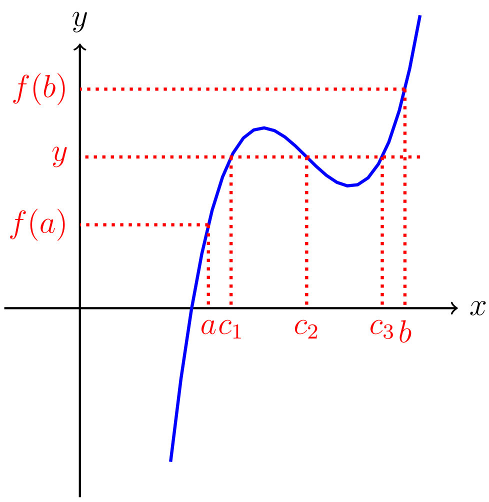


Figure 5.2: The Intermediate Value Theorem says that a continuous function must attain any given value y between $f(a)$ and $f(b)$ at least once. In this example there are three values of c for which $f(c) = y$ holds.

5.2.1 Example – Square root of 2.

Consider the function $f(x) = x^2$. Since $f(1) < 2$ and $f(2) = 4 > 2$ the intermediate value theorem with $a = 1$, $b = 2$, $y = 2$ tells us that there is a number c between 1 and 2 such that $f(c) = 2$, i.e. for which $c^2 = 2$. So the theorem tells us that the square root of 2 exists.

5.2.2 Example – The equation $\theta + \sin\theta = \frac{\pi}{2}$.

Consider the function $f(x) = x + \sin x$. It is a continuous function at all x, so from $f(0) = 0$ and $f(\pi) = \pi$ it follows that there is a number θ between 0 and π such that $f(\theta) = \pi/2$. In other words, the equation

$$\theta + \sin\theta = \frac{\pi}{2} \tag{5.5}$$

has a solution θ with $0 \le \theta \le \pi/2$. Unlike the previous example, where we knew the solution was $\sqrt{2}$, there is no simple formula for the solution to (5.5).

5.2.3 Example – Solving $1/x = 0$.

If we apply the intermediate value theorem to the function $f(x) = 1/x$ on the interval $[a, b] = [-1, 1]$, then we see that for any y between $f(a) = f(-1) = -1$ and $f(b) = f(1) = 1$ there is a number c in the interval $[-1, 1]$ such that $1/c = y$. For instance, we could choose $y = 0$ (that's between -1 and $+1$), and conclude that there is some c with $-1 \le c \le 1$ and $1/c = 0$.

But there is no such c, because $1/c$ is never zero! So we have done something wrong, and the mistake we made is that we overlooked that our function $f(x) = 1/x$ is not defined on the ***whole*** interval $-1 \le x \le 1$ because it is not defined at $x = 0$. *The moral: always check the hypotheses of a theorem before you use it!*

5.3 Finding sign changes of a function

The intermediate value theorem implies the following very useful fact.

Theorem 5.3.1. If f is continuous function on some interval $a < x < b$, and if $f(x) \ne 0$ for all x in this interval, then $f(x)$ is either positive for all $a < x < b$ or else it is negative for all $a < x < b$.

Proof. The theorem says that there can't be two numbers $a < x_1 < x_2 < b$ such that $f(x_1)$ and $f(x_2)$ have opposite signs. If there were two such numbers then the intermediate value theorem would imply that somewhere between x_1 and x_2 there was a c with $f(c) = 0$. But we are assuming that $f(c) \ne 0$ whenever $a < c < b$. $\square$

5.3.1 Example.

Consider

$$f(x) = (x-3)(x-1)^2(2x+1)^3.$$

The zeros of f (i.e. the solutions of $f(x)=0$) are $-\frac{1}{2}, 1, 3$. These numbers split the real line into four intervals

$$(-\infty, -\tfrac{1}{2}), \quad (-\tfrac{1}{2}, 1), \quad (1,3), \quad (3,\infty).$$

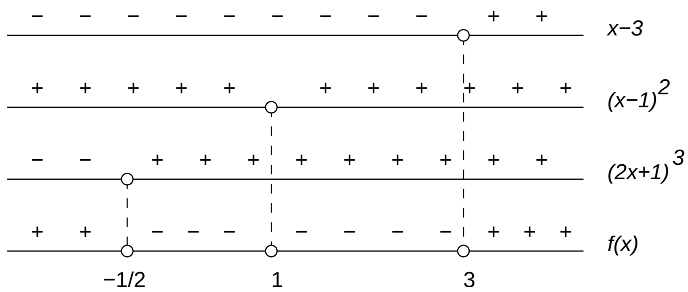
Theorem 5.3.1 tells us that $f(x)$ cannot change its sign in any of these intervals. For instance, $f(x)$ has the same sign for all x in the first interval $(\infty, -\frac{1}{2})$. Now we choose a number we like from this interval (e.g. -1) and find the sign of $f(-1)$: $f(-1) = (-4)(-2)^2(-3)^3$ is positive. Therefore $f(x) > 0$ for all x in the interval $(-\infty, -\frac{1}{2})$. In the same we find

$$\begin{aligned}
f(-1) = (-4)(-2)^2(-3)^3 > 0 &\implies f(x) > 0 \text{ for } x < -\tfrac{1}{2} \\
f(0) = (-3)(-1)^2(1)^3 < 0 &\implies f(x) < 0 \text{ for } -\tfrac{1}{2} < x < 1 \\
f(2) = (-1)(1)^2(5)^3 < 0 &\implies f(x) < 0 \text{ for } 1 < x < 3 \\
f(4) = (1)(3)^2(9)^3 > 0 &\implies f(x) > 0 \text{ for } x > 3.
\end{aligned}$$

If you know all the zeroes of a continuous function, then this method allows you to decide where the function is positive or negative. However, when the given function is factored into easy functions, as in this example, there is a different way of finding the signs of f. For each of the factors $x-3$, $(x-1)^2$ and $(2x+1)^3$ it is easy to determine the sign, for any given x. These signs can only change at a zero of the factor. Thus we have

- $x-3$ is positive for $x>3$ and negative for $x<3$;
- $(x-1)^2$ is always positive (except at $x=1$);
- $(2x+1)^3$ is positive for $x > -\frac{1}{2}$ and negative for $x < -\frac{1}{2}$.

Multiplying these signs we get the same conclusions as above. We can summarize this computation in the following diagram:



5.4 The Mean Value Theorem.

Theorem 5.4.1 (The Mean Value Theorem)**.** If f is a differentiable function on the interval $a \leq x \leq b$, then there is some number c, with $a < c < b$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

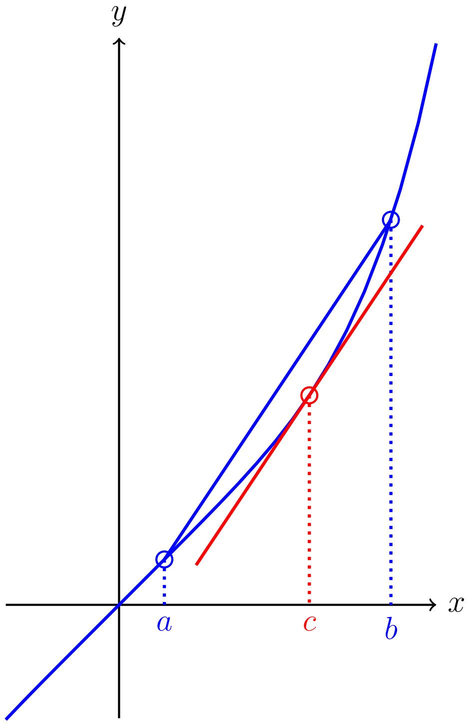


Figure 5.3: According to the Mean Value Theorem there always is some number c between a and b such that the tangent to the graph of f is parallel to the line segment connecting the two points $(a, f(a))$ and $(b, f(b))$. This is true for any choice of a and b; c depends on a and b of course.

For further elucidation consider watching YouTube by The Organic Chemistry Tutor

5.5 Increasing and decreasing functions

Here are four very similar definitions – look closely to see how they differ.

- A function is called ***increasing*** if $a < b$ implies $f(a) < f(b)$ for all numbers a and b in the domain of f.
- A function is called ***decreasing*** if $a < b$ implies $f(a) > f(b)$ for all numbers a and b in the domain of f.
- The function f is called ***non-decreasing*** if $a < b$ implies $f(a) \leq f(b)$ for all numbers a and b in the domain of f.

- The function f is called ***non-increasing*** if $a < b$ implies $f(a) \geq f(b)$ for all numbers a and b in the domain of f.

You can summarize these definitions as follows:

f **is ...**	**if for all** a **and** b **one has...**
Increasing:	$a < b \implies f(a) < f(b)$
Decreasing:	$a < b \implies f(a) > f(b)$
Non-increasing:	$a < b \implies f(a) \geq f(b)$
Non-decreasing:	$a < b \implies f(a) \leq f(b)$

The sign of the derivaitve of f tells you if f is increasing or not. More precisely:

Theorem 5.5.1. If a function is non-decreasing on an interval $a < x < b$ then $f'(x) \geq 0$ for all x in that interval.
If a function is non-increasing on an interval $a < x < b$ then $f'(x) \leq 0$ for all x in that interval.

Proof. For instance, if f is non-decreasing, then for any given x and any positive Δx one has $f(x + \Delta x) \geq f(x)$ and hence

$$\frac{f(x+\Delta x) - f(x)}{\Delta x} \geq 0.$$

Now let $\Delta x \searrow 0$ and you find that

$$f'(x) = \lim_{\Delta x \searrow 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} \geq 0.$$

□

What about the converse, i.e. if you know the sign of f' then what can you say about f?

Theorem 5.5.2. Suppose f is a differentiable function on an interval (a, b).
If $f'(x) > 0$ for all $a < x < b$, then f is increasing.
If $f'(x) < 0$ for all $a < x < b$, then f is decreasing.

Proof. We show that $f'(x) > 0$ for all x implies that f is increasing. Let $x_1 < x_2$ be two numbers between a and b. Then the Mean Value Theorem implies that there is some c between x_1 and x_2 such that

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1},$$

or

$$f(x_2) - f(x_1) = f'(c)(x_2 - x_1).$$

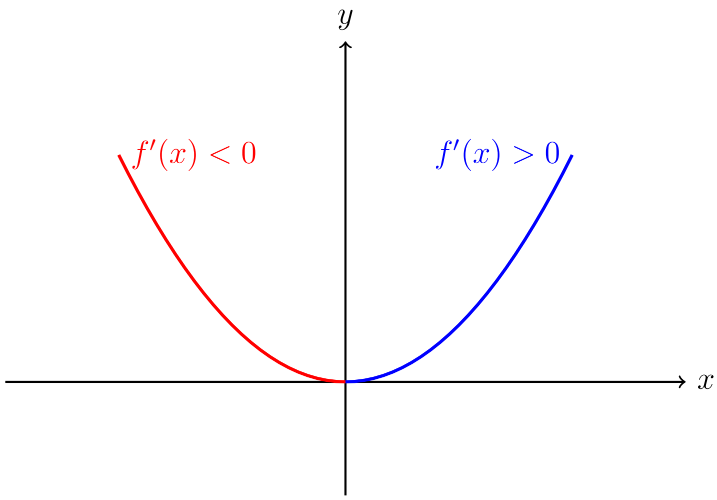
Since we know that $f'(c) > 0$ and $x_2 - x_1 > 0$ it follows that $f(x_2) - f(x_1) > 0$, i.e. $f(x_2) > f(x_1)$. □

5.6 Examples

Armed with these theorems we can now split the graph of any function into increasing and decreasing parts simply by computing the derivative $f'(x)$ and finding out where $f'(x) > 0$ and where $f'(x) < 0$ – i.e. we apply the method form the previous section to f' rather than f.

5.6.1 example: the parabola $y = x^2$.

The familiar graph of $f(x) = x^2$ consists of two parts, one decreasing and one increasing.



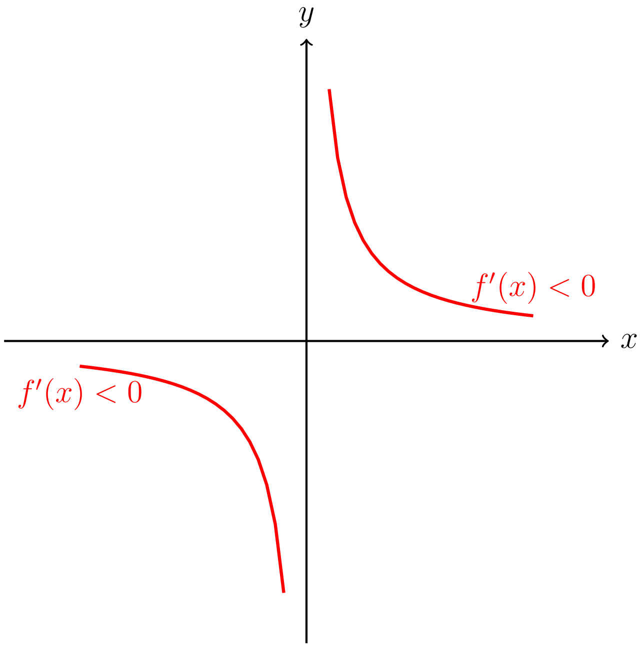
You can see this from the derivative which is

$$f'(x) = 2x \begin{cases} > 0 & \text{for } x > 0 \\ < 0 & \text{for } x < 0. \end{cases}$$

Therefore the function $f(x) = x^2$ is ***decreasing for*** $x < 0$ and ***increasing for*** $x > 0$***.***

5.6.2 example: the hyperbola $y = 1/x$.

The derivative of the function $f(x) = 1/x = x^{-1}$ is $f'(x) = -\frac{1}{x^2}$ which is always negative.



You would therefore think that this function is decreasing, or at least non-increasing: if $a < b$ then $1/a \geq 1/b$. But this isn't true if you take $a = -1$ and $b = 1$:

$$a = -1 < 1 = b, \text{ but } \frac{1}{a} = -1 < 1 = \frac{1}{b} \text{ !!}$$

The problem is that we used theorem 5.5.2, but it you carefully read that theorem then you see that it applies to functions ***that are defined on an interval***. The function in this example, $f(x) = 1/x$, is not defined on the interval $-1 < x < 1$ because it isn't defined at $x = 0$. That's why you can't conclude that the $f(x) = 1/x$ is increasing from $x = -1$ to $x = +1$.

On the other hand, the function is defined and differentiable on the interval $0 < x < \infty$, so theorem 5.5.2 tells us that $f(x) = 1/x$ is decreasing for $x > 0$. This means, that as long as x is positive, increasing x will decrease $1/x$.

5.6.3 example: a cubic function.

Consider the function

$$y = f(x) = x^3 - x.$$

Its derivative is

$$f'(x) = 3x^2 - 1.$$

We try to find out where f' is positive, and where it is negative by factoring $f'(x)$

$$f'(x) = 3\big(x^2 - \tfrac{1}{3}\big) = 3\big(x + \tfrac{1}{3}\sqrt{3}\big)\big(x - \tfrac{1}{3}\sqrt{3}\big)$$

from which you see that

$$\begin{aligned} f'(x) &> 0 \text{ for } x < -\tfrac{1}{3}\sqrt{3} \\ f'(x) &< 0 \text{ for } -\tfrac{1}{3}\sqrt{3} < x < \tfrac{1}{3}\sqrt{3} \\ f'(x) &> 0 \text{ for } x > \tfrac{1}{3}\sqrt{3} \end{aligned}$$

Therefore the function f is

$$\text{increasing on } \left(-\infty, -\tfrac{1}{3}\sqrt{3}\right), \quad \text{decreasing on } \left(-\tfrac{1}{3}\sqrt{3}, \tfrac{1}{3}\sqrt{3}\right), \quad \text{increasing on } \left(\tfrac{1}{3}\sqrt{3}, \infty\right).$$

At the two points $x = \pm\frac{1}{3}\sqrt{3}$ one has $f'(x) = 0$ so there the tangent will be horizontal. This leads us to the following picture of the graph of f:

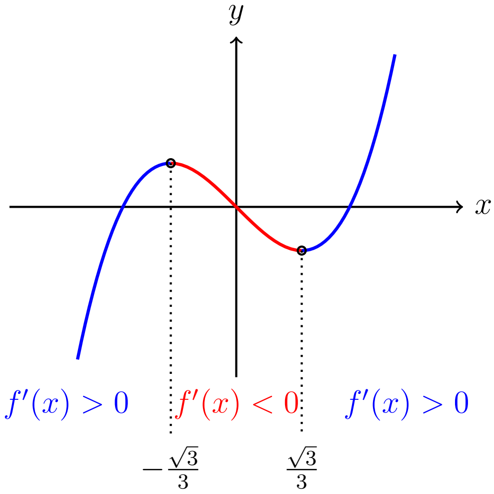


Figure 5.4: The graph of $f(x) = x^3 - x$.

5.6.4 A function whose tangent turns up and down infinitely often near the origin.

We end with a weird example. Somewhere in the mathematician's zoo of curious functions the following will be on exhibit. Consider the function

$$f(x) = \frac{x}{2} + x^2 \sin\frac{\pi}{x}.$$

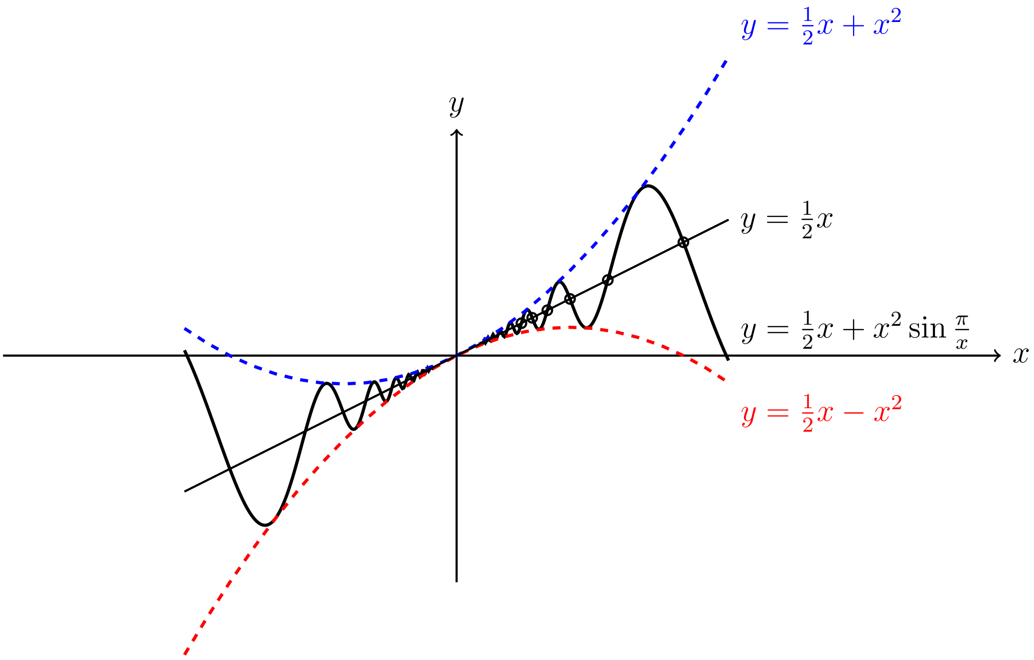


Figure 5.5: Positive derivative at a point ($x = 0$) does not mean that the function is "increasing near that point." The slopes at the intersection points alternate between $\frac{1}{2} - \pi$ and $\frac{1}{2} + \pi$.

For $x = 0$ this formula is undefined, and we are free to define $f(0) = 0$. This makes the function continuous at $x = 0$. In fact, this function is differentiable at $x = 0$, with derivative given by

$$f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{1}{2} + x \sin \frac{\pi}{x} = \frac{1}{2}.$$

(To find the limit apply the sandwich theorem to $-|x| \le x \sin \frac{\pi}{x} \le |x|$.)
So the slope of the tangent to the graph at the origin is positive ($\frac{1}{2}$), and one would *think* that the function should be increasing near $x = 0$ (i.e. bigger x gives bigger $f(x)$.) The point of this example is that this turns out not to be true.
To explain why not, we must compute the derivative of this function for $x \neq 0$. It is given by

$$f'(x) = \frac{1}{2} - \pi \cos \frac{\pi}{x} + 2x \sin \frac{\pi}{x}.$$

Now consider the sequence of intersection points $P_1, P_2, \dots$ of the graph with the line $y = x/2$. They are

$$P_k(x_k, y_k), \qquad x_k = \frac{1}{k}, \quad y_k = f(x_k).$$

For larger and larger k the points P_k tend to the origin (the x coordinate is $\frac{1}{k}$ which goes to 0 as $k \to \infty$). The slope of the tangent at P_k is given by

$$f'(x_k) = \frac{1}{2} - \pi \cos\frac{\pi}{1/k} + 2\frac{1}{k}\sin\frac{\pi}{1/k}$$
$$= \frac{1}{2} - \pi \underbrace{\cos k\pi}_{=(-1)^k} + \frac{2}{k}\underbrace{\sin k\pi}_{=0}$$
$$= \begin{cases} -\frac{1}{2} - \pi \approx -2.64159265358979\ldots & \text{for } k \text{ even} \\ \frac{1}{2} + \pi \approx +3.64159265358979\ldots & \text{for } k \text{ odd} \end{cases}$$

In other words, along the sequence of points P_k the slope of the tangent flip-flops between $\frac{1}{2} - \pi$ and $\frac{1}{2} + \pi$, i.e. between a positive and a negative number.

In particular, the slope of the tangent at the odd intersection points is negative, and so you would expect the function to be decreasing there. In other words we see that ***even though the derivative at $x = 0$ of this function is positive, there are points on the graph arbitrarily close to the origin where the tangent has negative slope.***

5.7 Maxima and Minima

A function has a ***global maximum*** at some a in its domain if $f(x) \le f(a)$ for all other x in the domain of f. Global maxima are sometimes also called “absolute maxima.”

A function has a ***local maximum*** at some a in its domain if there is a small $\delta > 0$ such that $f(x) \le f(a)$ for all x with $a - \delta < x < a + \delta$ which lie in the domain of f.

Every global maximum is a local maximum, but a local maximum doesn’t have to be a global maximum.

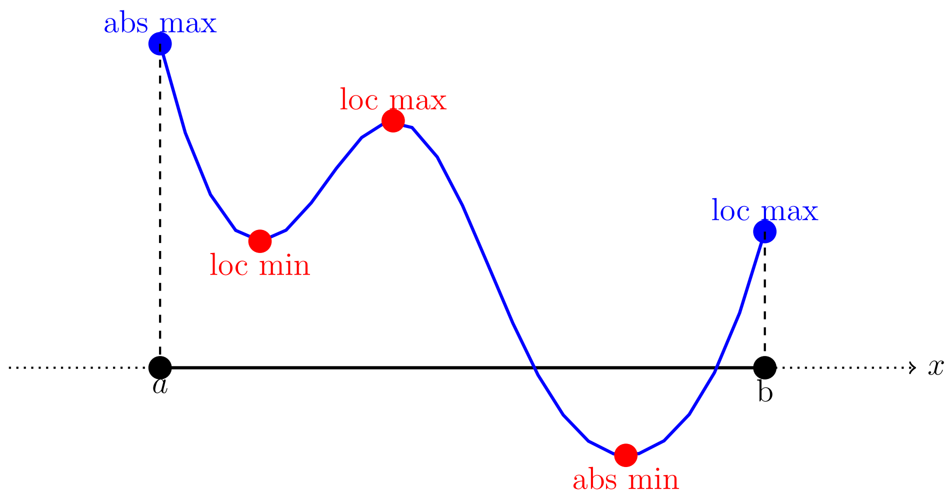


Figure 5.6: A function defined on a closed interval $[a, b]$ with one interior absolute minimum, another interior local minimum, an interior local maximum, and two local maxima on the boundary, one of which is in fact an absolute maximum.

5.7.1 Where to find local maxima and minima.

Any x value for which $f'(x) = 0$ is called a ***stationary point*** for the function f.

Theorem 5.7.1. Suppose f is a differentiable function on some interval $[a, b]$.
Every local maximum or minimum of f is either one of the end points of the interval $[a, b]$, or else it is a stationary point for the function f.

Proof. Suppose that f has a local maximum at x and suppose that x is not a or b. By assumption the left and right hand limits

$$f'(x) = \lim_{\Delta x \nearrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \text{ and } f'(x) = \lim_{\Delta x \searrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

both exist and they are equal.
Since f has a local maximum at x we have $f(x + \Delta x) - f(x) \leq 0$ if $-\delta < \Delta x < \delta$. In the first limit we also have $\Delta x < 0$, so that

$$\lim_{\Delta x \nearrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \leq 0$$

Hence $f'(x) \leq 0$.
In the second limit we have $\Delta x > 0$, so

$$\lim_{\Delta x \searrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \geq 0$$

which implies $f'(x) \geq 0$.
Thus we have shown that $f'(x) \leq 0$ and $f'(x) \geq 0$ at the same time. This can only be true if $f'(x) = 0$. □

5.7.2 How to tell if a stationary point is a maximum, a minimum, or neither.

If $f'(c) = 0$ then c is a stationary point (by definition), and it might be local maximum or a local minimum. You can tell what kind of stationary point c is by looking at the signs of $f'(x)$ for x near c.

Theorem 5.7.2. If in some small interval $(c - \delta, c + \delta)$ you have $f'(x) < 0$ for $x < c$ and $f'(x) > 0$ for $x > c$ then f has a local minimum at $x = c$.
If in some small interval $(c - \delta, c + \delta)$ you have $f'(x) > 0$ for $x < c$ and $f'(x) < 0$ for $x > c$ then f has a local maximum at $x = c$.

The reason is simple: if f increases to the left of c and decreases to the right of c then it has a maximum at c. More precisely:
if $f'(x) > 0$ for x between $c - \delta$ and c, then f is increasing for $c - \delta < x < c$ and therefore $f(x) < f(c)$ for x between $c - \delta$ and c.
If in addition $f'(x) < 0$ for $x > c$ then f is decreasing for x between c and $c + \delta$, so that $f(x) < f(c)$ for those x.
Combine these two facts and you get $f(x) \leq f(c)$ for $c - \delta < x < c + \delta$.

5.7.3 Example – local maxima and minima of $f(x) = x^3 - x$.

In §5.6.3 we had found that the function $f(x) = x^3 - x$ is decreasing when $-\infty < x < -\frac{1}{3}\sqrt{3}$, and also when $\frac{1}{3}\sqrt{3} < x < \infty$, while it is increasing when $-\frac{1}{3}\sqrt{3} < x < \frac{1}{3}\sqrt{3}$. It follows that the function has a local minimum at $x = -\frac{1}{3}\sqrt{3}$, and a local maximum at $x = \frac{1}{3}\sqrt{3}$.

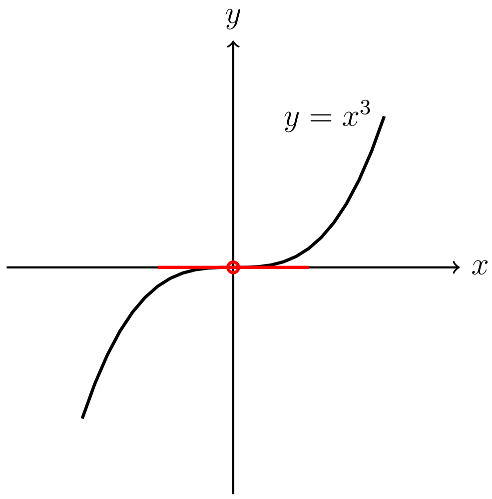
Neither the local maximum nor the local minimum are global max or min since

$$\lim_{x\to-\infty} f(x) = +\infty \text{ and } \lim_{x\to\infty} f(x) = -\infty.$$

5.7.4 A stationary point that is neither a maximum nor a minimum.

If you look for stationary points of the function $f(x) = x^3$ you find that there's only one, namely $x = 0$. The derivative $f'(x) = 3x^2$ does not change sign at $x = 0$, so the test in Theorem 5.7.2 does not tell us anything.

And in fact, $x = 0$ is neither a local maximum nor a local minimum since $f(x) < f(0)$ for $x < 0$ and $f(x) > 0$ for $x > 0$.



5.8 Must there always be a maximum?

Theorem 5.7.1 is very useful since it tells you how to find (local) maxima and minima. The following theorem is also useful, but in a different way. It doesn't say how to find maxima or minima, but it tells you that they do exist, and hence that you are not wasting your time trying to find a maximum or minimum.

Theorem 5.8.1. Let f be continuous function defined on the ***closed*** interval $a \le x \le b$. Then f attains its maximum and also its minimum somewhere in this interval. In other words there exist real numbers c and d such that

$$f(c) \le f(x) \le f(d)$$

whenever $a \le x \le b$.

The proof of this theorem requires a more careful definition of the real numbers than we have given in Chapter 1, and we will take the theorem for granted.

5.9 Examples – functions with and without maxima or minima

In the following three example we explore what can happen if some of the hypotheses in Theorem 5.8.1 are not met.

5.9.1 Question:

Does the function

$$f(x) = \begin{cases} x & \text{for } 0 \le x < 1 \\ 0 & \text{for } x = 1. \end{cases}$$

have a maximum on the interval $0 \le x \le 1$?

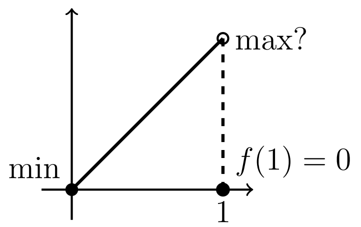


Figure 5.7: Does this function have a maximum?

5.9.2 Answer:

No. What would the maximal value be? Since

$$\lim_{x \nearrow 1} f(x) = \lim_{x \nearrow 1} x = 1$$

The maximal value cannot be less than 1. On the other hand the function is never larger than 1. So if there were a number a in the interval $[0, 1]$ such that $f(a)$ was the maximal value of f, then we would have $f(a) = 1$. If you now search the interval for numbers a with $f(a) = 1$, then you notice that such an a does not exist. Conclusion: this function does ***not*** attain its maximum on the interval $[0, 1]$.

What about Theorem 5.8.1? That theorem only applies to continuous functions, and the function f in this example is not continuous at $x = 1$. For at $x = 1$ one has

$$f(1) = 0 \neq 1 = \lim_{x \nearrow 1} f(x).$$

So all it takes for the Theorem to fail is that the function f be discontinuous at just one point.

5.9.3 Question:

Does the function

$$f(x) = \frac{1}{x^2}, \quad 1 \le x < \infty$$

have a maximum or minimum?

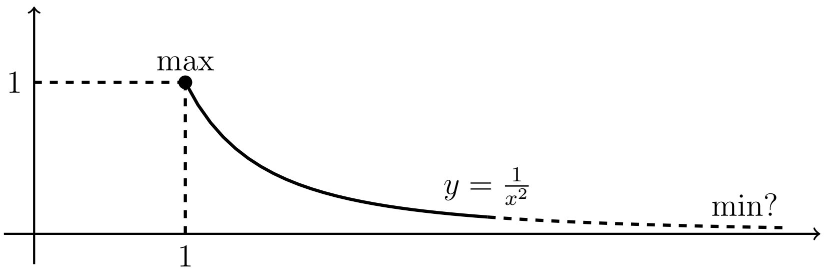


Figure 5.8: Does this function have a minimum?

5.9.4 Answer:

The function has a maximum at $x = 1$, but it has no minimum.

Concerning the maximum: if $x \ge 1$ then $f(x) = 1/x^2 \le 1$, while $f(1) = 1$. Hence $f(x) \le f(1)$ for all x in the interval $[1, \infty)$ and that is why f attains its maximum at $x = 1$.

If we look for a minimal value of f then we note that $f(x) \ge 0$ for all x in the interval $[1, \infty)$, and also that

$$\lim_{x \to \infty} f(x) = 0,$$

so that ***if*** f attains a minimum at some a with $1 \le a < \infty$, then the minimal value $f(a)$ must be zero. However, the equation $f(a) = 0$ has no solution – f does not attain its minimum.

Why does Theorem 5.8.1 not apply?

In this example the function f is continuous on the whole interval $[1, \infty)$, but this interval is not a closed interval, i.e. it is not of the form $[a, b]$ (it does not include its endpoints).

5.10 General method for sketching the graph of a function

Given a differentiable function f defined on some interval $a \le x \le b$, you can find the increasing and decreasing parts of the graph, as well as all the local maxima and minima by following this procedure:

1. find all solutions of $f'(x) = 0$ in the interval $[a, b]$: these are called the *critical* or *stationary* points for f.

2. find the sign of $f'(x)$ at all other points
3. each stationary point at which $f'(x)$ actually changes sign is a local maximum or local minimum. Compute the function value $f(x)$ at each stationary point.
4. compute the function values at the endpoints of the interval, i.e. compute $f(a)$ and $f(b)$.
5. the absolute maximum is attained at the stationary point or the boundary point with the highest function value; the absolute minimum occurs at the boundary or stationary point with the smallest function value.

If the interval is unbounded, i.e. if the function is defined for $-\infty < x < \infty$ then you can't compute the values $f(a)$ and $f(b)$, but instead you should compute $\lim_{x\to\pm\infty} f(x)$.

5.10.1 Example – the graph of a rational function.

Let's "sketch the graph" of the function

$$f(x) = \frac{x(1-x)}{1+x^2}.$$

By looking at the signs of numerator and denominator we see that

$$f(x) > 0 \text{ for } 0 < x < 1$$
$$f(x) < 0 \text{ for } x < 0 \text{ and also for } x > 1.$$

We compute the derivative of f

$$f'(x) = \frac{1-2x-x^2}{\left(1+x^2\right)^2}.$$

Hence $f'(x) = 0$ holds if and only if

$$1 - 2x - x^2 = 0$$

and the solutions to this quadratic equation are $-1 \pm \sqrt{2}$. These two roots will appear several times and it will shorten our formulas if we abbreviate

$$A = -1 - \sqrt{2} \text{ and } B = -1 + \sqrt{2}.$$

To see if the derivative changes sign we factor the numerator and denominator. The denominator is always positive, and the numerator is

$$-x^2 - 2x + 1 = -(x^2 + 2x - 1) = -(x - A)(x - B).$$

Therefore

$$f'(x) \begin{cases} < 0 & \text{for } x < A \\ > 0 & \text{for } A < x < B \\ < 0 & \text{for } x > B \end{cases}$$

It follows that f is decreasing on the interval $(-\infty, A)$, increasing on the interval (A, B) and decreasing again on the interval (B, ∞). Therefore

A is a local minimum, and B is a local maximum.

Are these global maxima and minima?

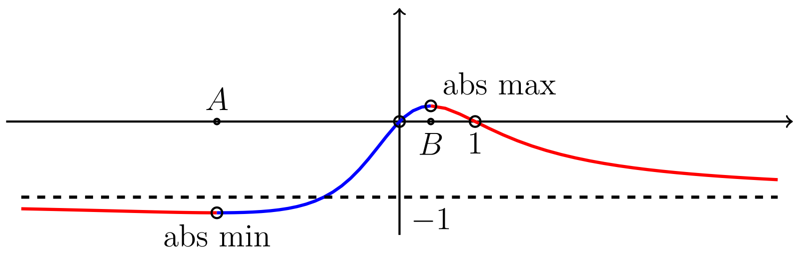


Figure 5.9: The graph of $f(x) = (x - x^2)/(1 + x^2)$

Since we are dealing with an unbounded interval we must compute the limits of $f(x)$ as $x \to \pm\infty$. You find

$$\lim_{x\to\infty} f(x) = \lim_{x\to-\infty} f(x) = -1.$$

Since f is decreasing between $-\infty$ and A, it follows that

$$f(A) \le f(x) < -1 \text{ for } -\infty < x \le A.$$

Similarly, f is decreasing from B to $+\infty$, so

$$-1 < f(x) \le f(-1 + \sqrt{2}) \text{ for } B < x < \infty.$$

Between the two stationary points the function is increasing, so

$$f(-1 - \sqrt{2}) \le f(x) \le f(B) \text{ for } A \le x \le B.$$

From this it follows that $f(x)$ is the smallest it can be when $x = A = -1 - \sqrt{2}$ and at its largest when $x = B = -1 + \sqrt{2}$: the local maximum and minimum which we found are in fact a global maximum and minimum.

5.11 Convexity, Concavity and the Second Derivative

By definition, a function f is ***convex*** on some interval $a < x < b$ if the line segment connecting any pair of points on the graph lies *above* the piece of the graph between those two points.

The function is called ***concave*** if the line segment connecting any pair of points on the graph lies *below* the piece of the graph between those two points.

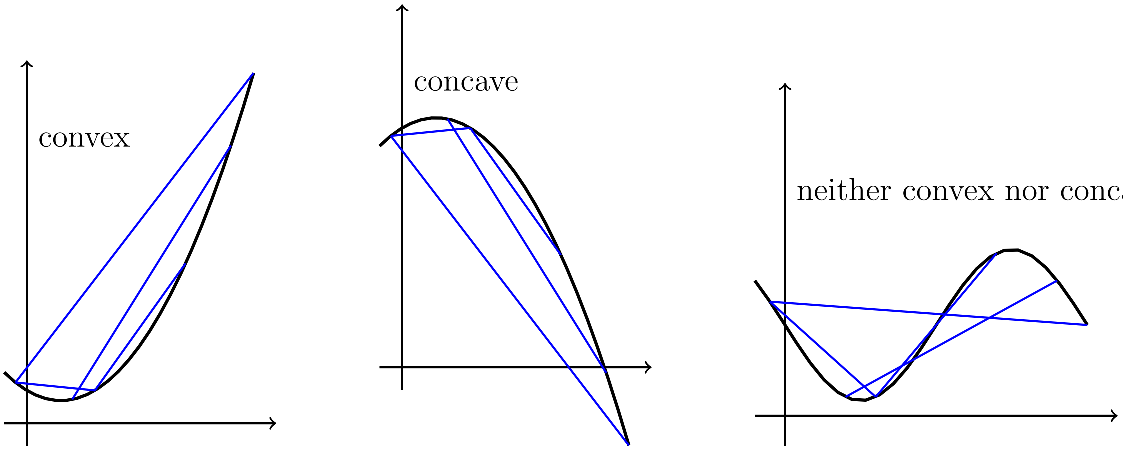


Figure 5.10: If a graph is convex then **all** chords lie above the graph. If a graph is concave then **all** chords lie below the graph.

Instead of "convex" and "concave" one often says "concave up" or "concave down" respectively.
A point on the graph of f where $f''(x)$ changes sign is called an ***inflection point***.

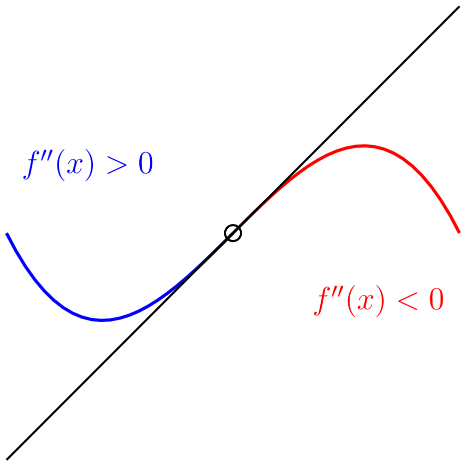


Figure 5.11: At an inflection point the tangent crosses the graph.

You can use the second derivative to tell if a function is concave or convex.

Theorem 5.11.1. If a function f is convex on some interval $a < x < b$ then $f''(x) \geq 0$ for all x on that interval.

Theorem 5.11.2. If $f''(x) \geq 0$ for all x on some interval $a < x < b$ then f is convex on that interval.

Theorem 5.11.3. A function f is convex on some interval $a < x < b$ if and only if the derivative $f'(x)$ is a nondecreasing function on that interval.

To gain further insight to convexity intervals and inflection points consider watching YouTube by Brightstorm .

5.11.1 Example – the cubic function $f(x) = x^3 - x$.

The second derivative of the function $f(x) = x^3 - x$ is

$$f''(x) = 6x$$

which is positive for $x > 0$ and negative for $x < 0$. Hence, in the graph in §5.6.3, the origin is an inflection point, and the piece of the graph where $x > 0$ is convex, while the piece where $x < 0$ is concave.

5.11.2 The second derivative test.

In §5.7.2 we saw how you can tell if a stationary point is a local maximum or minimum by looking at the sign changes of $f'(x)$. There is another way of distinguishing between local maxima and minima which involves computing the second derivative.

Theorem 5.11.4. If c is a stationary point for a function f, and if $f''(c) < 0$ then f has a local maximum at $x = c$.
If $f''(c) > 0$ then f has a local minimum at c.

The theorem doesn't say what happens when $f''(c) = 0$. In that case you must go back to checking the signs of the first derivative near the stationary point.
The basic reason why this theorem is true is that if c is a stationary point with $f''(c) > 0$ then "$f'(x)$ is increasing near $x = c$" and hence $f'(x) < 0$ for $x < c$ and $f'(x) > 0$ for $x > c$. So the function f is decreasing for $x < c$ and increasing for $x > c$, and therefore it reaches a local minimum at $x = c$.

5.11.3 Example – that cubic function again.

Consider the function $f(x) = x^3 - x$ from §5.6.3 and §5.11.1. We had found that this function has two stationary points, namely at $x = \pm\frac{1}{3}\sqrt{3}$. By looking at the sign of $f'(x) = 3x^2 - 1$ we concluded that $-\frac{1}{3}\sqrt{3}$ is a local maximum while $+\frac{1}{3}\sqrt{3}$ is a local minimum. Instead of looking at $f'(x)$ we could also have computed $f''(x)$ at $x = \pm\frac{1}{3}\sqrt{3}$ and applied the second derivative test. Here is how it goes:
Since $f''(x) = 6x$ we have

$$f''\bigl(-\tfrac{1}{3}\sqrt{3}\bigr) = -2\sqrt{3} < 0 \text{ and } f''\bigl(\tfrac{1}{3}\sqrt{3}\bigr) = 2\sqrt{3} > 0.$$

Therefore f has a local maximum at $-\frac{1}{3}\sqrt{3}$ and a local minimum at $\frac{1}{3}\sqrt{3}$.

5.11.4 When the second derivative test doesn't work.

Usually the second derivative test will work, but sometimes a stationary point c has $f''(c) = 0$. In this case the second derivative test gives no information at all. The figure below shows you the graphs of three functions, all three of which have a stationary point at $x = 0$. In all three cases the second derivative vanishes at $x = 0$ so the second derivative test says nothing. As you can see, the stationary point can be a local maximum, a local minimum, or neither.

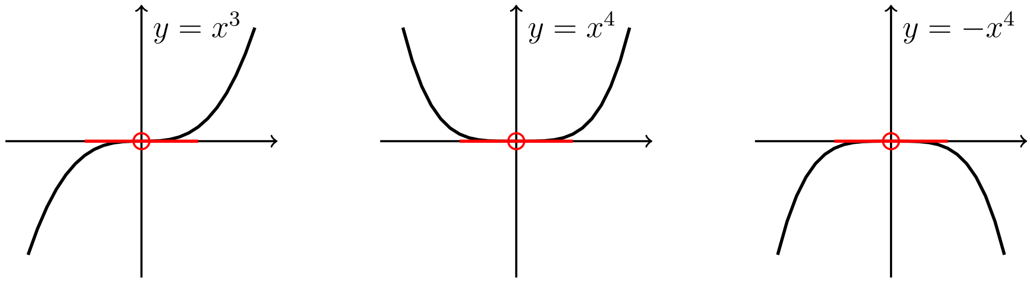


Figure 5.12: Three functions for which the second derivative test doesn't work.

5.12 Proofs of some of the theorems

5.12.1 Proof of the Mean Value Theorem.

Let m be the slope of the chord connecting the points $(a, f(a))$ and $(b, f(b))$, i.e.

$$m = \frac{f(b) - f(a)}{b - a},$$

and consider the function

$$g(x) = f(x) - f(a) - m(x - a).$$

This function is continuous (since f is continuous), and g attains its maximum and minimum at two numbers $c_{\min}$ and $c_{\max}$.

There are now two possibilities: either at least one of $c_{\min}$ or $c_{\max}$ is an interior point, or else both $c_{\min}$ and $c_{\max}$ are endpoints of the interval $a \le x \le b$.

Consider the first case: one of these two numbers is an interior point, i.e. if $a < c_{\min} < b$ or $a < c_{\max} < b$, then the derivative of g must vanish at $c_{\min}$ or $c_{\max}$. If one has $g'(c_{\min}) = 0$, then one has

$$0 = g'(c_{\min}) = f'(c_{\min}) - m, \text{i.e. } m = f'(c_{\min}).$$

The definition of m implies that one gets

$$f'(c_{\min}) = \frac{f(b) - f(a)}{b - a}.$$

If $g'(c_{\max}) = 0$ then one gets $m = f'(c_{\max})$ and hence

$$f'(c_{\max}) = \frac{f(b) - f(a)}{b - a}.$$

We are left with the remaining case, in which both $c_{\min}$ and $c_{\max}$ are end points. To deal with this case note that at the endpoints one has

$$g(a) = 0 \text{ and } g(b) = 0.$$

Thus the maximal and minimal values of g are both zero! This means that $g(x) = 0$ for all x, and thus that $g'(x) = 0$ for all x. Therefore we get $f'(x) = m$ for ***all*** x, and not just for some c.

5.12.2 Proof of Theorem 5.5.1.

If f is a non-decreasing function and if it is differentiable at some interior point a, then we must show that $f'(a) \geq 0$.
Since f is non-decreasing, one has $f(x) \geq f(a)$ for all $x > a$. Hence one also has

$$\frac{f(x) - f(a)}{x - a} \geq 0$$

for all $x > a$. Let $x \searrow a$, and you get

$$f'(a) = \lim_{x \searrow a} \frac{f(x) - f(a)}{x - a} \geq 0.$$

5.12.3 Proof of Theorem 5.5.2.

Suppose f is a differentiable function on an interval $a < x < b$, and suppose that $f'(x) \geq 0$ on that interval. We must show that f is non-decreasing on that interval, i.e. we have to show that if $x_1 < x_2$ are two numbers in the interval (a, b), then $f(x_1) \geq f(x_2)$. To prove this we use the Mean Value Theorem: given x_1 and x_2 the Mean Value Theorem hands us a number c with $x_1 < c < x_2$, and

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}.$$

We don't know where c is exactly, but it doesn't matter because we do know that wherever c is we have $f'(c) \geq 0$. Hence

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} \geq 0.$$

Multiply with $x_2 - x_1$ (which we are allowed to do since $x_2 > x_1$ so $x_2 - x_1 > 0$) and you get

$$f(x_2) - f(x_1) \geq 0,$$

as claimed.

5.13 Optimization

Often a problem can be phrased as

For which value of x in the interval $a \leq x \leq b$ is $f(x)$ the largest?

In other words you are given a function f on an interval $[a, b]$ and you must find all global maxima of f on this interval.
If the function is continuous then according to theorem 5.8.1 there always is at least one x in the interval $[a, b]$ which maximizes $f(x)$.
If f is differentiable then we know what to do: any local maximum is either a stationary point or one of the end points a and b. Therefore you can find the global maxima by following this recipe:

1. Find all stationary points of f;
2. Compute $f(x)$ at each stationary point you found in step (1);
3. Compute $f(a)$ and $f(b)$;
4. The global maxima are those stationary- or endpoints from steps (2) and (3) which have the largest function value.

Usually there is only one global maximum, but sometimes there can be more.

If you have to *minimize* rather than *maximize* a function, then you must look for global minima. The same recipe works (of course you should look for the smallest function value instead of the largest in step 4.)

The difficulty in optimization problems frequently lies not with the calculus part, but rather with setting up the problem. Choosing which quantity to call x and finding the function f is half the job.

5.13.1 Example – The rectangle with largest area and given perimeter.

Which rectangle has the largest area, among all those rectangles for which the total length of the sides is 1?

Solution: If the sides of the rectangle have lengths x and y, then the total length of the sides is

$$L = x + x + y + y = 2(x + y)$$

and the area of the rectangle is

$$A = xy.$$

So are asked to find the largest possible value of $A = xy$ provided $2(x + y) = 1$. The lengths of the sides can also not be negative, so x and y must satisfy $x \geq 0$, $y \geq 0$.

We now want to turn this problem into a question of the form "maximize a function over some interval." The quantity which we are asked to maximize is A, but it depends on two variables x and y instead of just one variable. However, the variables x and y are not independent since we are only allowed to consider rectangles with $L = 1$. From this equation we get

$$L = 1 \implies y = \tfrac{1}{2} - x.$$

Hence we must find the maximum of the quantity

$$A = xy = x\big(\tfrac{1}{2} - x\big)$$

The values of x which we are allowed to consider are only limited by the requirements $x \geq 0$ and $y \geq 0$, i.e. $x \leq \frac{1}{2}$. So we end up with this problem:

Find the maximum of the function $f(x) = x\big(\frac{1}{2} - x\big)$ *on the interval* $0 \leq x \leq \frac{1}{2}$.

Before we start computing anything we note that the function f is a polynomial so that it is differentiable, and hence continuous, and also that the interval $0 \le x \le \frac{1}{2}$ is closed. Therefore the theory guarantees that there is a maximum and our recipe will show us where it is.

The derivative is given by

$$f'(x) = \tfrac{1}{2} - 2x,$$

and hence the only stationary point is $x = \frac{1}{4}$. The function value at this point is

$$f\tfrac{1}{4}) = \tfrac{1}{4}(\tfrac{1}{2} - \tfrac{1}{4}) = \tfrac{1}{16}.$$

At the endpoints one has $x = 0$ or $x = \frac{1}{2}$, which corresponds to a rectangle one of whose sides has length zero. The area of such rectangles is zero, and so this is not the maximal value we are looking for.

We conclude that the largest area is attained by the rectangle whose sides have lengths

$$x = \tfrac{1}{4}, \text{ and } y = \tfrac{1}{2} - \tfrac{1}{4} = \tfrac{1}{4},$$

i.e. by a square with sides $\frac{1}{4}$.

5.14 PROBLEMS

TANGENTS AND NORMALS

211. Where does the normal to the graph of $y = x^2$ at the point (1,1) intersect the x-axis? †383

212. Where does the tangent to the graph of $y = x^2$ at the point (a, a^2) intersect the x-axis? †383

213. Where does the normal to the graph of $y = x^2$ at the point (a, a^2) intersect the x-axis? †383

214. Where does the normal to the graph of $y = \sqrt{x}$ at the point $(a, \sqrt{a})$ intersect the x-axis? †383

215. Does the graph of $y = x^4 - 2x^2 + 2$ have any horizontal tangents? If so, where?

Does the graph of the same function have any vertical tangents?

Does it have vertical normals?

Does it have horizontal normals?

216. At some point $(a, f(a))$ on the graph of $f(x) = -1 + 2x - x^2$ the tangent to this graph goes through the origin. Which point is it?

217. Find equations for the tangent and normal lines

to the curve . . .	**at the point. . .**
(a) $y = 4x/(1 + x^2)$	$(1, 2)$
(b) $y = 8/(4 + x^2)$	$(2, 1)$
(c) $y^2 = 2x + x^2$	$(2, 2)$
(d) $xy = 3$	$(1, 3)$

218.

The function

$$f(x) = \frac{x^2 + |x|}{x}$$

satisfies $f(-1) = -2$ and $f(+1) = +2$, so, by the Intermediate Value Theorem, there should be some value c between -1 and $+1$ such that $f(c) = 0$. ***True or False?*** †383

CRITICAL POINTS

219. What does the Intermediate Value Theorem say?

220. What does the Mean Value Theorem say?

221.

If $f(a) = 0$ ***and*** $f(b) = 0$ ***then there is a*** c ***between*** a ***and*** b ***such that*** $f'(c) = 0$***.*** Show that this follows from the Mean Value Theorem. (Help! A proof! Relax: this one is not difficult. Make a drawing of the situation, then read the Mean Value Theorem again.)

222. What is a stationary point?

223. How can you tell if a local maximum is a global maximum?

224. If $f''(a) = 0$ then the graph of f has an inflection point at $x = a$. ***True or False?*** †383

225. What is an inflection point? †383

226. Give an example of a function for which $f'(0) = 0$ even though the function f has neither a local maximum or a local minimum at $x = 0$.

227. Draw four graphs of functions, one for each of the following four combinations

$$f' > 0 \text{ and } f'' > 0 \qquad f' > 0 \text{ and } f'' < 0$$
$$f' < 0 \text{ and } f'' > 0 \qquad f' < 0 \text{ and } f'' < 0$$

228. Which of the following combinations are possible:

$$f'(x) > 0 \text{ and } f''(x) = 0 \text{ for all } x$$
$$f'(x) = 0 \text{ and } f''(x) > 0 \text{ for all } x$$

†383

Sketch the graph of the following functions. To accomplish this you should

find where f, f' and f'' are positive or negative

find all stationary points

decide which stationary points are local maxima or minima decide which local max/minima are in fact global max/minima

find all inflection points

find "horizontal asymptotes," i.e. compute the limits $\lim_{x\to\pm\infty} f(x)$ when appropriate.

229. $y = x^3 + 2x^2$ †383

230. $y = x^3 - 4x^2$ †383

231. $y = x^4 + 27x$ †383

232. $y = x^4 - 27x$ †383

233. $y = x^4 + 2x^2 - 3$ †384

234. $y = x^4 - 5x^2 + 4$ †384

235. $y = x^5 + 16x$ †384

236. $y = x^5 - 16x$ †384

237. $y = \dfrac{x}{x+1}$ †384

238. $y = \dfrac{x}{1+x^2}$ †384

239. $y = \dfrac{x^2}{1+x^2}$ †384

240. $y = \dfrac{1+x^2}{1+x}$ †384

241. $y = x + \dfrac{1}{x}$ †384

242. $y = x - \dfrac{1}{x}$ †384

243. $y = x^3 + 2x^2 + x$ †384

244. $y = x^3 + 2x^2 - x$ †384

245. $y = x^4 - x^3 - x$ †384

246. $y = x^4 - 2x^3 + 2x$ †384

247. $y = \sqrt{1+x^2}$ †385

248. $y = \sqrt{1-x^2}$ †385

249. $y = \sqrt[4]{1+x^2}$ †385

250. $y = \dfrac{1}{1+x^4}$ †385

The following functions are periodic, i.e. they satisfy $f(x+L) = f(x)$ for all x, where the constant L is called the period of the function. The graph of a periodic function repeats itself indefinitely to the left and to the right. It therefore has infinitely many (local) minima and maxima, and infinitely many inflections points. Sketch the graphs of the following functions as in the previous problem, but only list those "interesting points" that lie in the interval $0 \le x < 2\pi$.

251. $y = \sin x$

252. $y = \sin x + \cos x$ †385

253. $y = \sin x + \sin^2 x$ †385

254. $y = 2\sin x + \sin^2 x$

255. $y = 4\sin x + \sin^2 x$

256. $y = 2\cos x + \cos^2 x$

257. $y = \dfrac{4}{2+\sin x}$

258. $y = \big(2+\sin x\big)^2$

Find the domain and sketch the graphs of each of the following functions

259. $y = \arcsin x$

260. $y = \arctan x$

261. $y = 2\arctan x - x$

262. $y = \arctan(x^2)$

263. $y = 3\arcsin(x) - 5x$

264. $y = 6\arcsin(x) - 10x^2$

In the following two problems it is not possible to solve the equation $f'(x) = 0$, but you can tell something from the second derivative.

265. Show that the function

$$f(x) = x \arctan x$$

is convex. Then sketch the graph of f.

266. Show that the function

$$g(x) = x \arcsin x$$

is convex. Then sketch the graph of g.

For each of the following functions use the derivative to decide if they are increasing, decreasing or neither on the indicated intervals

267. $f(x) = \dfrac{x}{1+x^2} \quad 10 < x < \infty$

268. $f(x) = \dfrac{2+x^2}{x^3-x} \quad 1 < x < \infty$

269. $f(x) = \dfrac{2+x^2}{x^3-x} \quad 0 < x < 1$

270. $f(x) = \dfrac{2+x^2}{x^3-x} \quad 0 < x < \infty$

OPTIMIZATION

271. By definition, the perimeter of a rectangle is the sum of the lengths of its four sides. Which rectangle, of all those whose perimeter is 1, has the smallest area? Which one has the largest area? †385

272. Which rectangle of area 100in^2 minimizes its height plus two times its length? †385

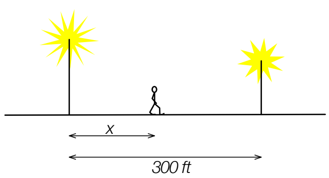
273. You have 1 yard of string from which you make a circular wedge with radius R and opening angle θ. Which choice of θ and R will give you the wedge with the largest area? Which choice leads to the smallest area?

[A circular wedge is the figure consisting of two radii of a circle and the arc connecting them. So the yard of string is used to form the two radii and the arc.] †385

274.

(The lamp post problem)

In a street two lamp posts are 300 feet apart. The light intensity at a distance d from the first lamp post is $1000/d^2$, the light intensity at distance d from the second (weaker) lamp post is $125/d^2$ (in both cases the light intensity is inversely proportional to the square of the distance to the light source).



The *combined light intensity* is the sum of the two light intensities coming from both lamp posts.

(a) If you are in between the lamp posts, at distance x feet from the stronger light, then give a formula for the combined light intensity coming from both lamp posts as a function of x.

(b) What is the darkest spot between the two lights, i.e. where is the combined light intensity the smallest?

†385

275. **(a)** You have a sheet of metal with area 100 in^2 from which you are to make a cylindrical soup can. If r is the radius of the can and h its height, then which h and r will give you the can with the largest volume?

(b) If instead of making a plain cylinder you replaced the flat top and bottom of the cylinder with two spherical caps, then (using the same 100in^2 os sheet metal), then which choice of radius and height of the cylinder give you the container with the largest volume?

(c) Suppose you only replace the top of the cylinder with a spherical cap, and leave the bottom flat, then which choice of height and radius of the cylinder result in the largest volume?

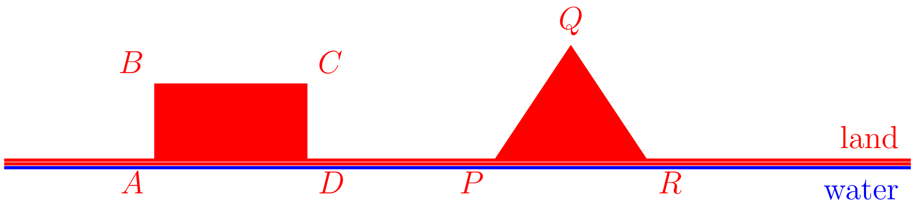
†386

276. A triangle has one vertex at the origin $O(0,0)$, another at the point $A(2a,0)$ and the third at $(a, a/(1+a^3))$. What are the largest and smallest areas this triangle can have if $0 \le a < \infty$?

277.

Queen Dido's problem

According to tradition Dido was the founder and first Queen of Carthage. When she arrived on the north coast of Africa (∼800BC) the locals allowed her to take as much land as could be enclosed with the hide of one ox. She cut the hide into thin strips and put these together to form a length of 100 yards[1].



(a) If Dido wanted a rectangular region, then how wide should she choose it to enclose as much area as possible (the coastal edge of the boundary doesn't count, so in this problem the length $AB + BC + CD$ is 100 yards.)

(b) If Dido chose a region in the shape of an isosceles triangle PQR, then how wide should she make it to maximize its area (again, don't include the coast in the perimiter: $PQ + QR$ is 100 yards long, and $PQ = QR$.)

278. The product of two numbers x, y is 16. We know $x \ge 1$ and $y \ge 1$. What is the greatest possible sum of the two numbers?

[1] I made that number up. For the rest start at http://en.wikipedia.org/wiki/Dido

279. What are the smallest and largest values that $(\sin x)(\sin y)$ can have if $x + y = \pi$ and if x and y are both nonnegative?

280. What are the smallest and largest values that $(\cos x)(\cos y)$ can have if $x + y = \frac{\pi}{2}$ and if x and y are both nonnegative?

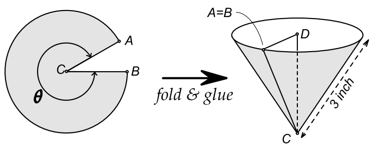
281. **(a)** What are the smallest and largest values that $\tan x + \tan y$ can have if $x + y = \frac{\pi}{2}$ and if x and y are both nonnegative?

(b) What are the smallest and largest values that $\tan x + 2\tan y$ can have if $x + y = \frac{\pi}{2}$ and if x and y are both nonnegative?

282. The cost per hour of fuel to run a locomotive is $v^2/25$ dollars, where v is speed (in miles per hour), and other costs are \$100 per hour regardless of speed. What is the speed that minimizes cost per mile ?

283.

Bronwyn is in need of coffee. Emily has a circular filter with 3 inch radius. She cuts out a wedge and glues the two edges AC and BC together to make a conical filter to hold the ground coffee. The volume V of the coffee cone depends the angle θ of the piece of filter paper Emily made.



(a) Find the volume in terms of the angle θ. (Hint: how long is the circular arc AB on the left? How long is the circular top of the cone on the right? If you know that you can find the radius $AD = BD$ of the top of the cone, and also the height CD of the cone.)

(b) Which angle θ maximizes the volume V?

Chapter 6

Implicit Derivatives and Related Rate Problems

6.1 Differentiating implicitly defined functions

6.1.1 Implicitly defined functions

When we say that the function $y = f(x)$ is ***implicitly defined*** by an equation in x and y we mean that if we substitute $f(x)$ for y in that equation, we get an equation (in x) that holds for all values of x. In this case, we can find the derivative by differentiating the equation and solving for the derivative.

6.1.2 Example

The function $y = \sqrt{1-x^2}$ is implicitly defined by the equation $x^2 + y^2 = 1$ (with the additional condition that $y \geq 0$). We can find the derivative explicitly via

$$\frac{dy}{dx} = -\frac{x}{\sqrt{1-x^2}}$$

but it is easier to view $x^2 + y^2$ as a (constant) function of x, differentiate to get

$$0 = \frac{d}{dx}(x^2 + y^2) = 2x + 2y\frac{dy}{dx},$$

and then solve to get

$$\frac{dy}{dx} = -\frac{x}{y} = -\frac{x}{\sqrt{1-x^2}}.$$

6.1.3 Example

Here is a more complicated example. A differentiable function $y = f(x)$ is implicitly defined by the equation

$$y^2 + 3xy + 7x^2 - 17 = 0. \qquad (\dagger)$$

and satisfies $f(1) = 2$. To find $f'(1)$ we can differentiate (†) and solve:

$$2y\frac{dy}{dx} + 3x\frac{dy}{dx} + 3y + 14x = 0$$

(we used the product rule when we differentiated $3xy$) so

$$\frac{dy}{dx} = -\frac{3y + 14x}{2y + 3x}. \tag{†'}$$

Then

$$f'(1) = \left.\frac{dy}{dx}\right|_{x=1} = -\left.\frac{3y + 14x}{2y + 3x}\right|_{(x,y)=(1,2)} = -\frac{6+14}{4+3}.$$

Another (harder) way is to find an explicit formula for y by using the quadratic formula:

$$y = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A} = \frac{-3x \pm \sqrt{9x^2 - 4(7x^2 - 17)}}{2}$$

where $A = 1$, $B = 3x$, and $C = 7x^2 - 17$. Because $f(1) = 2$ we must take the plus sign on the right and we see that $y = f(x)$ is explicitly defined by

$$y = \frac{-3x + \sqrt{9x^2 - 4(7x^2 - 17)}}{2} = \frac{-3x + \sqrt{68 - 19x^2}}{2} \tag{‡}$$

We can find $f'(x)$ by differentiating (‡):

$$\frac{dy}{dx} = -\frac{3}{2} - \frac{19x}{2\sqrt{68 - 19x^2}}. \tag{‡'}$$

In even more complicated examples, it will be impossible (not merely difficult) to find a formula for the implicitly defined function. Nonetheless we can still compute the derivative.

6.1.4 Equation for the tangent to a curve

A typical problem asks you to find an equation for the tangent line to a curve at a point on the curve. For example, to find the equation for the tangent line to the graph of (†) at the point $(x, y) = (1, 2)$ we calculate the slope by implicit differentiation as before:

$$m = \left.\frac{dy}{dx}\right|_{(x,y)=(1,2)} = -\left.\frac{3y + 14x}{2y + 3x}\right|_{(x,y)=(1,2)} = -\frac{20}{7}.$$

Then, since the value of the derivative at a point is the slope of the tangent line at that point, the equation of the tangent line is

$$y = 2 - \frac{20}{7}(x - 1)$$

where we have used the point slope equation

$$y = y_0 + m(x - x_0)$$

for the equation of the line of slope m through the point (x_0, y_0).

6.2 Inverse Functions

6.2.1 Vertical and Horizontal line tests for functions

A graph of an equation of form $y = f(x)$ satisfies the ***Vertical Line Test***: every vertical line $x = a$ intersects the graph in at most one point. If a is in the domain of f, then the vertical line $x = a$ intersects the graph $y = f(x)$ in the point $P(a, f(a))$; if a is not in the domain of f, then the vertical line $x = a$ does not intersect the graph $y = f(x)$ at all. A graph of an equation of form $x = g(y)$ satisfies the ***Horizontal Line Test***: every horizontal line $y = b$ intersects the graph in at most one point. If b is in the domain of g, then the horizontal line $y = b$ intersects the graph in the point $P(g(b), b)$; if b is not in the domain of g, then the line $y = b$ does not intersect the graph $x = g(y)$ at all.

Definition 6.2.1. When the graphs $y = f(x)$ and $x = g(y)$ are the same, i.e. when

$$y = f(x) \iff x = g(y)$$

we say that f and g are ***inverse functions*** and write $g = f^{-1}$. Thus

$$\text{domain}(f^{-1}) = \text{range}(f), \qquad \text{range}(f^{-1}) = \text{domain}(f),$$

and

$$y = f(x) \iff x = f^{-1}(y) \tag{\#}$$

for x in the domain of f and y in the range of f.

6.2.2 Example

The graph $y = x^2$ does not satisfy the horizontal line test since the horizontal line $y = 9$ intersects the graph in the two points $(-3, 9)$ and $(3, 9)$. Therefore this graph cannot be written in the form $x = g(y)$. However, if we restrict the the domain to $x \geq 0$ the resulting graph does have the form $x = g(y)$:

$$\text{For } x \geq 0: \qquad y = x^2 \iff x = \sqrt{y}.$$

Let $f(x) = x^2$ (with the domain artificially restricted to $x \geq 0$); then $f^{-1}(y) = \sqrt{y}$. Thus $f(3) = 9$ so by (#) $f^{-1}(9) = 3$. Hence $f(f^{-1}(9)) = 9$ and $f^{-1}(f(3)) = 3$. In general:

$$\sqrt{x^2} = x, \qquad (\sqrt{y})^2 = y$$

for $x \geq 0$. But there is nothing special about this example:

6.2.3 Cancellation Equations

. If the graph $y = f(x)$ satisfies the horizontal line test (so that the inverse function f^{-1} is defined) then

$$f^{-1}(f(x)) = x \qquad \text{and} \qquad f(f^{-1}(y)) = y$$

for x in $\text{domain}(f) = \text{range}(f^{-1})$ and y in $\text{range}(f) = \text{domain}(f^{-1})$.

To see this choose x and let $y = f(x)$. Then $x = f^{-1}(y)$ by Equation (#) in Definition 6.2.1. Substituting the former in the latter gives $x = f^{-1}(f(x))$. Reversing the roles of f and f^{-1} proves the other cancellation equation.

Theorem 6.2.1 (***Inverse Function Theorem***)**.** Suppose that f and g are inverse functions, that f is differentiable, and that $f'(x) \neq 0$ for all x. Then g is differentiable and

$$g'(y) = \frac{1}{f'(g(y))}.$$

Proof: The fact that g is differentiable is normally proved in more advanced courses like Math 521. Assuming this we prove the formula for $g'(y)$ as follows. By the Cancellation Equations of 6.2.3 we have

$$f(g(y)) = y.$$

Differentiate with respect to y and use the Chain Rule to get

$$f'(g(y))g'(y) = 1.$$

Now divide both sides by $f'(g(y))$.

We can also write the Inverse Function Theorem as

$$(f^{-1})'(y) = \frac{1}{f'(f^{-1}(y))}.$$

If we use this notation, we don't need a name for the inverse function.

A handy way to summarize the formula $(f^{-1})'(y) = 1/f'(f^{-1}(y))$ from Theorem 6.2.1 is with Leibniz notation:

$\frac{dx}{dy} = \left(\frac{dy}{dx}\right)^{-1}.$

For example, for $x > 0$ and $y = x^2$ we have $x = \sqrt{y} = y^{\frac{1}{2}}$ so

$$\frac{dy}{dx} = 2x, \qquad \frac{dx}{dy} = \frac{1}{2x} = \frac{1}{2\sqrt{y}},$$

in agreement with the power rule

$$\frac{d}{dy}y^{\frac{1}{2}} = \frac{dx}{dy} = \frac{1}{2\sqrt{y}} = \frac{1}{2}y^{\frac{1}{2}-1}.$$

6.2.4 Example

We find the inverse function $g = f^{-1}$ of the function

$$f(x) = x^3 + 1$$

and its derivative. Since $y = x^3 + 1 \iff x = (y-1)^{1/3}$, the inverse function is $g(y) = (y-1)^{1/3}$ so

$$g'(y) = \frac{(y-1)^{-2/3}}{3}.$$

The following calculation confirms that $g'(y) = 1/f'(g(y))$:

$$\frac{1}{f'(g(y))} = \frac{1}{3g(y)^2} = \frac{1}{3\big((y-1)^{1/3}\big)^2} = \frac{1}{3(y-1)^{2/3}} = \frac{(y-1)^{-2/3}}{3}.$$

6.2.5 Derivatives of the inverse trigonometric functions

One has

$$\begin{aligned}
\frac{d \arcsin x}{dx} &= \frac{1}{\sqrt{1-x^2}} \\
\frac{d \arccos x}{dx} &= \frac{-1}{\sqrt{1-x^2}} \\
\frac{d \arctan x}{dx} &= \frac{1}{1+x^2}
\end{aligned}$$

6.2.6 Inverse Trigonometric Functions

The trig functions sine, cosine, tangent etc. do not satisfy the horizontal line test: they are periodic. The inverse trig functions are defined by artificially restricting the domain of the corresponding trig function. When we do this we get

- If $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$ then $y = \sin\theta \iff \theta = \sin^{-1}(y)$
- If $0 \leq \theta \leq \pi$ then $x = \cos\theta \iff \theta = \cos^{-1}(y)$
- If $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$ then $u = \tan\theta \iff \theta = \tan^{-1}(y)$.

Using these we can differentiate the inverse trig functions.
$\frac{d}{dy}\sin^{-1}(y) = \frac{1}{\sqrt{1-y^2}}$.

Proof: Let $y = \sin\theta$ so $\theta = \sin^{-1}(y)$. Then

$$\frac{d\theta}{dy} = \left(\frac{dy}{d\theta}\right)^{-1} = \frac{1}{\cos\theta} = \frac{1}{\sqrt{1-\sin^2\theta}} = \frac{1}{\sqrt{1-y^2}}$$

where we used the Pythogorean Theorem $\cos^2\theta + \sin^2\theta = 1$.

$\frac{d}{dx}\cos^{-1}(x) = -\frac{1}{\sqrt{1-x^2}}$.

Proof: Let $x = \cos\theta$ so $\theta = \cos^{-1}(y)$. Then

$$\frac{d\theta}{dx} = \left(\frac{dx}{d\theta}\right)^{-1} = -\frac{1}{\sin\theta} = -\frac{1}{\sqrt{1-\cos^2\theta}} = -\frac{1}{\sqrt{1-x^2}}$$

where we used the Pythogorean Theorem as before.

$\frac{d}{du}\tan^{-1}(u) = \frac{1}{1+u^2}$.

PROOF: Let $u = \tan\theta$ so $\theta = \tan^{-1}(u)$. Then

$$\frac{d\theta}{du} = \left(\frac{du}{d\theta}\right)^{-1} = \frac{1}{\sec^2\theta} = \frac{1}{1+\tan^2\theta} = \frac{1}{1+u^2}$$

where we used the Pythogorean Theorem in the form $1 + \tan^2\theta = \sec^2\theta$. This follows from $\cos^2\theta + \sin^2\theta = 1$ by dividing by $\cos^2\theta$. (Recall that the secant function is the reciprocal of the cosine.)

6.2.7 Example

We calculate the derivative of $y = \sin\sqrt{1+x^2}$ using the Chain Rule:

$$\frac{dy}{dx} = \left(\cos\sqrt{1+x^2}\right) \cdot \left(\frac{1}{2\sqrt{1+x^2}}\right) \cdot 2x.$$

6.3 Parametric Equations

Definition 6.3.1. A pair of equations

$$x = f(t), \qquad y = g(t)$$

assigns to each value of t a corresponding point $P(x, y)$. The set of these points is called a ***parametric curve*** and the equations are called ***parametric equations*** for the curve. The variable t is called the ***parameter*** and we say that the equations "***trace out***" or ***parameterize*** the curve. Often t has the interpretation of time and the parametric equations describe the position of a moving particle at time t, i.e. the point corresponding to to the parameter value t is the position of the particle at time t. Parameters other than time are also used. The following examples show that sometimes (but not always) we can eliminate the parameter and find an equation of the form

$$F(x, y) = 0$$

which describes the curve.

6.3.1 Rectilinear Motion.

Here's a parametric curve:

$$x = 1 + t, \qquad y = 2 + 3t.$$

Both x and y are linear functions of time (i.e. the parameter t), so every time t increases by an amount Δt (every time Δt seconds go by) the first component x increases by Δt, and the second component y increases by $3\Delta t$. The point at $P(x, y)$ moves horizontally to the left with speed 1, and it moves vertically upwards with speed 3.

Which curve is traced out by these equations? In this example we can find out by eliminating the parameter, i.e. solving one of the two equations for t, and substituting the value of t you find in the other equation. Here you can solve $x = 1 + t$ for t, with result $t = x - 1$. From there you find that

$$y = 2 + 3t = 2 + 3(x - 1) = 3x - 1.$$

So for any t the point $P(x, y)$ is on the line $y = 3x - 1$. This particular parametric curve traces out a straight line with equation $y = 3x - 1$, going from left to right.

6.3.2 Rectilinear Motion (More Generally).

Any constants x_0, y_0, a, b such that either $a \neq 0$ or $b \neq 0$ give parametric equations

$$x = x_0 + a(t - t_0), \qquad y = y_0 + b(t - t_0) \tag{$*$}$$

which trace out the line

$$a(y - y_0) = b(x - x_0). \tag{$**$}$$

(Both sides equal $ab(t - t_0)$.) At time $t = t_0$ the point $P(x, y)$ is at $P_0(x_0, y_0)$. The values corresponding to Example 6.3.1 are $t_0 = 0$, $x_0 = 1$, $y_0 = 2$, $a = 1$, $b = 3$.

6.3.3 Going back and forth on a straight line.

Consider

$$x = x_0 + a \sin t, \qquad y = y_0 + b \sin t.$$

At each moment in time the point whose motion is described by this parametric curve is on the straight line with equation ($*$) as in Example 6.3.2. However, instead of moving along the line from one end to the other, the point at $P(x, y)$ keeps moving back and forth along the line ($**$) between the point P_1 corresponding to time $t = \pi/2$ and the point P_2 corresponding to time $t = 3\pi/2$.

6.3.4 Motion along a graph.

Let $y = f(x)$ be some function of one variable (defined for x in some interval) and consider the parametric curve given by

$$x = t, \qquad y = f(t).$$

At any moment in time the point $P(x, y)$ has x coordinate equal to t, and $y = f(t) = f(x)$, since $x = t$. So this parametric curve describes motion on the graph of $y = f(x)$ in which the horizontal coordinate increases at a constant rate.

6.3.5 The standard parametrization of a circle.

The parametric equations

$$x = \cos\theta, \quad y = \sin\theta$$

satisfy

$$x^2 + y^2 = \cos^2\theta + \sin^2\theta = 1,$$

so that $P(x, y)$ always lies on the unit circle. As θ increases from $-\infty$ to $+\infty$ the point will move around the circle, going around infinitely often. The point runs around the circle in the *counterclockwise direction,* which is the mathematician's favorite way of running around in circles. The more general equations

$$x = a\cos t, \quad y = b\sin t.$$

parameterize the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

6.3.6 Another parametrization of a circle.

The equations

$$x = \frac{1-t^2}{1+t^2}, \quad y = \frac{2t}{1+t^2}$$

also parameterize the unit circle. To see this divide both sides of the identity[1]

$$(1-t^2)^2 + (2t)^2 = (1+t^2)^2$$

by $(1+t^2)^2$ to get $x^2 + y^2 = 1$. However the point $Q(-1, 0)$ is left out since $y = 0$ only when $t = 0$ and $x = 1 \neq -1$ when $t = 0$.

6.3.7 A parametrization of a hyperbola.

The functions

$$\sinh(t) = \frac{e^t - e^{-t}}{2}, \qquad \cosh(t) = \frac{e^t + e^{-t}}{2}$$

are called the ***hyperbolic sine*** and ***hyperbolic cosine*** respectively. This is because the equations

$$x = \cosh(t), \qquad y = \sinh(t),$$

parameterise the part of the hyperbola

$$x^2 - y^2 = 1$$

which to the right of the y-axis.

[1] $(1-t^2)^2 + (2t)^2 = (1-2t^2+t^4) + 4t^2 = 1 + 2t^2 + t^4 = (1+t^2)^2$

6.3.8 Slope of the tangent to a parametric curve

For parametric equations as in Definition 6.3.1 the chain rule gives

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$$

so dividing gives the formula

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

We can use this formula to find the slope of the tangent line at a point on the curve. The following example illustrates this.

6.3.9 Tangent to a circle

The point $P_0\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$ lies on the unit circle $x^2 + y^2 = 1$. This point corresponds to the parameter value $\theta = \pi/6$ in the standard parameterization $x = \cos\theta$, $y = \sin\theta$ of Example 6.3.5. Since

$$\frac{dx}{d\theta} = -\sin\theta, \qquad \frac{dy}{d\theta} = \cos\theta$$

we get

$$\left.\frac{dx}{d\theta}\right|_{\theta=\pi/6} = -\frac{\sqrt{3}}{2}, \qquad \left.\frac{dy}{d\theta}\right|_{\theta=\pi/6} = \frac{1}{2},$$

and so the slope of the tangent line at P_0 is

$$m = \left.\frac{dy}{dx}\right|_{\theta=\pi/6} = \left.\frac{dy/d\theta}{dx/d\theta}\right|_{\theta=\pi/6} = -\frac{1}{\sqrt{3}}.$$

The point slope equation $y = y_0 + m(x - x_0)$ for the tangent line is

$$y = \frac{1}{2} - \frac{1}{\sqrt{3}}\left(x - \frac{\sqrt{3}}{2}\right).$$

Let $P_0(x_0, y_0)$ be a point on a parametric curve corresponding to a parameter value $t = t_0$ and let

$$a = \left.\frac{dx}{dt}\right|_{t=t_0} \qquad \text{and} \qquad b = \left.\frac{dy}{dt}\right|_{t=t_0}.$$

Then

$$x = x_0 + a(t - t_0), \qquad y = y_0 + b(t - t_0),$$

are parametric equations for the tangent line to the curve at P_0. This is because the point slope equation for the tangent line is

$$y = y_0 + m(x - x_0), \qquad \text{where} \qquad m = \left.\frac{dy}{dx}\right|_{(x,y)=(x_0,y_0)}.$$

and the slope is $m = b/a$. (See Example 6.3.2.)

Before tackling this chapter's exercise problems the reader should consider watching YouTube by 3Blue1Brown .

6.4 PROBLEMS

IMPLICIT DIFFERENTIATION

284. Check that the two formulas (†′) and (‡′) for dy/dx in Example 6.1.3 are actually equal.

285. Find an equation for the tangent line to the curve

$$x^2 + xy - y^2 = 1$$

at the point $P_0(2, 3)$. Answer: $y - 3 = \frac{7}{4}(x - 2)$

286. The point $P(1, 2)$ lies on the curve

$$y^5 + 3xy + 7x^5 - 45 = 0.$$

Find equations for the tangent line at P via the method of Example 6.1.4. In this case you must use implicit differentiation: there is no analog of Equation (‡).

287. Find equations for the tangent line and the normal line to the curve $x^3 + y^3 = 9xy$ at the point $(x, y) = (2, 4)$. Hint: The slope of the normal line is the negative reciprocal of the slope of the tangent line.

INVERSE FUNCTIONS

288. Find the inverse function to $f(x) = 3x + 6$.

289. Find the inverse function to $f(x) = 7 + 5x^3$. Then find its derivative.

290. Does the function $f(x) = x^3 - x$ have an inverse? (i.e. does it satisfy the horizontal line test?) Hint: Factor $x^3 - x$ and draw the graph.)

291. Find the inverse function to $f(x) = \sqrt{1 - x^2}$ where the domain is artificially restricted to the interval $0 \le x \le 1$. Draw a graph.

292. Let $f(x) = x^5 + x$ and $g(y) = f^{-1}(y)$. What is $f(1)$? $g(2)$? $f(2)$? $g(34)$? Find $f'(1)$, $g'(2)$, $f'(2)$, and $g'(34)$. Warning: Don't try to find a formula for $g(y)$.

293. Assume that $y = f(x)$. With the information given below you can find dx/dy for some values of y. Which values of y and what are the corresponding values of dx/dy?

$$f(3) = 4, \quad f(5) = 6, \quad f'(3) = 1, \quad f'(4) = 2, \quad f'(5) = 3, \quad f'(6) = 4.$$

294. (***i***) For which constants c does is the function defined by

$$f(x) = \begin{cases} 3x & \text{for } 0 \le x < 1; \\ 4x - c & \text{for } 1 \le x, \end{cases}$$

have an inverse function? (Hint: Horizontal Line Test.)

(***ii***) For which value of c is $f(x)$ continuous? (iii) Draw a graph of $y = f(x)$ for this value of c. (iv) Find a formula (like the formula for $f(x)$) for the inverse function $x = f^{-1}(y)$.

TRIGONOMETRIC FUNCTIONS

295. Find $\dfrac{d}{d\theta}\cot\theta$ and $\dfrac{d}{dv}\cot^{-1}(v)$.

296. Find $\dfrac{d}{d\theta}\sec\theta$ and $\dfrac{d}{dw}\sec^{-1}(w)$.

297. Find the second derivative of $\tan\theta$ with respect to θ.

298. In each of the following, find dy/dx.

(a) $y = \sin x$. (b) $y = (\sin x)^{-1}$. (c) $y = \sin(x^{-1})$. (d) $y = \sin^{-1}(x)$.

299. Consider the following functions

$$f_1(x) = \sin(x^2), \quad f_2(x) = (\sin x)^2, \quad f_3(x) = (\sin x)x,$$

$$f_4(x) = \sin^2 x, \quad f_5(x) = \sin(\sin x).$$

Which (if any) of these functions are the same? Evaluate the derivative of each of them. Use parentheses to make absolutely certain the order of evaluation is unambiguous. When you use the Chain Rule to differentiate a composition $f \circ g$ say which function plays the role of g and which plays the role of f.

300. Find the limit. Distinguish between an infinite limit and one which doesn't exist. (Give reasons!)

(**a**) $\displaystyle\lim_{x\to 0}\frac{\sin 3x}{x}$. (**b**) $\displaystyle\lim_{x\to\infty}\frac{\sin 3x}{x}$. (**c**) $\displaystyle\lim_{x\to 0+}\frac{\sin 3}{x}$.

(**d**) $\displaystyle\lim_{h\to 0}\frac{\sin(3+h)-\sin 3}{h}$. (**e**) $\displaystyle\lim_{x\to 3}\frac{\sin x-\sin 3}{x-3}$.

PARAMETRIC EQUATIONS

301. Confirm Example 6.3.7 by showing that

$$\big(\cosh(t)\big)^2 - \big(\sinh(t)\big)^2 = 1.$$

This is analogous to the Pythogorean Theorem

$$\big(\cos(t)\big)^2 + \big(\sin(t)\big)^2 = 1.$$

Also show that the hyperbolic sine and hyperbolic cosine are derivatives of each other. Thus we have the analogous equations

$$\frac{d}{dt}\sinh(t) = \cosh(t), \qquad \frac{d}{dt}\cosh(t) = \sinh(t),$$

$$\frac{d}{dt}\sin(t) = \cos(t), \qquad \frac{d}{dt}\cos(t) = -\sin(t).$$

Note the signs!

302. The point $P_0(-\frac{3}{5}, \frac{4}{5})$ lies on the unit circle $x^2 + y^2 = 1$. In the parameterization of Example 6.3.6 it corresponds to the parameter value $t = 2$. Use this parameterization to find the equation of the tangent line at this point. Then find the (same) equation using $y = \sqrt{1 - x^2}$.

303. Consider the parameterization

$$x = \frac{1 - t^2}{1 + t^2}, \quad y = \frac{2t}{1 + t^2}$$

of the unit circlefrom 6.3.6. For which value of t is $(x, y) = (1, 0)$? $(0, 1)$? $(0, -1)$? $(\frac{3}{5}, \frac{4}{5})$? $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$? Is there a value of t for which $(x, y) = (-1, 0)$?

304. Let $f(x) = \sqrt{(a + x)(b + x)}$ where a and b are constants. Show that

$$f''(x) = -\frac{(b - a)^2}{4f(x)^3}.$$

305. Find all points on the parabola with the equation $y = x^2 - 1$ such that the normal line at the point goes through the origin.

306. (***i***) Find c so that function

$$f(x) = \begin{cases} x + c & \text{for } x < 1 \\ 3^x & \text{for } x \geq 1 \end{cases}$$

is continuous. (***ii***) Draw a crude graph of the equation $y = f(x)$. (***iii***) Give a formula (like the above formula for $f(x)$) for the inverse function $x = f^{-1}(y)$ of the function $y = f(x)$.

Chapter 7

Exponentials and Logarithms (naturally)

In this chapter we first recall some facts about exponentials (x^y with $x > 0$ and y arbitrary): they should be familiar from algebra, or "precalculus." What is new is perhaps the definition of x^y when y is not a fraction: e.g., $2^{3/4}$ is the 4th root of the third power of 2 ($\sqrt[4]{2^3}$), but what is $2^{\sqrt{2}}$?

Then we ask "what is the derivative of $f(x) = a^x$?" The answer leads us to the famous number $e \approx 2.718\,281\,828\,459\,045\,235\,360\,287\,471\,352\,662\,497\,757\,247\,093\,699\,95\cdots$.

Finally, we compute the derivative of $f(x) = \log_a x$, and we look at things that "grow exponentially."

7.1 Exponents

Here we go over the definition of x^y when x and y are arbitrary real numbers, with $x > 0$.

For any real number x and any positive integer $n = 1, 2, 3, \ldots$ one defines

$$x^n = \overbrace{x \cdot x \cdot \cdots \cdot x}^{n \text{ times}}$$

and, if $x \neq 0$,

$$x^{-n} = \frac{1}{x^n}.$$

One defines $x^0 = 1$ for any $x \neq 0$.

To define $x^{p/q}$ for a general fraction $\frac{p}{q}$ one must assume that the number x is positive. One then defines

$$x^{p/q} = \sqrt[q]{x^p}. \tag{7.1}$$

This does not tell us how to define x^a is the exponent a is not a fraction. One can define x^a for irrational numbers a by taking limits. For example, to define $2^{\sqrt{2}}$, we look at the sequence of numbers you get by truncating the decimal expansion of $\sqrt{2}$, i.e.

$$a_1 = 1, \qquad a_2 = 1.4 = \tfrac{14}{10}, \qquad a_3 = 1.41 = \tfrac{141}{100}, \qquad a_4 = 1.414 = \tfrac{1414}{1000}, \qquad \cdots.$$

Each a_n is a fraction, so that we know what 2^{a_n} is, e.g. $2^{a_4} = \sqrt[1000]{2^{1414}}$. Our definition of $2^{\sqrt{2}}$ then is

$$2^{\sqrt{2}} = \lim_{n\to\infty} 2^{a_n},$$

i.e. we define $2^{\sqrt{2}}$ as the limit of the sequence of numbers

$$2,\ \sqrt[10]{2^{14}},\ \sqrt[100]{2^{141}},\ \sqrt[1000]{2^{1414}},\ldots$$

(See table 7.1.)

x	2^x
1.0000000000	**2**.*000000000000*
1.4000000000	**2.6***39015821546*
1.4100000000	**2.6***57371628193*
1.4140000000	**2.66***4749650184*
1.4142000000	**2.6651***19088532*
1.4142100000	**2.6651***37561794*
1.4142130000	**2.66514***3103798*
1.4142135000	**2.665144***027466*
⋮	⋮

Table 7.1: Approximating $2^{\sqrt{2}}$. Note that as x gets closer to $\sqrt{2}$ the quantity 2^x appears to converge to some number. This limit is our definition of $2^{\sqrt{2}}$.

Here one ought to prove that this limit exists, and that its value does not depend on the particular choice of numbers a_n tending to a. We will not go into these details in this course.

It is shown in precalculus texts that the exponential functions satisfy the following properties:

$$\boxed{x^a x^b = x^{a+b}, \qquad \frac{x^a}{x^b} = x^{a-b}, \qquad \left(x^a\right)^b = x^{ab}} \tag{7.2}$$

provided a and b are fractions. One can show that these properties still hold if a and b are real numbers (not necessarily fractions.) Again, we won't go through the proofs here.

Now instead of considering x^a as a function of x we can pick a positive number a and consider the function $f(x) = a^x$. This function is defined for all real numbers x (as long as the base a is positive.).

7.1.1 The trouble with powers of negative numbers.

The cube root of a negative number is well defined. For instance $\sqrt[3]{-8} = -2$ because $(-2)^3 = -8$. In view of the definition (7.1) of $x^{p/q}$ we can write this as

$$(-8)^{1/3} = \sqrt[3]{(-8)^1} = \sqrt[3]{-8} = -2.$$

But there is a problem: since $\frac{2}{6} = \frac{1}{3}$ you would think that $(-8)^{2/6} = (-8)^{1/3}$. However our definition (7.1) tells us that

$$(-8)^{2/6} = \sqrt[6]{(-8)^2} = \sqrt[6]{+64} = +2.$$

Another example:

$$(-4)^{1/2} = \sqrt{-4} \text{ is not defined}$$

but, even though $\frac{1}{2} = \frac{2}{4}$,

$$(-4)^{2/4} = \sqrt[4]{(-4)^2} = \sqrt[4]{+16} = 2 \text{ is defined.}$$

There are two ways out of this mess:

1. avoid taking fractional powers of negative numbers
2. when you compute $x^{p/q}$ first simplify the fraction by removing common divisors of p and q.

The safest is just not to take fractional powers of negative numbers.

Given that fractional powers of negative numbers cause all these headaches it is not surprising that we didn't try to define x^a for negative x if a is irrational. For example, $(-8)^\pi$ is not defined[1].

7.2 Logarithms

Briefly, $y = \log_a x$ is the inverse function to $y = a^x$. This means that, by definition,

$$y = \log_a x \iff x = a^y.$$

In other words, $\log_a x$ is the answer to the question "for which number y does one have $x = a^y$?" The number $\log_a x$ is called ***the logarithm with base a of x.*** In this definition both a and x must be positive.

For instance,

$$2^3 = 8, \quad 2^{1/2} = \sqrt{2}, \quad 2^{-1} = \frac{1}{2}$$

so

$$\log_2 8 = 3, \quad \log_2\bigl(\sqrt{2}\bigr) = \frac{1}{2}, \quad \log_2 \frac{1}{2} = -1.$$

Also:

$$\log_2(-3) \text{ doesn't exist}$$

because there is no number y for which $2^y = -3$ (2^y is always positive) and

$$\log_{-3} 2 \text{ doesn't exist either}$$

because $y = \log_{-3} 2$ would have to be some real number which satisfies $(-3)^y = 2$, and we don't take non-integer powers of negative numbers.

[1] There is a definition of $(-8)^\pi$ which uses complex numbers. You will see later in the text

7.3 Properties of logarithms

In general one has

$$\log_a a^x = x, \text{ and } a^{\log_a x} = x.$$

There is a subtle difference between these formulas: the first one holds for all real numbers x, but the second only holds for $x > 0$, since $\log_a x$ doesn't make sense for $x \le 0$.

Again, one finds the following formulas in precalculus texts:

$$\boxed{\begin{aligned} \log_a xy &= \log_a x + \log_a y \\ \log_a \frac{x}{y} &= \log_a x - \log_a y \\ \log_a x^y &= y \log_a x \\ \log_a x &= \frac{\log_b x}{\log_b a} \end{aligned}} \tag{7.3}$$

They follow from (7.2).

7.4 Graphs of exponential functions and logarithms

Figure 7.1 shows the graphs of some exponential functions $y = a^x$ with different values of a, and figure 7.2 shows the graphs of $y = \log_2 x$, $y = \log_3 x$, $\log_{1/2} x$, $\log_{1/3}(x)$ and $y = \log_{10} x$. Can you tell which is which? (Yes, you can.)

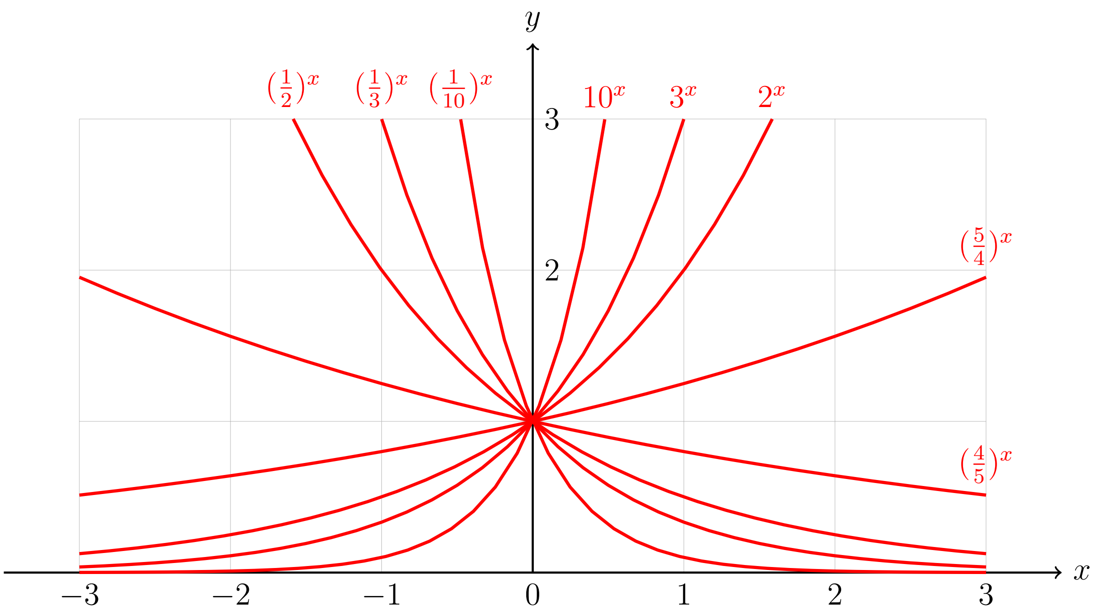


Figure 7.1: The graphs of $y = 2^x, 3^x, 10^x, (4/5)^x, (1/2)^x, (1/3)^x), (1/10)^x$ and $y = (5/4)^x$. Could you have figured out which was which?

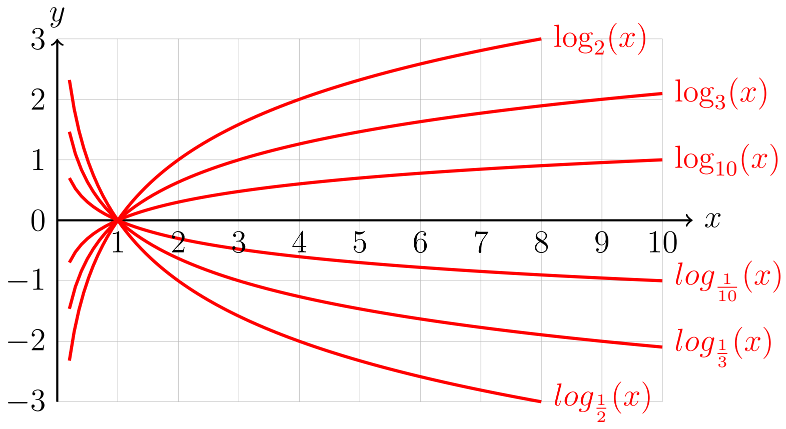


Figure 7.2: Graphs of some logarithms. Each curve is the graph of a function $y = \log_a x$ for various values of $a > 0$. Could you have figured out which was which?

From algebra/precalc recall:

$$\boxed{\text{If } a > 1 \text{ then } f(x) = a^x \text{ is an increasing function.}}$$

and

$$\boxed{\text{If } 0 < a < 1 \text{ then } f(x) = a^x \text{ is a decreasing function.}}$$

In other words, for $a > 1$ it follows from $x_1 < x_2$ that $a^{x_1} < a^{x_2}$; if $0 < a < 1$, then $x_1 < x_2$ implies $a^{x_1} > a^{x_2}$.

7.5 The derivative of a^x and the definition of e

To begin, we try to differentiate the function $y = 2^x$:

$$\begin{aligned} \frac{d2^x}{dx} &= \lim_{\Delta x \to 0} \frac{2^{x+\Delta x} - 2^x}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{2^x 2^{\Delta x} - 2^x}{\Delta x} \\ &= \lim_{\Delta x \to 0} 2^x \frac{2^{\Delta x} - 1}{\Delta x} \\ &= 2^x \lim_{\Delta x \to 0} \frac{2^{\Delta x} - 1}{\Delta x}. \end{aligned}$$

So if we assume that the limit

$$\lim_{\Delta x \to 0} \frac{2^{\Delta x} - 1}{\Delta x} = C$$

exists then we have

$$\frac{d2^x}{dx} = C2^x. \tag{7.4}$$

On your calculator you can compute $\frac{2^{\Delta x}-1}{\Delta x}$ for smaller and smaller values of Δx, which leads you to suspect that the limit actually exists, and that $C \approx 0.693\,147\ \ldots$. One can in fact prove that the limit exists, but we will not do this here.

For a slick presentation on this topic consider watching YouTube by 3Blue1Brown .

Once we know (7.4) we can compute the derivative of a^x for any other positive number a. To do this we write $a = 2^{\log_2 a}$, and hence

$$a^x = \left(2^{\log_2 a}\right)^x = 2^{x\cdot\log_2 a}.$$

By the chain rule we therefore get

$$\begin{aligned}\frac{da^x}{dx} &= \frac{d2^{x\cdot\log_2 a}}{dx}\\ &= C\ 2^{x\cdot\log_2 a}\ \frac{dx\cdot\log_2 a}{dx}\\ &= (C\log_2 a)\ 2^{x\cdot\log_2 a}\\ &= (C\log_2 a)\ a^x.\end{aligned}$$

So the derivative of a^x is just some constant times a^x, the constant being $C\log_2 a$. This is essentially our formula for the derivative of a^x, but one can make the formula look nicer by introducing a special number, namely, we define

$$e = 2^{1/C} \text{ where } C = \lim_{\Delta x\to 0}\frac{2^{\Delta x}-1}{\Delta x}.$$

One has

$$e \approx 2.718\,281\,818\,459\ \cdots$$

This number is special because if you set $a = e$, then

$$C\log_2 a = C\log_2 e = C\log_2 2^{1/C} = C\cdot\frac{1}{C} = 1,$$

and therefore the derivative of the function $y = e^x$ is

$$\boxed{\frac{de^x}{dx} = e^x.} \tag{7.5}$$

Read that again: the function e^x is its own derivative!

The logarithm with base e is called the ***Natural Logarithm***, and is written

$$\ln x = \log_e x.$$

Thus we have

$$\boxed{e^{\ln x} = x \qquad \ln e^x = x} \tag{7.6}$$

where the second formula holds for all real numbers x but the first one only makes sense for $x > 0$.

For any positive number a we have $a = e^{\ln a}$, and also

$$a^x = e^{x\ln a}.$$

By the chain rule you then get

$$\boxed{\frac{da^x}{dx} = a^x\ln a.} \tag{7.7}$$

7.6 Derivatives of Logarithms

Since the natural logarithm is the inverse function of $f(x) = e^x$ we can find its derivative by implicit differentiation. Here is the computation (which you should do yourself)
The function $f(x) = \log_a x$ satisfies

$$a^{f(x)} = x$$

Differentiate both sides, and use the chain rule on the left,

$$(\ln a)a^{f(x)} f'(x) = 1.$$

Then solve for $f'(x)$ to get

$$f'(x) = \frac{1}{(\ln a)a^{f(x)}}.$$

Finally we remember that $a^{f(x)} = x$ which gives us the derivative of a^x

$$\frac{da^x}{dx} = \frac{1}{x \ln a}.$$

In particular, the natural logarithm has a very simple derivative, namely, since $\ln e = 1$ we have

$$\boxed{\frac{d \ln x}{dx} = \frac{1}{x}.} \tag{7.8}$$

7.7 Limits involving exponentials and logarithms

Theorem 7.7.1. Let r be any real number. Then, if $a > 1$,

$$\lim_{x\to\infty} x^r a^{-x} = 0,$$

i.e.

$$\lim_{x\to\infty} \frac{x^r}{a^x} = 0.$$

This theorem says that *any exponential will beat any power of x as $x \to \infty$.* For instance, as $x \to \infty$ both x^{1000} and $(1.001)^x$ go to infinity, but

$$\lim_{x\to\infty} \frac{x^{1000}}{(1.001)^x} = 0,$$

so, in the long run, for very large x, 1.001^x will be much larger than 1000^x.

Proof when $a = e$. We want to show $\lim_{x\to\infty} x^r e^{-x} = 0$. To do this consider the function $f(x) = x^{r+1}e^{-x}$. Its derivative is

$$f'(x) = \frac{dx^{r+1}e^{-x}}{dx} = \big((r+1)x^r - x^{r+1}\big)e^{-x} = (r+1-x)x^r e^{-x}.$$

Therefore $f'(x) < 0$ for $x > r+1$, i.e. $f(x)$ is decreasing for $x > r+1$. It follows that $f(x) < f(r+1)$ for all $x > r+1$, i.e.

$$x^{r+1}e^{-x} < (r+1)^{r+1}e^{-(r+1)} \text{ for } x > r+1.$$

Divide by x, abbreviate $A = (r+1)^{r+1}e^{-(r+1)}$, and we get

$$0 < x^r e^{-x} < \frac{A}{x} \text{ for all } x > r+1.$$

The Sandwich Theorem implies that $\lim_{x\to\infty} x^r e^{-x} = 0$, which is what we had promised to show.

□

Here are some related limits:

$$\begin{aligned} a > 1 &\implies \lim_{x\to\infty} \frac{a^x}{x^r} = \infty \quad (D.N.E.) \\ m > 0 &\implies \lim_{x\to\infty} \frac{\ln x}{x^m} = 0 \\ m > 0 &\implies \lim_{x\to 0} x^m \ln x = 0 \end{aligned}$$

The second limit says that even though $\ln x$ becomes infinitely large as $x \to \infty$, it is always much less than any power x^m with $m > 0$ real. To prove it you set $x = e^t$ and then $t = s/m$, which leads to

$$\lim_{x\to\infty} \frac{\ln x}{x^m} \overset{x=e^t}{=} \lim_{t\to\infty} \frac{t}{e^{mt}} \overset{t=s/m}{=} \frac{1}{m} \lim_{t\to\infty} \frac{s}{e^s} = 0.$$

The third limit follows from the second by substituting $x = 1/y$ and using $\ln \frac{1}{x} = -\ln x$.

7.8 Exponential growth and decay

A quantity X which depends on time t is said to grow or decay exponentially if it is given by

$$X(t) = X_0 e^{kt}. \tag{7.9}$$

The constant X_0 is the value of $X(t)$ at time $t = 0$ (sometimes called "the initial value of X").

The derivative of an exponentially growing quantity, i.e. its rate of change with time, is given by $X'(t) = X_0\, k e^{kt}$ so that

$$\frac{dX(t)}{dt} = kX(t). \tag{7.10}$$

In words, *for an exponentially growing quantity the rate of change is always proportional to the quantity itself.* The proportionality constant is k and is sometimes called "the relative growth rate."

This property of exponential functions completely describes them, by which I mean that any function which satisfies (7.10) automatically satisfies (7.9). To see that this is true, suppose you have a function $X(t)$ for which $X'(t) = kX(t)$ holds at all times t. Then

$$\begin{aligned}\frac{dX(t)e^{-kt}}{dt} &= X(t)\frac{de^{-kt}}{dt} + \frac{dX(t)}{dt}e^{-kt} \\ &= -kX(t)e^{-kt} + X'(t)e^{-kt} \\ &= (X'(t) - kX(t))e^{-kt} \\ &= 0.\end{aligned}$$

It follows that $X(t)e^{-kt}$ does not depend on t. At $t = 0$ one has

$$X(t)e^{-kt} = X(0)e^{0} = X_0$$

and therefore we have

$$X(t)e^{-kt} = X_0 \text{ for all } t.$$

Multiply with e^{kt} and we end up with

$$X(t) = X_0e^{kt}.$$

7.8.1 Half time and doubling time.

If $X(t) = X_0e^{kt}$ then one has

$$X(t+T) = X_0e^{kt+kT} = X_0e^{kt}e^{kT} = e^{kT}X(t).$$

In words, after time T goes by an exponentially growing (decaying) quantity changes by a factor e^{kT}. If $k > 0$, so that the quantity is actually growing, then one calls

$$T = \frac{\ln 2}{k}$$

the ***doubling time*** for X because $X(t)$ changes by a factor $e^{kT} = e^{\ln 2} = 2$ every T time units: $X(t)$ doubles every T time units.
If $k < 0$ then $X(t)$ is decaying and one calls

$$T = \frac{\ln 2}{-k}$$

the ***half life*** because $X(t)$ is reduced by a factor $e^{kT} = e^{-\ln 2} = \frac{1}{2}$ every T time units.

To see this concept in action consider watching YouTube by 3Blue1Brown on the Covid 19 outbreak.

7.8.2 Determining X_0 and k.

The general exponential growth/decay function (7.9) contains only two constants, X_0 and k, and if you know the values of $X(t)$ at two different times then you can compute these constants.

Suppose that you know

$$X_1 = X(t_1) \text{ and } X_2 = X(t_2).$$

Then we have

$$X_0 e^{kt_1} = X_1 \text{ and } X_2 = X_0 e^{kt_2}$$

in which t_1, t_2, X_1, X_2 are given and k and X_0 are unknown. One first finds k from

$$\frac{X_1}{X_2} = \frac{X_0 e^{kt_1}}{X_0 e^{kt_2}} = e^{k(t_1 - t_2)} \implies \ln \frac{X_1}{X_2} = k(t - 1 - t_2)$$

which implies

$$k = \frac{\ln X_1 - \ln X_2}{t_1 - t_2}.$$

Once you have computed k you can find X_0 from

$$X_0 = \frac{X_1}{e^{kt_1}} = \frac{X_2}{e^{kt_2}}.$$

(both expressions should give the same result.)

7.9 PROBLEMS

GRAPHS OF EXP AND LOG

Sketch the graphs of the following functions.

Hint: for some of these you have to solve something like $e^{4x} - 3e^{3x} + e^x = 0$, then call $w = e^x$, and you get a polynomial equation for w, namely $w^4 - 3w^3 + w = 0$.

307. $y = e^x$

308. $y = e^{-x}$

309. $y = e^x + e^{-2x}$ †386

310. $y = e^{3x} - 4e^x$ †386

311. $y = \dfrac{e^x}{1 + e^x}$

312. $y = \dfrac{2e^x}{1 + e^{2x}}$

313. $y = xe^{-x}$

314. $y = \sqrt{x} e^{-x/4}$

315. $y = x^2 e^{x+2}$

316. $y = e^{x/2} - x$

317. $y = \ln \sqrt{x}$

318. $y = \ln \dfrac{1}{x}$

319. $y = x \ln x$

320. $y = \dfrac{-1}{\ln x}$ $\quad (0 < x < \infty, x \neq 1)$

321. $y = (\ln x)^2$ $\quad (x > 0)$

322. $y = \dfrac{\ln x}{x}$ $\quad (x > 0)$

323. $y = \ln \sqrt{\dfrac{1 + x}{1 - x}}$ $\quad (|x| < 1)$

324. $y = \ln\bigl(1 + x^2\bigr)$

325. $y = \ln\bigl(x^2 - 3x + 2\bigr) \quad (x > 2)$

326. $y = \ln\cos x \quad (|x| < \frac{\pi}{2})$

327. The function $f(x) = e^{-x^2}$ plays a central in statistics and its graph is called ***the bell curve*** (because of its shape). Sketch the graph of f.

328. Sketch the part of the graph of the function

$$f(x) = e^{-\frac{1}{x}}$$

with $x > 0$.

Find the limits

$$\lim_{x \searrow 0} \frac{f(x)}{x^n} \quad \text{and} \quad \lim_{x\to\infty} f(x)$$

where n can be any positive integer (hint: substitute $x = \ldots$?)

329. A ***damped oscillation*** is a function of the form

$$f(x) = e^{-ax}\cos bx \text{ or } f(x) = e^{-ax}\sin bx$$

where a and b are constants.

Sketch the graph of $f(x) = e^{-x}\sin 10x$ (i.e. find zeroes, local max and mins, inflection points) and draw (with pencil on paper) the piece of the graph with $0 \le x \le 2\pi$.

This function has many local maxima and minima. *What is the ratio between the function values at two consecutive local maxima?* (Hint: the answer does not depend on which pair of consecutive local maxima you consider.)

330. Find the inflection points on the graph of $f(x) = (1 + x)\ln x$ $(x > 0)$.

331. **(a)** If x is large, which is bigger: 2^x or x^2?
(b) The graphs of $f(x) = x^2$ and $g(x) = 2^x$ intersect at $x = 2$ (since $2^2 = 2^2$). How many more intersections do these graphs have (with $-\infty < x < \infty$)?

LIMITS OF EXP AND LOG FUNCTIONS.

Find the following limits.

332. $\displaystyle\lim_{x\to\infty} \frac{e^x - 1}{e^x + 1}$

333. $\displaystyle\lim_{x\to\infty} \frac{e^x - x^2}{e^x + x}$

334. $\displaystyle\lim_{x\to\infty} \frac{2^x}{3^x - 2^x}$

335. $\displaystyle\lim_{x\to\infty} \frac{e^x - x^2}{e^{2x} + e^{-x}}$

336. $\displaystyle\lim_{x\to\infty} \frac{e^{-x} - e^{-x/2}}{\sqrt{e^x + 1}}$

337. $\displaystyle\lim_{x\to\infty} \frac{\sqrt{x + e^{4x}}}{e^{2x} + x}$

338. $\displaystyle\lim_{x\to\infty} \frac{e^{\sqrt{x}}}{\sqrt{e^x + 1}}$

339. $\displaystyle\lim_{x\to\infty} \ln(1 + x) - \ln x$

340. $\displaystyle\lim_{x\to\infty} \frac{\ln x}{\ln x^2}$

341. $\displaystyle\lim_{x\to 0} x\ln x$

342. $\displaystyle\lim_{x\to\infty} \frac{\ln x}{\sqrt{x} + \ln x}$

343. $\displaystyle\lim_{x\to 0} \frac{\ln x}{\sqrt{x} + \ln x}$

MISCELLANEOUS PROBLEMS

344. Find the tenth derivative of xe^x.

345. For which real number x is $2^x - 3^x$ the largest?

346. Find $\dfrac{dx^x}{dx}$, $\dfrac{dx^{x^x}}{dx}$, and $\dfrac{d(x^x)^x}{dx}$. (Hint: $x^x = e^?$.)

347. Let $y = (x+1)^2(x+3)^4(x+5)^6$ and $u = \ln y$. Find du/dx.

Hint: Use the fact that ln converts multiplication to addition before you differentiate. It will simplify the calculation.

348. After 3 days a sample of radon-222 decayed to 58% of its original amount.

(a) What is the half life of radon-222?

(b) How long would it take the sample to decay to 10% of its original amount?

349. Polonium-210 has a half life of 140 days.

(a) If a sample has a mass of 200 mg find a formula for the mass that remains after t days.

(b) Find the mass after 100 days.

(c) When will the mass be reduced to 10 mg?

(d) Sketch the graph of the mass as a function of time.

350. Current agricultural experts believe that the world's farms can feed about 10 billion people. The 1950 world population was 2.517 billion and the 1992 world population was 5.4 billion. When can we expect to run out of food?

351. The AC*ME* company runs two ads on Sunday mornings. One says that "when this baby is old enough to vote, the world will have one billion new mouths to feed" and the other says "in thirty six years, the world will have to set eight billion places at the table." What does AC*ME* think the population of the world is at present? How fast does AC*ME* think the population is increasing? Use units of billions of people so you can write 8 instead of $8,000,000,000$. (Hint: $36 = 2 \times 18$.)

352. The population of California grows exponentially at an instantaneous rate of 2% per year. The population of California on January 1, 2000 was 20,000,000.

(a) Write a formula for the population $N(t)$ of California t years after January 1, 2000.

(b) Each Californian consumes pizzas at the rate of 70 pizzas per year. At what rate is California consuming pizzas t years after 1990?

(c) How many pizzas were consumed in California from January 1, 2005 to January 1, 2009?

353. The population of the country of Farfarawaÿ grows exponentially.

(a) If its population in the year 1980 was 1,980,000 and its population in the year 1990 was 1,990,000, what is its population in the year 2000?

(b) How long will it take the population to double? (Your answer may be expressed in terms of exponentials and natural logarithms.)

354. The *hyperbolic functions* are defined by

$$\sinh x = \frac{e^x - e^{-x}}{2},$$
$$\cosh x = \frac{e^x + e^{-x}}{2},$$
$$\tanh x = \frac{\sinh x}{\cosh x}.$$

(a) Prove the following identities

$$\cosh^2 x - \sinh^2 x = 1$$
$$\cosh 2x = \cosh^2 x + \sinh^2 x$$
$$\sinh 2x = 2 \sinh x \cosh x.$$

(b) Show that

$$\frac{d \sinh x}{dx} = \cosh x,$$
$$\frac{d \cosh x}{dx} = \sinh x,$$
$$\frac{d \tanh x}{dx} = \frac{1}{\cosh^2 x}.$$

(c) Sketch the graphs of the three hyperbolic functions.

Chapter 8

The Integral

In this chapter we define the integral of a function on some interval $[a, b]$. The most common interpretation of the integral is in terms of the area under the graph of the given function, so that is where we begin.

8.1 Area under a Graph

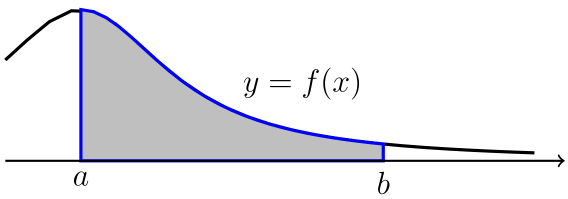


Figure 8.1: area under a graph

Let f be a function which is defined on some interval $a \leq x \leq b$ and assume it is positive, i.e. assume that its graph lies above the x axis. *How large is the area of the region caught between the x axis, the graph of $y = f(x)$ and the vertical lines $y = a$ and $y = b$?*

You can try to compute this area by approximating the region with many thin rectangles. Look at figure 8.2 before you read on. To make the approximating region you choose a *partition* of the interval $[a, b]$, i.e. you pick numbers $x_1 < \cdots < x_n$ with

$$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b.$$

These numbers split the interval $[a, b]$ into n sub-intervals

$$[x_0, x_1], \quad [x_1, x_2], \quad \ldots, \quad [x_{n-1}, x_n]$$

whose lengths are

$$\Delta x_1 = x_1 - x_0, \quad \Delta x_2 = x_2 - x_1, \quad \ldots, \quad \Delta x_n = x_n - x_{n-1}.$$

In each interval we choose a point c_k, i.e. in the first interval we choose $x_0 \le c_1 \le x_1$, in the second interval we choose $x_1 \le c_2 \le x_2$, ..., and in the last interval we choose some number $x_{n-1} \le c_n \le x_n$. See figure 8.2.

We then define n rectangles: the base of the k^{th} rectangle is the interval $[x_{k-1}, x_k]$ on the x-axis, while its height is $f(c_k)$ (here k can be any integer from 1 to n.)

The area of the k^{th} rectangle is of course the product of its height and width, i.e. its area is $f(c_k)\Delta x_k$. Adding these we see that the total area of the rectangles is

$$R = f(c_1)\Delta x_1 + f(c_2)\Delta x_2 + \cdots + f(c_n)\Delta x_n. \tag{8.1}$$

This kind of sum is called a ***Riemann sum.***

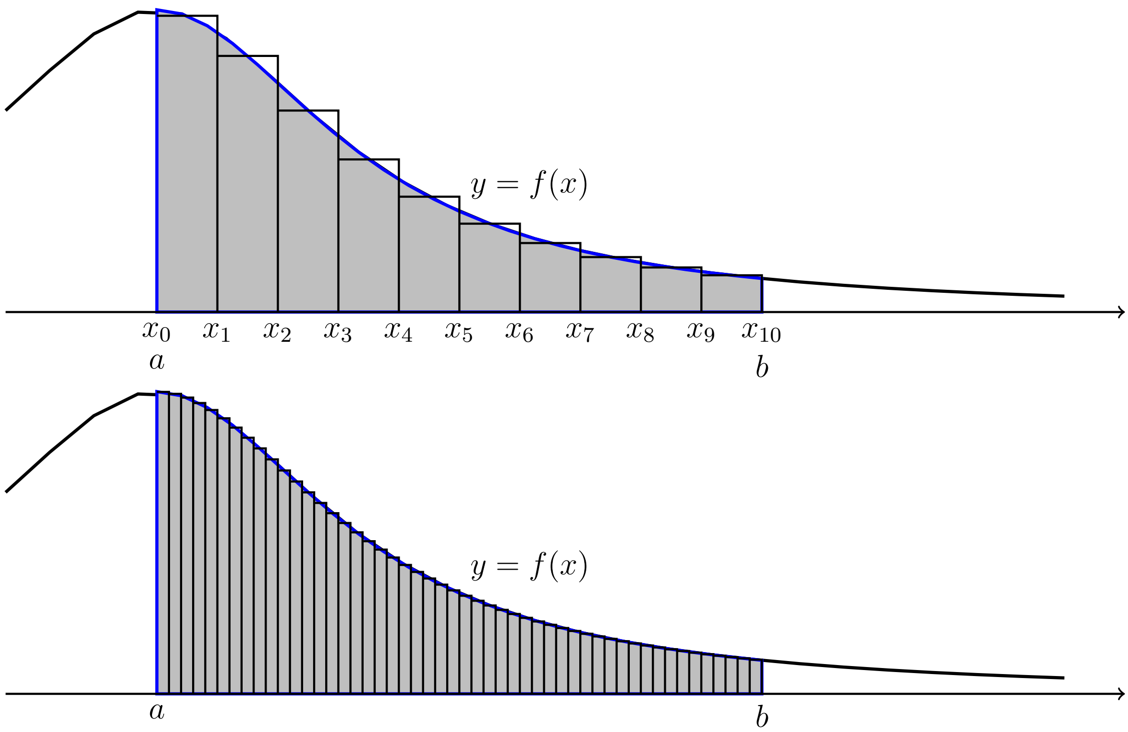


Figure 8.2: TOP: A Riemann sum in which the interval $a < x < b$ has been cut up into ten smaller intervals. In each of those intervals a point c_i has been chosen at random, and the resulting rectangles with heights $f(c_1)$, ..., $f(c_6)$ were drawn. The total area under the graph of the function is roughly equal to the total area of the rectangles. BOTTOM: Refining the partition. After adding more partition points the combined area of the rectangles will be a better approximation of the area under the graph of the function f.

If the partition is sufficiently fine then one would expect this sum, i.e. the total area of all rectangles to be a good approximation of the area of the region under the graph. Replacing the partition by a finer partition, with more division points, should improve the approximation. So you would expect the area to be the limit of Riemann-sums like R "as the partition becomes finer and finer." A precise formulation of the definition goes like this:

Definition 8.1.1. If f is a function defined on an interval $[a, b]$, then we say that

$$\int_a^b f(x)dx = I,$$

i.e. the integral of "$f(x)$ from $x = a$ to b" equals I, if for every $\varepsilon > 0$ one can find a $\delta > 0$ such that

$$\Big| f(c_1)\Delta x_1 + f(c_2)\Delta x_2 + \cdots + f(c_n)\Delta x_n - I \Big| < \varepsilon$$

holds for every partition all of whose intervals have length $\Delta x_k < \delta$.

8.2 When f changes its sign

If the function f is not necessarily positive everywhere in the interval $a \le x \le b$, then we still define the integral in exactly the same way: as a limit of Riemann sums whose mesh size becomes smaller and smaller. However the interpretation of the integral as "the area of the region between the graph and the x-axis" has a twist to it.

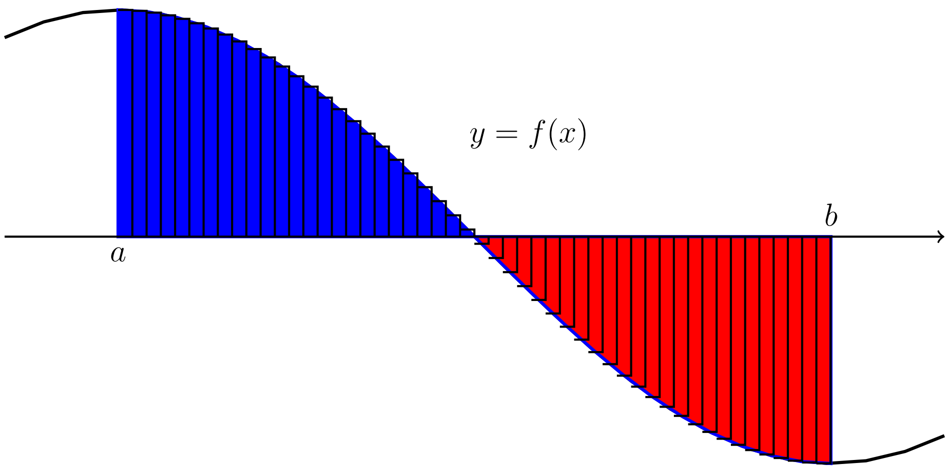


Figure 8.3: Illustrating a Riemann sum for a function whose sign changes. Always remember that areas are positive numbers. The Riemann-sum corresponding to this picture is the total area of the rectangles above the x-axis *minus* the total area of the rectangles below the x-axis.

Let f be some function on an interval $a \le x \le b$, and form the Riemann sum

$$R = f(c_1)\Delta x_1 + f(c_2)\Delta x_2 + \cdots + f(c_n)\Delta x_n$$

that goes with some partition, and some choice of c_k.

When f can be positive or negative, then the terms in the Riemann sum can also be positive or negative. If $f(c_k) > 0$ then the quantity $f(c_k)\Delta x_k$ is the area of the corresponding rectangle, but if $f(c_k) < 0$ then $f(c_k)\Delta x_k$ is a negative number, namely *minus* the area of the corresponding rectangle. The Riemann sum is therefore the area of the rectangles above the x-axis *minus* the area below the axis and above the graph.

Taking the limit over finer and finer partitions, we conclude that

$$\int_a^b f(x)dx = \begin{array}{c}\text{area above the } x\text{-axis, below the graph} \\ \textbf{\textit{minus}} \text{ the area below the } x\text{-axis, above the graph.}\end{array}$$

8.3 The Fundamental Theorem of Calculus

The reader may want to listen to this YouTube by 3Blue1Brown before continuing with this section.

Definition 8.3.1. A function F is called an ***antiderivative*** of f on the interval $[a, b]$ if one has $F'(x) = f(x)$ for all x with $a < x < b$.

For instance, $F(x) = \frac{1}{2}x^2$ is an antiderivative of $f(x) = x$, but so is $G(x) = \frac{1}{2}x^2 + 2008$.

Theorem 8.3.1. If f is a function whose integral $\int_a^b f(x)dx$ exists, and if F is an antiderivative of f on the interval $[a, b]$, then one has

$$\int_a^b f(x)dx = F(b) - F(a). \tag{8.2}$$

Because of this theorem the expression on the right appears so often that various abbreviations have been invented. We will abbreviate

$$F(b) - F(a) \stackrel{\text{def}}{=} \big[F(x)\big]_{x=a}^{b} = \big[F(x)\big]_{a}^{b}.$$

8.3.1 Terminology.

In the integral

$$\int_a^b f(x)\,dx$$

the numbers a and b are called the *bounds of the integral*, the function $f(x)$ which is being integrated is called *the integrand*, and the variable x is *integration variable*.

The integration variable is a *dummy variable*. If you systematically replace it with another variable, the resulting integral will still be the same. For instance,

$$\int_0^1 x^2\,dx = \left[\tfrac{1}{3}x^3\right]_{x=0}^{1} = \tfrac{1}{3},$$

and if you replace x by φ you still get

$$\int_0^1 \varphi^2\,d\varphi = \left[\tfrac{1}{3}\varphi^3\right]_{\varphi=0}^{1} = \tfrac{1}{3}.$$

Another way to appreciate that the integration variable is a dummy variable is to look at the Fundamental Theorem again:

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

The right hand side tells you that the value of the integral depends on a and b, and has absolutely nothing to do with the variable x.

8.4 The indefinite integral

The fundamental theorem tells us that in order to compute the integral of some function f over an interval $[a,b]$ you should first find an antiderivative F of f. In practice, much of the effort required to find an integral goes into finding the antiderivative. In order to simplify the computation of the integral

$$\int_a^b f(x)dx = F(b) - F(a) \tag{8.3}$$

the following notation is commonly used for the antiderivative:

$$F(x) = \int f(x)dx. \tag{8.4}$$

For instance,

$$\int x^2\, dx = \frac{1}{3}x^3, \quad \int \sin 5x\, dx = -\tfrac{1}{5}\cos 5x, \quad \text{etc}\ldots$$

The integral which appears here does not have the integration bounds a and b. It is called an ***indefinite integral***, as opposed to the integral in (10.1) which is called a ***definite integral***. You use the indefinite integral if you expect the computation of the antiderivative to be a lengthy affair, and you do not want to write the integration bounds a and b all the time.

It is important to distinguish between the two kinds of integrals. Here is a list of differences:

INDEFINITE INTEGRAL	DEFINITE INTEGRAL
$\int f(x)dx$ is a function of x.	$\int_a^b f(x)dx$ is a number.
By definition $\int f(x)dx$ is *any function of x whose derivative is $f(x)$.*	$\int_a^b f(x)dx$ was defined in terms of Riemann sums and can be interpreted as "area under the graph of $y = f(x)$", at least when $f(x) > 0$.
x is not a dummy variable, for example, $\int 2xdx = x^2 + C$ and $\int 2tdt = t^2 + C$ are functions of different variables, so they are not equal.	x is a dummy variable, for example, $\int_0^1 2xdx = 1$, and $\int_0^1 2tdt = 1$, so $\int_0^1 2xdx = \int_0^1 2tdt$.

8.4.1 You can always check the answer.

Suppose you want to find an antiderivative of a given function $f(x)$ and after a long and messy computation which you don't really trust you get an "answer", $F(x)$. You can then throw away the dubious computation and differentiate the $F(x)$ you had found. If

$F'(x)$ turns out to be equal to $f(x)$, then your $F(x)$ is indeed an antiderivative and your computation isn't important anymore.

For example, suppose that we want to find $\int \ln x \, dx$. Emily's cousin Jesse says it might be $F(x) = x \ln x - x$. Let's see if he's right:

$$\frac{d}{dx}\big(x \ln x - x\big) = x \cdot \frac{1}{x} + 1 \cdot \ln x - 1 = \ln x.$$

Who knows how Jesse thought of this[1], but it doesn't matter: he's right! We now know that $\int \ln x dx = x \ln x - x + C$.

8.4.2 About "$+C$".

Let $f(x)$ be a function defined on some interval $a \le x \le b$. If $F(x)$ is an antiderivative of $f(x)$ on this interval, then for any constant C the function $\tilde{F}(x) = F(x) + C$ will also be an antiderivative of $f(x)$. So one given function $f(x)$ has many different antiderivatives, obtained by adding different constants to one given antiderivative.

Theorem 8.4.1. If $F_1(x)$ and $F_2(x)$ are antiderivatives of the same function $f(x)$ on some interval $a \le x \le b$, then there is a constant C such that $F_1(x) = F_2(x) + C$.

Proof. Consider the difference $G(x) = F_1(x) - F_2(x)$. Then $G'(x) = F_1'(x) - F_2'(x) = f(x) - f(x) = 0$, so that $G(x)$ must be constant. Hence $F_1(x) - F_2(x) = C$ for some constant. □

It follows that there is some ambiguity in the notation $\int f(x) \, dx$. Two functions $F_1(x)$ and $F_2(x)$ can both equal $\int f(x) \, dx$ without equaling each other. When this happens, they (F_1 and F_2) differ by a constant. This can sometimes lead to confusing situations, e.g. you can check that

$$\int 2 \sin x \cos x \, dx = \sin^2 x$$

$$\int 2 \sin x \cos x \, dx = -\cos^2 x$$

are both correct. (Just differentiate the two functions $\sin^2 x$ and $-\cos^2 x$!) These two answers look different until you realize that because of the trig identity $\sin^2 x + \cos^2 x = 1$ they really only differ by a constant: $\sin^2 x = -\cos^2 x + 1$.

To avoid this kind of confusion we will from now on never forget to include the "arbitrary constant $+C$" in our answer when we compute an antiderivative.

Table 8.1 lists a number of antiderivatives which you should know. All of these integrals should be familiar from the differentiation rules we have learned so far, except for for the integrals of $\tan x$ and of $\frac{1}{\cos x}$. You can check those by differentiation (using $\ln \frac{a}{b} = \ln a - \ln b$ simplifies things a bit).

[1] He took math and learned to integrate by parts.

$$\int f(x)\,dx = F(x) + C$$

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \qquad \text{for all } n \neq -1$$

$$\int \frac{1}{x}\,dx = \ln|x| + C \qquad \text{(Note the absolute values)}$$

$$\int e^x\,dx = e^x + C$$

$$\int a^x\,dx = \frac{a^x}{\ln a} + C \qquad \text{(don't memorize: use } a^x = e^{x\ln a}\text{)}$$

$$\int \sin x\,dx = -\cos x + C$$

$$\int \cos x\,dx = \sin x + C$$

$$\int \tan x\,dx = -\ln|\cos x| + C \qquad \text{(Note the absolute values)}$$

$$\int \frac{1}{1+x^2}\,dx = \arctan x + C$$

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin x + C$$

The following integral is also useful, but not as important as the ones above:

$$\int \frac{dx}{\cos x} = \frac{1}{2}\ln\frac{1+\sin x}{1-\sin x} + C \text{ for } -\frac{\pi}{2} < x < \frac{\pi}{2}.$$

Table 8.1: The list of the standard integrals everyone should know

8.5 Properties of the Integral

Just as we had a list of properties for the limits and derivatives of sums and products of functions, the integral has similar properties.

Suppose we have two functions $f(x)$ and $g(x)$ with antiderivatives $F(x)$ and $G(x)$, respectively. Then we know that

$$\frac{d}{dx}\big\{F(x) + G(x)\big\} = F'(x) + G'(x) = f(x) + g(x),$$

in words, $F + G$ is an antiderivative of $f + g$, which we can write as

$$\int \big\{f(x) + g(x)\big\}\,dx = \int f(x)\,dx + \int g(x)\,dx. \tag{8.5}$$

Similarly, $\frac{d}{dx}\big(cF(x)\big) = cF'(x) = cf(x)$ implies that

$$\int cf(x)\,dx = c\int f(x)\,dx \tag{8.6}$$

if c is a constant.

These properties imply analogous properties for the definite integral. For any pair of functions on an interval $[a, b]$ one has

$$\int_a^b \big[f(x) + g(x)\big]\,dx = \int_a^b f(x)dx + \int_a^b g(x)dx, \tag{8.7}$$

and for any function f and constant c one has

$$\int_a^b cf(x)dx = c\int_a^b f(x)dx. \tag{8.8}$$

Definite integrals have one other property for which there is no analog in indefinite integrals: if you split the interval of integration into two parts, then the integral over the whole is the sum of the integrals over the parts. The following theorem says it more precisely.

Theorem 8.5.1. Given $a < c < b$, and a function on the interval $[a, b]$ then

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx. \tag{8.9}$$

Proof. Let F be an antiderivative of f. Then

$$\int_a^c f(x)dx = F(c) - F(a) \text{ and } \int_c^b f(x)dx = F(b) - F(a),$$

so that

$$\begin{aligned}\int_a^b f(x)dx &= F(b) - F(a)\\ &= F(b) - F(c) + F(c) - F(a)\\ &= \int_a^c f(x)dx + \int_c^b f(x)dx.\end{aligned}$$

□

So far we have always assumed theat $a < b$ in all indefinitie integrals $\int_a^b \ldots$. The fundamental theorem suggests that when $b < a$, we should define the integral as

$$\int_a^b f(x)dx = F(b) - F(a) = -\big(F(a) - F(b)\big) = -\int_b^a f(x)dx. \tag{8.10}$$

For instance,

$$\int_1^0 xdx = -\int_0^1 xdx = -\tfrac{1}{2}.$$

8.6 The definite integral as a function of its integration bounds

Consider the expression

$$I = \int_0^x t^2\, dt.$$

What does I depend on? To see this, you calculate the integral and you find

$$I = \left[\tfrac{1}{3}t^3\right]_0^x = \tfrac{1}{3}x^3 - \tfrac{1}{3}0^3 = \tfrac{1}{3}x^3.$$

So the integral depends on x. It does not depend on t, since t is a "dummy variable" (see §8.3.1 where we already discussed this point.)

In this way you can use integrals to define new functions. For instance, we could define

$$I(x) = \int_0^x t^2 dt,$$

which would be a roundabout way of defining the function $I(x) = x^3/3$. Again, since t is a dummy variable we can replace it by any other variable we like. Thus

$$I(x) = \int_0^x \alpha^2\, d\alpha$$

defines the same function (namely, $I(x) = \frac{1}{3}x^3$).

The previous example does not define a new function ($I(x) = x^3/3$). An example of a *new* function defined by an integral is the "error-function" from statistics. It is given by

$$\operatorname{erf}(x) \stackrel{\text{def}}{=} \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, dt, \tag{8.11}$$

so $\operatorname{erf}(x)$ is the area of the shaded region in figure 8.4.

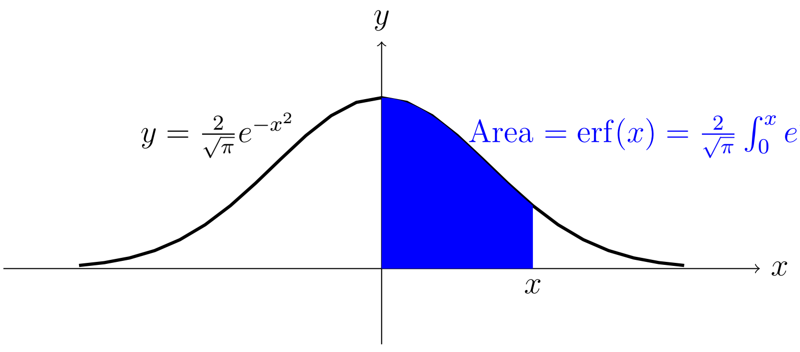


Figure 8.4: Definition of the Error function.

The integral in (8.11) cannot be computed in terms of the standard functions (square and higher roots, sine, cosine, exponential and logarithms). Since the integral in (8.11) occurs very often in statistics (in relation with the so-called normal distribution) it has been given a name, namely, "$\operatorname{erf}(x)$".

How do you differentiate a function that is defined by an integral? The answer is simple, for if $f(x) = F'(x)$ then the fundamental theorem says that

$$\int_a^x f(t)\, dt = F(x) - F(a),$$

and therefore

$$\frac{d}{dx}\int_a^x f(t)\, dt = \frac{d}{dx}\{F(x) - F(a)\} = F'(x) = f(x),$$

i.e.

$$\frac{d}{dx}\int_a^x f(t)\, dt = f(x).$$

A similar calculation gives you

$$\frac{d}{dx}\int_x^b f(t)\, dt = -f(x).$$

So what is the derivative of the error function? We have

$$\begin{aligned}\operatorname{erf}'(x) &= \frac{d}{dx}\Big\{\frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\, dt\Big\}\\ &= \frac{2}{\sqrt{\pi}}\frac{d}{dx}\int_0^x e^{-t^2}\, dt\\ &= \frac{2}{\sqrt{\pi}}e^{-x^2}.\end{aligned}$$

8.7 Method of substitution

The chain rule says that

$$\frac{dF(G(x))}{dx} = F'(G(x)) \cdot G'(x),$$

so that

$$\int F'(G(x)) \cdot G'(x)\, dx = F(G(x)) + C.$$

8.7.1 Example.

Consider the function $f(x) = 2x\sin(x^2 + 3)$. It does not appear in the list of standard antiderivatives we know by heart. But we do notice[2] that $2x = \frac{d}{dx}(x^2 + 3)$. So let's call $G(x) = x^2 + 3$, and $F(u) = -\cos u$, then

$$F(G(x)) = -\cos(x^2 + 3)$$

and

$$\frac{dF(G(x))}{dx} = \underbrace{\sin(x^2 + 3)}_{F'(G(x))} \cdot \underbrace{2x}_{G'(x)} = f(x),$$

so that

$$\int 2x\sin(x^2 + 3)\, dx = -\cos(x^2 + 3) + C. \tag{8.12}$$

[2] You *will* start noticing things like this after doing several examples.

8.7.2 Leibniz' notation for substitution.

The most transparent way of computing an integral by substitution is by following Leibniz and introduce new variables. Thus to do the integral

$$\int f(G(x))G'(x)\,dx$$

where $f(u) = F'(u)$, we introduce the substitution $u = G(x)$, and agree to write

$$du = dG(x) = G'(x)\,dx.$$

Then we get

$$\int f(G(x))G'(x)\,dx = \int f(u)\,du = F(u) + C.$$

At the end of the integration we must remember that u really stands for $G(x)$, so that

$$\int f(G(x))G'(x)\,dx = F(u) + C = F(G(x)) + C.$$

As an example, let's do the integral (8.12) using Leibniz' notation. We want to find

$$\int 2x\sin(x^2+3)\,dx$$

and decide to substitute $z = x^2 + 3$ (the substitution variable doesn't always have to be called u). Then we compute

$$dz = d\big(x^2+3\big) = 2x\,dx \text{ and } \sin(x^2+3) = \sin z,$$

so that

$$\int 2x\sin(x^2+3)\,dx = \int \sin z\,dz = -\cos z + C.$$

Finally we get rid of the substitution variable z, and we find

$$\int 2x\sin(x^2+3)\,dx = -\cos\big(x^2+3\big) + C.$$

When we do integrals in this calculus class, we always get rid of the substitution variable because it is a variable we invented, and which does not appear in the original problem. But if you are doing an integral which appears in some longer discussion of a real-life (or real-lab) situation, then it may be that the substitution variable actually has a meaning (e.g. "the effective stoichiometric modality of CQF self-inhibition") in which case you may want to skip the last step and leave the integral in terms of the (meaningful) substitution variable.

8.7.3 Substitution for definite integrals.

For definite integrals the chain rule

$$\frac{d}{dx}\big(F(G(x))\big) = F'(G(x))G'(x) = f(G(x))G'(x)$$

implies

$$\int_a^b f(G(x))G'(x)\,dx = F(G(b)) - F(G(a)).$$

which you can also write as

$$\int_{x=a}^{b} f(G(x))G'(x)\,dx = \int_{u=G(a)}^{G(b)} f(u)\,du. \tag{8.13}$$

8.7.4 Example of substitution in a definite integral.

Let's compute

$$\int_0^1 \frac{x}{1+x^2}\,dx,$$

using the substitution $u = G(x) = 1 + x^2$. Since $du = 2x\,dx$, the associated ***indefinite*** integral is

$$\int \underbrace{\frac{1}{1+x^2}}_{\frac{1}{u}} \underbrace{x\,dx}_{\frac{1}{2}du} = \tfrac{1}{2}\int \frac{1}{u}\,du.$$

To find the definite integral you must compute the new integration bounds $G(0)$ and $G(1)$ (see equation (10.3).) If x runs between $x = 0$ and $x = 1$, then $u = G(x) = 1 + x^2$ runs between $u = 1 + 0^2 = 1$ and $u = 1 + 1^2 = 2$, so the definite integral we must compute is

$$\int_0^1 \frac{x}{1+x^2}\,dx = \tfrac{1}{2}\int_1^2 \frac{1}{u}\,du, \tag{8.14}$$

which is in our list of memorable integrals. So we find

$$\int_0^1 \frac{x}{1+x^2}\,dx = \tfrac{1}{2}\int_1^2 \frac{1}{u}\,du = \tfrac{1}{2}\big[\ln u\big]_1^2 = \tfrac{1}{2}\ln 2.$$

Sometimes the integrals in (8.14) are written as

$$\int_{x=0}^1 \frac{x}{1+x^2}\,dx = \tfrac{1}{2}\int_{u=1}^2 \frac{1}{u}\,du,$$

to emphasize (and remind yourself) to which variable the bounds in the integral refer.

8.8 PROBLEMS

RIEMANN SUMS

355. What is a Riemann sum of a function $y = f(x)$?

356. Let f be the function $f(x) = 1 - x^2$.

Draw the graph of $f(x)$ with $0 \le x \le 2$.

Compute the Riemann-sum for the partition

$$0 < \tfrac{1}{3} < 1 < \tfrac{3}{2} < 2$$

of the interval $[a, b] = [0, 2]$ if you choose each c_k to be the left endpoint of the interval it belongs to. Draw the corresponding rectangles (add them to your drawing of the graph of f).

Then compute the Riemann-sum you get if you choose the c_k to be the right endpoint of the interval it belongs to. Make a new drawing of the graph of f and include the rectangles corresponding to the right endpoint Riemann-sum. †386

357. Look at figure 8.2 (top). Which choice of intermediate points $c_1, \ldots, c_6$ leads to the smallest Riemann sum? Which choice would give you the largest Riemann-sum?

(Note: in this problem you're not allowed to change the division points x_i, only the points c_i in between them.) †386

ANTIDERIVATIVES

Find an antiderivative $F(x)$ for each of the following functions $f(x)$. Finding antiderivatives involves a fair amount of guess work, but with experience it gets easier to guess antiderivatives.

358. $f(x) = 2x + 1$

359. $f(x) = 1 - 3x$

360. $f(x) = x^2 - x + 11$

361. $f(x) = x^4 - x^2$

362. $f(x) = 1 + x + \dfrac{x^2}{2} + \dfrac{x^3}{3} + \dfrac{x^4}{4}$

363. $f(x) = \dfrac{1}{x}$

364. $f(x) = e^x$

365. $f(x) = \dfrac{2}{x}$

366. $f(x) = e^{2x}$

367. $f(x) = \dfrac{1}{2 + x}$

368. $f(x) = \dfrac{e^x - e^{-x}}{2}$

369. $f(x) = \dfrac{1}{1 + x^2}$

370. $f(x) = \dfrac{e^x + e^{-x}}{2}$

371. $f(x) = \dfrac{1}{\sqrt{1 - x^2}}$

372. $f(x) = \sin x$

373. $f(x) = \dfrac{2}{1 - x}$

374. $f(x) = \cos x$

375. $f(x) = \cos 2x$

376. $f(x) = \sin(x - \pi/3)$

377. $f(x) = \sin x + \sin 2x$

378. $f(x) = 2x(1 + x^2)^5$

INTEGRATING BETWEEN THE LINES

In each of the following exercises you should compute the area of the indicated region, and also of the smallest enclosing rectangle with horizontal and vertical sides.

Before computing anything draw the region.

379. The region between the vertical lines $x = 0$ and $x = 1$, and between the x-axis and the graph of $y = x^3$.

380. The region between the vertical lines $x = 0$ and $x = 1$, and between the x-axis and the graph of $y = x^n$ (here $n > 0$, draw for $n = \frac{1}{2}, 1, 2, 3, 4$).

381. The region above the graph of $y = \sqrt{x}$, below the line $y = 2$, and between the vertical lines $x = 0$, $x = 4$.

382. The region above the x-axis and below the graph of $f(x) = x^2 - x^3$.

383. The region above the x-axis and below the graph of $f(x) = 4x^2 - x^4$.

384. The region above the x-axis and below the graph of $f(x) = 1 - x^4$.

385. The region above the x-axis, below the graph of $f(x) = \sin x$, and between $x = 0$ and $x = \pi$.

386. The region above the x-axis, below the graph of $f(x) = 1/(1 + x^2)$ (a curve known as *Maria Agnesi's witch*), and between $x = 0$ and $x = 1$.

387. The region between the graph of $y = 1/x$ and the x-axis, and between $x = a$ and $x = b$ (here $0 < a < b$ are constants, e.g. choose $a = 1$ and $b = \sqrt{2}$ if you have something against either letter a or b.)

388. The region above the x-axis and below the graph of

$$f(x) = \frac{1}{1+x} + \frac{x}{2} - 1.$$

389. Compute

$$\int_0^1 \sqrt{1 - x^2}dx$$

without finding an antiderivative for $\sqrt{1 - x^2}$ (you can find such an antiderivative, but it's not easy. This integral is the area of some region: which region is it, and what is that area?)

390. Compute these integrals without finding antiderivatives.

$$I = \int_0^{1/2} \sqrt{1 - x^2}\, dx$$

$$J = \int_{-1}^{1} |1 - x|\, dx$$

$$K = \int_{-1}^{1} |2 - x|\, dx$$

DIFFERENTIATING INTEGRALS:

391. $\dfrac{d}{dx}\displaystyle\int_0^x \left(1+t^2\right)^4\,dt$

392. $\dfrac{d}{dx}\displaystyle\int_x^1 \ln z\,dz$

393. $\dfrac{d}{dt}\displaystyle\int_0^t \frac{dx}{1+x^2}$

394. $\dfrac{d}{dt}\displaystyle\int_0^{1/t} \frac{dx}{1+x^2}$

395. $\dfrac{d}{dx}\displaystyle\int_x^{2x} s^2 ds$

396. $\dfrac{d}{dq}\displaystyle\int_{-q}^{q} \frac{dx}{1-x^2}$ [Which values of q are allowed here?]

397. $\dfrac{d}{dt}\displaystyle\int_0^{t^2} e^{2x} dx$

398. You can see the graph of the error function at

http://en.wikipedia.org/wiki/Error_function

(a) Compute the second derivative of the error function. How many inflection points does the graph of the error function have?

(b) The graph of the error function on Wikipedia shows that $\operatorname{erf}(x)$ is negative when $x < 0$. But the error function is defined as an integral of a positive function so it should be positive. Is Wikipedia wrong? Explain. †386

INDEFINITE INTEGRALS:

399. $\displaystyle\int \left\{6x^5 - 2x^{-4} - 7x\right\} dx$

400. $\displaystyle\int \left\{+3/x - 5 + 4e^x + 7^x\right\} dx$

401. $\displaystyle\int (x/a + a/x + x^a + a^x + ax)\, dx$

402. $\displaystyle\int \left\{\sqrt{x} - \sqrt[3]{x^4} + \frac{7}{\sqrt[3]{x^2}} - 6e^x + 1\right\} dx$

403. $\displaystyle\int \left\{2^x + \left(\tfrac{1}{2}\right)^x\right\} dx$

DEFINITE INTEGRALS

404. $\displaystyle\int_{-2}^{4} (3x-5)\, dx$

405. $\displaystyle\int_1^4 x^{-2}\, dx$ (hm...)

406. $\displaystyle\int_1^4 t^{-2}\, dt$ **(!)**

407. $\displaystyle\int_1^4 x^{-2}\, dt$ **(!!!)**

408. $\displaystyle\int_0^1 (1-2x-3x^2)\, dx$

409. $\displaystyle\int_1^2 (5x^2-4x+3)\, dx$

410. $\displaystyle\int_{-3}^{0} (5y^4-6y^2+14)\, dy$

411. $\displaystyle\int_0^1 (y^9-2y^5+3y)\, dy$

412. $\displaystyle\int_0^4 \sqrt{x}\, dx$

413. $\displaystyle\int_0^1 x^{3/7}\, dx$

414. $\int_1^3 \left(\frac{1}{t^2} - \frac{1}{t^4}\right) dt$

415. $\int_1^2 \frac{t^6 - t^2}{t^4}\, dt$

416. $\int_1^2 \frac{x^2 + 1}{\sqrt{x}}\, dx$

417. $\int_0^2 (x^3 - 1)^2\, dx$

418. $\int_0^1 u(\sqrt{u} + \sqrt[3]{u})\, du$

419. $\int_1^2 (x + 1/x)^2\, dx$

420. $\int_3^3 \sqrt{x^5 + 2}\, dx$

421. $\int_1^{-1} (x - 1)(3x + 2)\, dx$

422. $\int_1^4 (\sqrt{t} - 2/\sqrt{t})\, dt$

423. $\int_1^8 \left(\sqrt[3]{r} + \frac{1}{\sqrt[3]{r}}\right) dr$

424. $\int_{-1}^0 (x + 1)^3\, dx$

425. $\int_{-5}^{-2} \frac{x^4 - 1}{x^2 + 1}\, dx$

426. $\int_1^e \frac{x^2 + x + 1}{x}\, dx$

427. $\int_4^9 \left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)^2 dx$

428. $\int_0^1 \left(\sqrt[4]{x^5} + \sqrt[5]{x^4}\right) dx$

429. $\int_1^8 \frac{x - 1}{\sqrt[3]{x^2}}\, dx$

430. $\int_{\pi/4}^{\pi/3} \sin t\, dt$

431. $\int_0^{\pi/2} (\cos\theta + 2\sin\theta)\, d\theta$

432. $\int_0^{\pi/2} (\cos\theta + \sin 2\theta)\, d\theta$

433. $\int_{2\pi/3}^{\pi} \frac{\tan x}{\cos x}\, dx$

434. $\int_{\pi/3}^{\pi/2} \frac{\cot x}{\sin x}\, dx$

435. $\int_1^{\sqrt{3}} \frac{6}{1 + x^2}\, dx$

436. $\int_0^{0.5} \frac{dx}{\sqrt{1 - x^2}}$

437. $\int_4^8 (1/x)\, dx$

438. $\int_{\ln 3}^{\ln 6} 8e^x\, dx$

439. $\int_8^9 2^t\, dt$

440. $\int_{-e^2}^{-e} \frac{3}{x}\, dx$

441. $\int_{-2}^3 |x^2 - 1|\, dx$

442. $\int_{-1}^2 |x - x^2|\, dx$

443. $\int_{-1}^2 (x - 2|x|)\, dx$

444. $\int_0^2 (x^2 - |x - 1|)\, dx$

445. $\int_0^2 f(x)\, dx$ where $f(x) = \begin{cases} x^4 & \text{if } 0 \le x < 1, \\ x^5, & \text{if } 1 \le x \le 2. \end{cases}$

446. $\int_{-\pi}^{\pi} f(x)\, dx$ where

$$f(x) = \begin{cases} x, & \text{if } -\pi \le x \le 0, \\ \sin x, & \text{if } 0 < x \le \pi. \end{cases}$$

447. Compute

$$I = \int_0^2 2x\left(1+x^2\right)^3 dx$$

in two different ways:
(a) Expand $(1+x^2)^3$, multiply with $2x$, and integrate each term.
(b) Use the substitution $u = 1 + x^2$.

448. Compute

$$I_n = \int 2x\left(1+x^2\right)^n dx.$$

449. If $f'(x) = x - 1/x^2$ and $f(1) = 1/2$ find $f(x)$.

450. Sketch the graph of the curve $y = \sqrt{x+1}$ and determine the area of the region enclosed by the curve, the x-axis and the lines $x = 0$, $x = 4$.

451. Find the area under the curve $y = \sqrt{6x+4}$ and above the x-axis between $x = 0$ and $x = 2$. Draw a sketch of the curve.

452. Graph the curve $y = 2\sqrt{1-x^2}$, $x \in [0, 1]$, and find the area enclosed between the curve and the x-axis. (Don't evaluate the integral, but compare with the area under the graph of $y = \sqrt{1-x^2}$.)

453. Determine the area under the curve $y = \sqrt{a^2 - x^2}$ and between the lines $x = 0$ and $x = a$.

454. Graph the curve $y = 2\sqrt{9-x^2}$ and determine the area enclosed between the curve and the x-axis.

455. Graph the area between the curve $y^2 = 4x$ and the line $x = 3$. Find the area of this region.

456. Find the area bounded by the curve $y = 4 - x^2$ and the lines $y = 0$ and $y = 3$.

457. Find the area enclosed between the curve $y = \sin 2x$, $0 \le x \le \pi/4$ and the axes.

458. Find the area enclosed between the curve $y = \cos 2x$, $0 \le x \le \pi/4$ and the axes.

459. Graph $y^2 + 1 = x$, and find the area enclosed by the curve and the line $x = 2$.

460. Find the area of the region bounded by the parabola $y^2 = 4x$ and the line $y = 2x$.

461. Find the area bounded by the curve $y = x(2 - x)$ and the line $x = 2y$.

462. Find the area bounded by the curve $x^2 = 4y$ and the line $x = 4y - 2$.

463. Calculate the area of the region bounded by the parabolas $y = x^2$ and $x = y^2$.

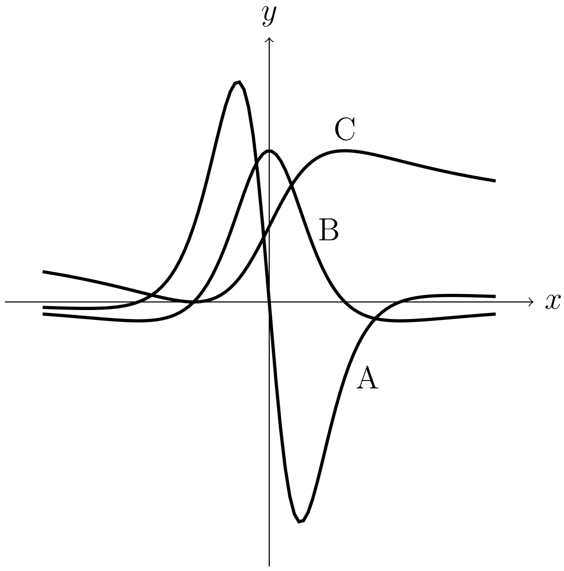
464. Find the area of the region included between the parabola $y^2 = x$ and the line $x + y = 2$.

465. Find the area of the region bounded by the curves $y = \sqrt{x}$ and $y = x$.

466.

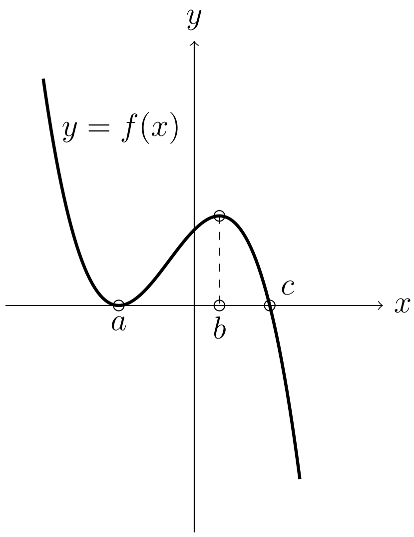
Emily asks her assistant, Kate, to produce graphs of a function $f(x)$, its derivative $f'(x)$ and an antiderivative $F(x)$ of $f(x)$.

Unfortunately Kate simply labelled the graphs "A," "B," and "C," and now she doesn't remember which graph is f, which is f' and which is F. In the diagram below, identify which graph is which ***and explain your answer.***



467.

Below is the graph of a function $y = f(x)$.



The function $F(x)$ (graph not shown) is an antiderivative of $f(x)$. Which among the following statements true?

(a) $F(a) = F(c)$
(b) $F(b) = 0$
(c) $F(b) > F(c)$
(d) The graph of $y = F(x)$ has ***two*** inflection points?

INTEGRATION BY SUBSTITUTION

468. $\displaystyle\int_1^2 \frac{u\,du}{1+u^2}$

469. $\displaystyle\int_0^5 \frac{x\,dx}{\sqrt{x+1}}$

470. $\displaystyle\int_1^2 \frac{x^2\,dx}{\sqrt{2x+1}}$

471. $\displaystyle\int_0^5 \frac{s\,ds}{\sqrt[3]{s+2}}$

472. $\displaystyle\int_1^2 \frac{x\,dx}{1+x^2}$

473. $\displaystyle\int_0^\pi \cos(\theta+\tfrac{\pi}{3})d\theta$

474. $\displaystyle\int \sin\frac{\pi+x}{5}dx$

475. $\displaystyle\int \frac{\sin 2x}{\sqrt{1+\cos 2x}}\,dx$

476. $\displaystyle\int_{\pi/4}^{\pi/3} \sin^2\theta\cos\theta\,d\theta$

477. $\displaystyle\int_2^3 \frac{1}{r\ln r}, dr$

478. $\displaystyle\int \frac{\sin 2x}{1+\cos^2 x}\,dx$

479. $\displaystyle\int \frac{\sin 2x}{1+\sin x}\,dx$

480. $\displaystyle\int_0^1 z\sqrt{1-z^2}\,dz$

481. $\displaystyle\int_1^2 \frac{\ln 2x}{x}\,dx$

482. $\displaystyle\int_{\xi=0}^{\sqrt{2}} \xi(1+2\xi^2)^{10}\,d\xi$

483. $\displaystyle\int_2^3 \sin\rho\big(\cos 2\rho)^4\,d\rho$

484. $\displaystyle\int \alpha e^{-\alpha^2}\,d\alpha$

485. $\displaystyle\int \frac{e^{\frac{1}{t}}}{t^2}\,dt$

Chapter 9

Applications of the integral

The integral appears as the answer to many different questions. In this chapter we will describe a number of "things which are an integral." In each example there is a quantity we want to compute, and which we can approximate through Riemann-sums. After letting the partition become arbitrarily fine we then find that the quantity we are looking for is given by an integral. The derivations are an important part of the subject.

9.1 Areas between graphs

Suppose you have two functions f and g on an interval $[a, b]$, one of which is always larger than the other, i.e. for which you know that $f(x) \le g(x)$ for all x in the interval $[a, b]$. Then the area of the region between the graphs of the two functions is

$$\text{Area} = \int_a^b \big(g(x) - f(x)\big)dx. \tag{9.1}$$

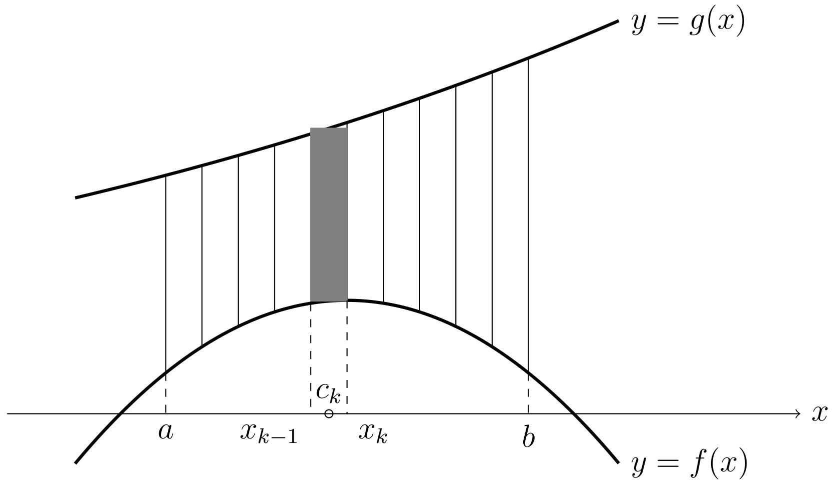


Figure 9.1: Finding the area between two graphs using Riemann-sums.

To get this formula you approximate the region by a large number of thin rectangles. Choose a partition $a = x_0 < x_1 < \cdots < x_n = b$ of the interval $[a, b]$; choose a number c_k in each interval $[x_{k-1}, x_k]$; form the rectangles

$$x_{k-1} \le x \le x_k, \qquad f(c_k) \le y \le g(c_k).$$

The area of this rectangle is

$$\text{width} \times \text{height} = \Delta x_k \times \big(g(c_k) - f(c_k)\big).$$

Hence the combined area of the rectangles is

$$R = \big(g(c_1) - f(c_1)\big)\Delta x_1 + \cdots + \big(g(c_n) - f(c_n)\big)\Delta x_n$$

which is just the Riemann-sum for the integral

$$I = \int_a^b (g(x) - f(x))dx.$$

So,

1. since the area of the region between the graphs of f and g is the limit of the combined areas of the rectangles,
2. and since this combined area is equal to the Riemann sum R,
3. and since the Riemann-sums R converge to the integral I,

we conclude that the area between the graphs of f and g is exactly the integral I.

9.2 Cavalieri's principle and volumes of solids

You can use integration to derive the formulas for volumes of spheres, cylinder, cones, and many many more solid objects in a systematic way. In this section we'll see the "method of slicing."

9.2.1 Example – Volume of a pyramid.

As an example let's compute the volume of a pyramid whose base is a square of side 1, and whose height is 1. Our strategy will be to divide the pyramid into thin horizontal slices whose volumes we can compute, and to add the volumes of the slices to get the volume of the pyramid.

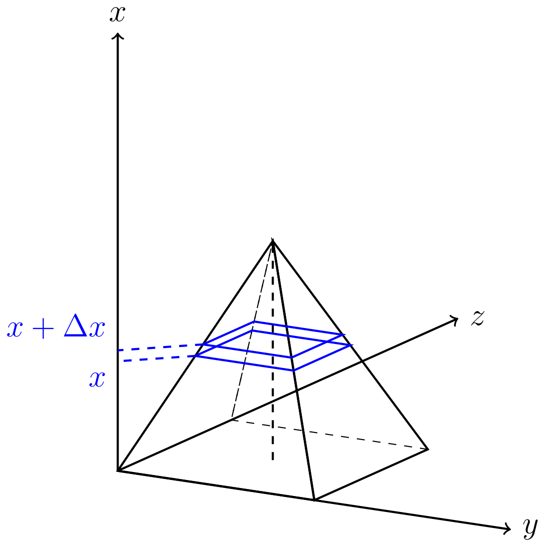


Figure 9.2: The slice at height x is a square with side $1-x$.

To construct the slices we choose a partition of the (height) interval $[0,1]$ into N subintervals, i.e. we pick numbers

$$0 = x_0 < x_1 < x_2 < \cdots < x_{N-1} < x_N = 1,$$

and as usual we set $\Delta x_k = x_k - x_{k-1}$, we define the mesh size of the partition to be the largest of the Δx_k.

The k^{th} slice consists of those points on the pyramid whose height is between x_{k-1} and x_k. The intersection of the pyramid with the plane at height x is a square, and by similarity the length of the side of this square is $1-x$. Therefore the bottom of the k^{th} slice is a square with side $1-x_{k-1}$, and its top is a square with side $1-x_k$. The height of the slice is $x_k - x_{k-1} = \Delta x_k$.

Thus the k^{th} slice *contains* a block of height Δx_k whose base is a square with sides $1-x_k$, and its volume must therefore be larger than $(1-x_k)^2\Delta x_k$. On the other hand the k^{th} slice *is contained in* a block of the same height whose base is a square with sides $1-x_{k-1}$. The volume of the slice is therefore not more than $(1-x_{k-1})^2\Delta x_k$. So we have

$$(1-x_k)^2\Delta x_k < \text{volume of } k^{\text{th}} \text{ slice} < (1-x_{k-1})^2\Delta x_k.$$

Therefore there is some c_k in the interval $[x_{k-1}, x_k]$ such that

$$\text{volume of } k^{\text{th}} \text{ slice} = (1-c_k)^2\Delta x_k.$$

Adding the volumes of the slices we find that the volume V of the pyramid is given by

$$V = (1-c_1)^2\Delta x_1 + \cdots + (1-c_N)^2\Delta x_N.$$

The right hand side in this equation is a Riemann sum for the integral

$$I = \int_0^1 (1-x)^2 dx$$

and therefore we have

$$I = \lim_{\dots} \left\{ (1-c_1)^2 \Delta x_1 + \cdots + (1-c_N)^2 \Delta x_N \right\} = V.$$

Compute the integral and you find that the volume of the pyramid is

$$V = \frac{1}{3}.$$

9.2.2 General case.

The "method of slicing" which we just used to compute the volume of a pyramid works for solids of any shape. The strategy always consists of dividing the solid into many thin (horizontal) slices, compute their volumes, and recognize that the total volume of the slices is a Riemann sum for some integral. That integral then is the volume of the solid.

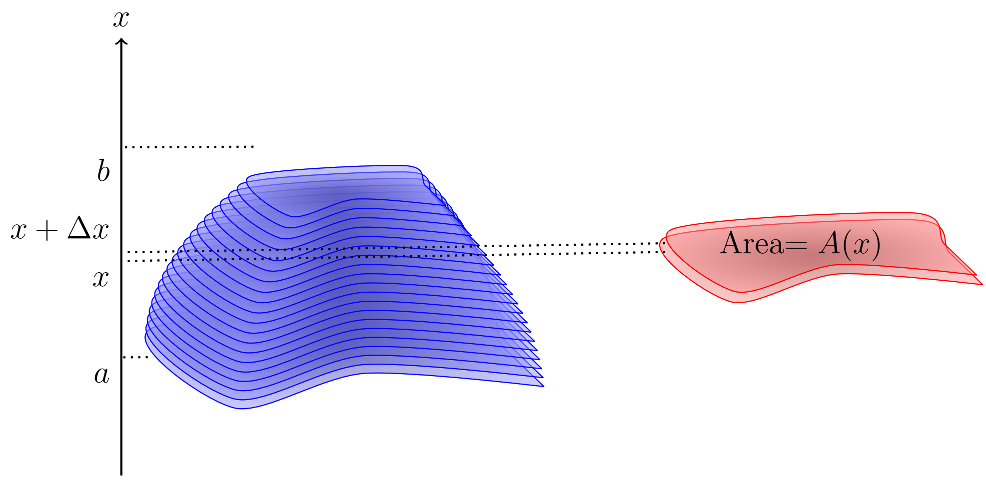


Figure 9.3: Slicing a solid to compute its volume. The volume of one slice is approximately the product of its thickness (Δx) and the area $A(x)$ of its top. Summing the volume $A(x)\Delta x$ over all slices leads approximately to the integral $\int_a^b A(x)dx$.

To be more precise, let a and b be the heights of the lowest and highest points on the solid, and let $a = x_0 < x_1 < x_2 < \ldots < x_{N-1} < x_N = b$ be a partition of the interval $[a, b]$. Such a partition divides the solid into N distinct slices, where slice number k consists of all points in the solid whose height is between x_{k-1} and x_k. The thickness of the k^{th} slice is $\Delta x_k = x_k - x_{k-1}$. If

$$A(x) = \text{area of the intersection of the solid with the plane at height } x.$$

then we can approximate the volume of the k^{th} slice by

$$A(c_k)\Delta x_k$$

where c_k is any number (height) between x_{k-1} and x_k.
The total volume of all slices is therefore approximately

$$V \approx A(c_1)\Delta x_1 + \cdots + A(c_N)\Delta x_N.$$

While this formula only holds approximately, we expect the approximation to get better as we make the partition finer, and thus

$$V = \lim_{\cdots} \bigl\{A(c_1)\Delta x_1 + \cdots + A(c_N)\Delta x_N\bigr\}. \tag{9.2}$$

On the other hand the sum on the right is a Riemann sum for the integral $I = \int_a^b A(x)dx$, so the limit is exactly this integral. Therefore we have

$$V = \int_a^b A(x)dx. \tag{9.3}$$

9.2.3 Cavalieri's principle.

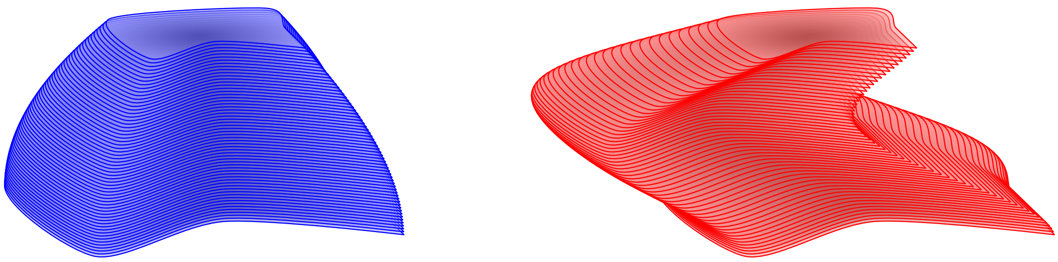


Figure 9.4: Cavalieri's principle. Both solids consist of a pile of horizontal slices. The solid on the right was obtained from the solid on the left by sliding some of the slices to the left and others to the right. This operation does not affect the volumes of the slices, and hence both solids have the same volume.

The formula (9.3) for the volume of a solid which we have just derived shows that the volume only depends on the areas $A(x)$ of the cross sections of the solid, and not on the particular shape these cross sections may have. This observation is older than calculus itself and goes back at least to Bonaventura Cavalieri (1598 – 1647) who said: *If the intersections of two solids with a horizontal plane always have the same area, no matter what the height of the horizontal plane may be, then the two solids have the same volume.*

This principle is often illustrated by considering a stack of coins: If you put a number of coins on top of each other then the total volume of the coins is just the sum of the volumes of the coins. If you change the shape of the pile by sliding the coins horizontally then the volume of the pile will still be the sum of the volumes of the coins, i.e. it doesn't change.

To see some examples of volume by slicing consider watching YouTube by Houstan Math Prep .

9.2.4 Solids of revolution.

In principle, formula (9.3) allows you to compute the volume of any solid, provided you can compute the areas $A(x)$ of all cross sections. One class of solids for which the areas of the cross sections are easy are the so-called "solids of revolution."

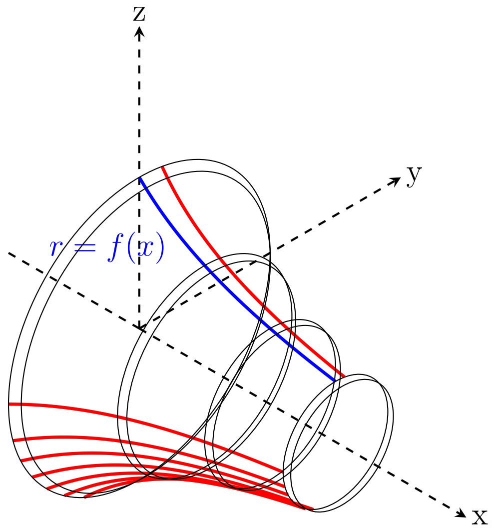


Figure 9.5: A solid of revolution consists of all points in three-dimensional space whose distance r to the x-axis satisfies $r \le f(x)$.

A solid of revolution is created by rotating (revolving) the graph of a positive function around the x-axis. More precisely, let f be a function which is defined on an interval $[a, b]$ and which is always positive ($f(x) > 0$ for all x). If you now imagine the x-axis floating in three dimensional space, then the solid of revolution obtained by rotating the graph of f around the x-axis consists of all points in three-dimensional space with $a \le x \le b$, and whose distance to the x-axis is no more than $f(x)$.

Yet another way of describing the solid of revolution is to say that the solid is the union of all discs which meet the x-axis perpendicularly and whose radius is given by $r = f(x)$.

If we slice the solid with planes perpendicular to the x-axis, then (9.3) tells us the volume of the solid. Each slice is a disc of radius $r = f(x)$ so that its area is $A(x) = \pi r^2 = \pi f(x)^2$. We therefore find that

$$V = \pi \int_a^b f(x)^2 dx. \tag{9.4}$$

9.3 Examples of volumes of solids of revolution

9.3.1 Problem 1: Revolve $\mathcal{R}$ around the y-axis .

Consider the solid obtained by revolving the region

$$\mathcal{R} = \{(x, y) \mid 0 \le x \le 2, \quad (x-1)^2 \le y \le 1\}$$

around the y-axis.

Solution: The region we have to revolve around the y-axis consists of all points above the parabola $y = (x-1)^2$ but below the line $y = 1$.

If we intersect the solid with a plane at height y then we get a ring shaped region, or "annulus", i.e. a large disc with a smaller disc removed. You can see it in the figure below: if you cut the region $\mathcal{R}$ horizontally at height y you get the line segment AB, and if you rotate this segment around the y-axis you get the grey ring region pictured below the graph. Call the radius of the outer circle r_{out} and the radius of the inner circle r_{in}. These radii are the two solutions of

$$y = (1 - r)^2$$

so they are

$$r_{\text{in}} = 1 - \sqrt{y}, \qquad r_{\text{out}} = 1 + \sqrt{y}.$$

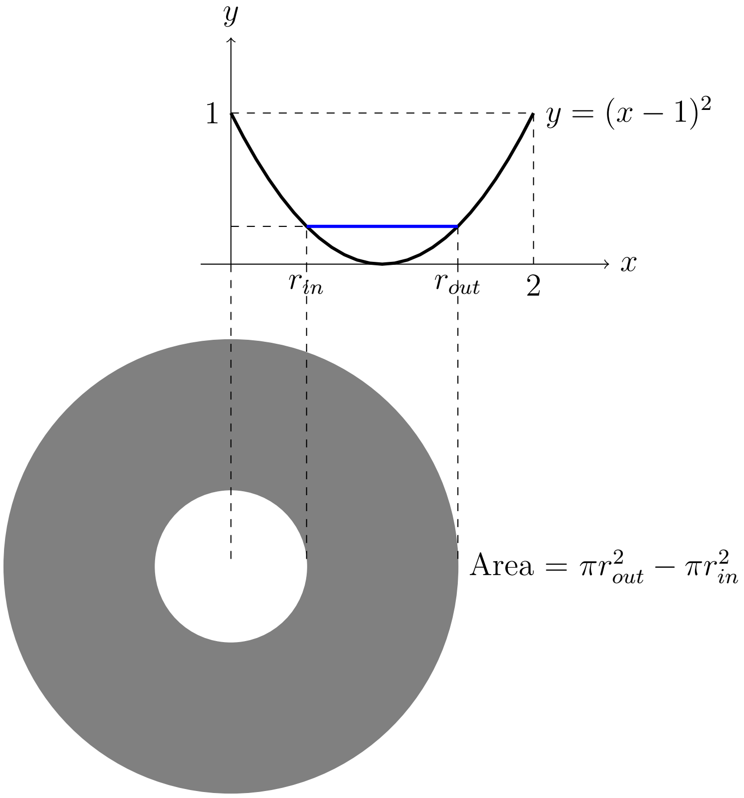


Figure 9.6: Computing the volume of the solid you get when you revolve the region $\mathcal{R}$ around the y-axis. A horizontal cross section of the solid is a "washer" with inner radius r_{in}, and outer radius r_{out}.

The area of the cross section is therefore given by

$$A(y) = \pi r_{\text{out}}^2 - \pi r_{\text{in}}^2 = \pi\big(1 + \sqrt{y}\big)^2 - \pi\big(1 - \sqrt{y}\big)^2 = 4\pi\sqrt{y}.$$

The y-values which occur in the solid are $0 \le y \le 1$ and hence the volume of the solid is given by

$$V = \int_0^1 A(y)dy = 4\pi \int_0^1 \sqrt{y}\, dy = 4\pi \times \tfrac{2}{3} = \frac{8\pi}{3}.$$

9.3.2 Problem 2: Revolve $\mathcal{R}$ around the line $x = -1$.

Find the volume of the solid of revolution obtained by revolving the same region $\mathcal{R}$ around the line $x = -1$.

Solution: The line $x = -1$ is vertical, so we slice the solid with horizontal planes. The height of each plane will be called y.

As before the slices are ring shaped regions but the inner and outer radii are now given by

$$r_{\text{in}} = 1 + x_{\text{in}} = 2 - \sqrt{y}, \qquad r_{\text{out}} = 1 + x_{\text{out}} = 2 + \sqrt{y}.$$

The volume is therefore given by

$$V = \int_0^1 \bigl(\pi r_{\text{out}}^2 - \pi r_{\text{in}}^2\bigr) dy = \pi \int_0^1 8\sqrt{y}\, dy = \frac{16\pi}{3}.$$

9.3.3 Problem 3: Revolve $\mathcal{R}$ around the line $y = 2$.

Compute the volume of the solid you get when you revolve the same region $\mathcal{R}$ around the line $y = 2$.

Solution: This time the line around which we rotate $\mathcal{R}$ is horizontal, so we slice the solid with planes perpendicular to the x-axis.

A typical slice is obtained by revolving the line segment AB about the line $y = 2$. The result is again an annulus, and from the figure we see that the inner and outer radii of the annulus are

$$r_{\text{in}} = 1, \qquad r_{\text{out}} = 2 - (1 - x)^2.$$

The area of the slice is therefore

$$A(x) = \pi\bigl\{2 - (1 - x)^2\bigr\}^2 - \pi 1^2 = \pi\left\{3 - 4(1 - x)^2 + (1 - x)^4\right\}.$$

The x values which occur in the solid are $0 \le x \le 2$, and so its volume is

$$\begin{aligned} V &= \pi \int_0^2 \left\{3 - 4(1 - x)^2 + (1 - x)^4\right\} dx \\ &= \pi \left[3x + \tfrac{4}{3}(1 - x)^3 - \tfrac{1}{5}(1 - x)^5\right]_0^2 \\ &= \tfrac{56}{15}\pi \end{aligned}$$

9.4 Volumes by cylindrical shells

Instead of slicing a solid with planes you can also try to decompose it into cylindrical shells.

The volume of a cylinder of height h and radius r is $\pi r^2 h$ (height times area base). Therefore the volume of a cylindrical shell of height h, (inner) radius r and thickness Δr is

$$\begin{aligned} \pi h(r + \Delta r)^2 - \pi h r^2 &= \pi h(2r + \Delta r)\Delta r \\ &\approx 2\pi h r \Delta r. \end{aligned}$$

Now consider the solid you get by revolving the region

$$\mathcal{R} = \{(x, y) \mid a \le x \le b, 0 \le y \le f(x)\}$$

around the y-axis. By partitioning the interval $a \le x \le b$ into many small intervals we can decompose the solid into many thin shells. The volume of each shell will approximately be given by $2\pi x f(x)\Delta x$. Adding the volumes of the shells, and taking the limit over finer and finer partitions we arrive at the following formula for the volume of the solid of revolution:

$$V = 2\pi \int_a^b x f(x)\, dx. \tag{9.5}$$

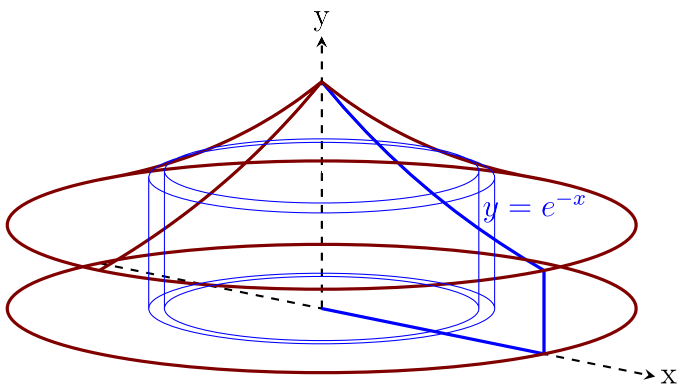


Figure 9.7: Computing the volume of a circus tent using cylindrical shells. This particular tent is obtained by rotating the graph of $y = e^{-x}$, $0 \le x \le 1$ around the y-axis.

If the region $\mathcal{R}$ is not the region under the graph, but rather the region between the graphs of two functions $f(x) \le g(x)$, then we get

$$V = 2\pi \int_a^b x\bigl\{g(x) - f(x)\bigr\}\, dx.$$

9.4.1 Example – The solid obtained by rotating $\mathcal{R}$ about the y-axis, again.

The region $\mathcal{R}$ from §9.3.1 can also be described as

$$\mathcal{R} = \bigl\{(x, y) \mid 0 \le x \le 2, f(x) \le y \le g(x)\bigr\},$$

where

$$f(x) = (x-1)^2 \text{ and } g(x) = 1.$$

The volume of the solid which we already computed in §9.3.1 is thus given by

$$\begin{aligned} V &= 2\pi \int_0^1 x\bigl\{1 - (x-1)^2\bigr\}\, dx \\ &= 2\pi \int_0^2 \bigl\{-x^3 + 2x^2\bigr\}\, dx \\ &= 2\pi\bigl[-\tfrac{1}{4}x^4 + \tfrac{2}{3}x^3\bigr]_0^2 \\ &= 8\pi/3, \end{aligned}$$

which coincides with the answer we found in §9.3.1.

9.5 Distance from velocity, velocity from acceleration

9.5.1 Motion along a line.

If an object is moving on a straight line, and if its position at time t is $x(t)$, then we had defined the velocity to be $v(t) = x'(t)$. Therefore the position is an antiderivative of the velocity, and the fundamental theorem of calculus says that

$$\int_{t_a}^{t_b} v(t)\,dt = x(t_b) - x(t_a), \tag{9.6}$$

or

$$x(t_b) = x(t_a) + \int_{t_a}^{t_b} v(t)\,dt.$$

In words, the integral of the velocity gives you the distance travelled of the object (during the interval of integration).

Equation (9.6) can also be obtained using Riemann sums. Namely, to see how far the object moved between times t_a and t_b we choose a partition $t_a = t_0 < t_1 < \cdots < t_N = t_b$. Let Δs_k be the distance travelled during the time interval (t_{k-1}, t_k). The length of this time interval is $\Delta t_k = t_k - t_{k-1}$. During this time interval the velocity $v(t)$ need not be constant, but if the time interval is short enough then we can estimate the velocity by $v(c_k)$ where c_k is some number between t_{k-1} and t_k. We then have

$$\Delta s_k = v(c_k)\Delta t_k$$

and hence the total distance travelled is the sum of the travel distances for all time intervals $t_{k-1} < t < t_k$, i.e.

$$\text{Distance travelled} \approx \Delta s_1 + \cdots + \Delta s_N = v(c_1)\Delta t_1 + \cdots + v(c_N)\Delta t_N.$$

The right hand side is again a Riemann sum for the integral in (9.6). As one makes the partition finer and finer you therefore get

$$\text{Distance travelled} = \int_{t_a}^{t_b} v(t)\,dt.$$

The return of the dummy. Often you want to write a formula for $x(t) = \cdots$ rather than $x(t_b) = \cdots$ as we did in (9.6), i.e. you want to say what the position is at time t, instead of at time t_a. For instance, you might want to express the fact that the position $x(t)$ is equal to the initial position $x(0)$ plus the integral of the velocity from 0 to t. To do this you cannot write

$$x(t) = x(0) + \int_0^t v(t)\,dt \quad \Leftarrow \quad \textbf{BAD FORMULA}$$

because the variable t gets used in two incompatible ways: the t in $x(t)$ on the left, and in the upper bound on the integral ($\int^t$) are the same, but they are not the same as the two

t's in $v(t)dt$. The latter is a dummy variable (see §3.7 and §8.3.1). To fix this formula we should choose a different letter or symbol for the integration variable. A common choice in this situation is to decorate the integration variable with a prime (t'), a tilde ($\tilde{t}$) or a bar ($\bar{t}$). So you can write

$$x(t) = x(0) + \int_0^t v(\bar{t})\, d\bar{t}$$

9.5.2 Velocity from acceleration.

The acceleration of the object is by definition the rate of change of its velocity,

$$a(t) = v'(t),$$

so you have

$$v(t) = v(0) + \int_0^t a(\bar{t}) d\bar{t}.$$

Conclusion: *If you know the acceleration $a(t)$ at all times t, and also the velocity $v(0)$ at time $t = 0$, then you can compute the velocity $v(t)$ at all times by integrating.*

9.5.3 Free fall in a constant gravitational field.

If you drop an object then it will fall, and as it falls its velocity increases. The object's motion is described by the fact that *its acceleration is constant.* This constant is called g and is about $9.8\text{m/sec}^2 \approx 32\text{ft/sec}^2$. If we dsignate the upward direction as positive then $v(t)$ is the upward velocity of the object, and this velocity is actually decreasing. Therefore the constant acceleration is negative: it is $-g$.

If you write $h(t)$ for the height of the object at time t then its velocity is $v(t) = h'(t)$, and its acceleration is $h''(t)$. Since the acceleration is constant you have the following formula for the velocity at time t:

$$v(t) = v(0) + \int_0^t (-g)\, d\bar{t} = v(0) - gt.$$

Here $v(0)$ is the velocity at time $t = 0$ (the "initial velocity").

To get the height of the object at any time t you must integrate the velocity:

$$\begin{aligned} h(t) &= h(0) + \int_0^t v(\bar{t})\, d\bar{t} && \text{(Note the use of the dummy } \bar{t}\text{)} \\ &= h(0) + \int_0^t \left\{v(0) - g\bar{t}\right\} d\bar{t} && \text{(use } v(\bar{t}) = v(0) - g\bar{t}\text{)} \\ &= h(0) + \left[v(0)\bar{t} - \tfrac{1}{2}g\bar{t}^2\right]_{\bar{t}=0}^{\bar{t}=t} \\ &= h(0) + v(0)t - \tfrac{1}{2}gt^2. \end{aligned}$$

For instance, if you launch the object upwards with velocity 5ft/sec from a height of 10ft, then you have

$$h(0) = 10\text{ft}, \quad v(0) = +5\text{ft/sec},$$

and thus

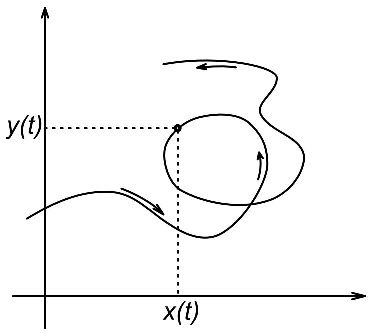
$$h(t) = 10 + 5t - 32t^2/2 = 10 + 5t - 16t^2.$$

The object reaches its maximum height when $h(t)$ has a maximum, which is when $h'(t) = 0$. To find that height you compute $h'(t) = 5 - 32t$ and conclude that $h(t)$ is maximal at $t = \frac{5}{32}$sec. The maximal height is then

$$h_{\max} = h(\tfrac{5}{32}) = 10 + \tfrac{25}{32} - \tfrac{25}{64} = 10\tfrac{25}{64}\text{ft}.$$

9.5.4 Motion in the plane – parametric curves.

To describe the motion of an object in the plane you could keep track of its x and y coordinates at all times t. This would give you *two* functions of t, namely, $x(t)$ and $y(t)$, both of which are defined on the same interval $t_0 \le t \le t_1$ which describes the duration of the motion you are describing. In this context a pair of functions $(x(t), y(t))$ is called ***a parametric curve.***



As an example, consider the motion described by

$$x(t) = \cos t, \quad y(t) = \sin t (0 \le t \le 2\pi).$$

In this motion the point $(x(t), y(t))$ lies on the unit circle since

$$x(t)^2 + y(t)^2 = \cos^2 t + \sin^2 t = 1.$$

As t increases from 0 to 2π the point $(x(t), y(t))$ goes around the unit circle exactly once, in the counter-clockwise direction.

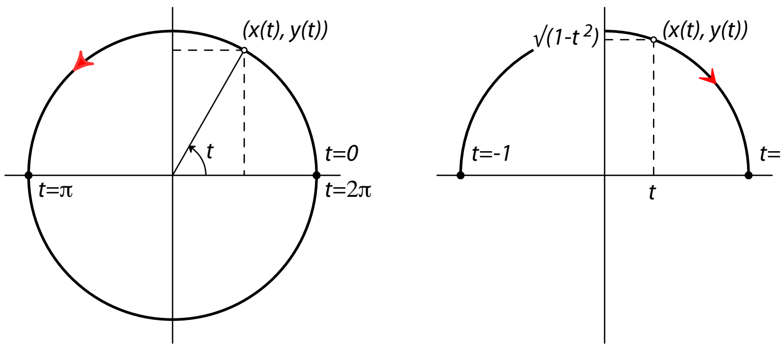


Figure 9.8: Two motions in the plane. On the left $x(t) = \cos t$, $y(t) = \sin t$ with $0 \le t \le 2\pi$, and on the right $x(t) = t$, $y(t) = \sqrt{(1-t^2)}$ with $-1 \le t \le 1$.

In another example one could consider

$$x(t) = t, \quad y(t) = \sqrt{1-t^2}, \quad (-1 \le t \le 1).$$

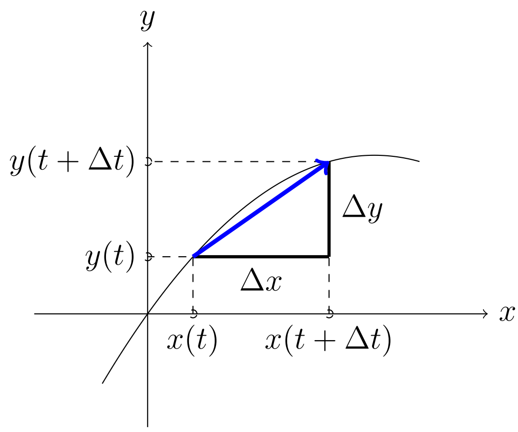
Here at all times the x and y coordinates satisfy

$$x(t)^2 + y(t)^2 = 1$$

again so that the point $(x(t), y(t)) = \left(t, \sqrt{1-t^2}\right)$ again lies on the unit circle. Unlike the previous example we now always have $y(t) \ge 0$ (since $y(t)$ is the square root of something), and unlike the previous example the motion is only defined for $-1 \le t \le 1$. As t increases from -1 to $+1$, $x(t) = t$ does the same, and hence the point $(x(t), y(t))$ moves along the *upper half of the unit circle* from the leftmost point to the rightmost point.

9.5.5 The velocity of an object moving in the plane.

We have seen that the velocity of an object which is moving along a line is the derivative of its position.



If the object is allowed to move in the plane, so that its motion is described by a parametric curve $(x(t), y(t))$, then we can differentiate both $x(t)$ and $y(t)$, which gives us $x'(t)$ and

$y'(t)$, and which leaves us with the following question: *At what speed is a particle moving if it is undergoing the motion $(x(t), y(t))$ $(t_a \le t \le t_b)$?*
To answer this question we consider a short time interval $(t, t + \Delta t)$. During this time interval the particle moves from $(x(t), y(t))$ to $(x(t+\Delta t), t(t+\Delta t))$. Hence it has traveled a distance

$$\Delta s = \sqrt{(\Delta x)^2 + (\Delta y)^2}$$

where

$$\Delta x = x(t + \Delta t) - x(t), \text{ and } \Delta y = y(t + \Delta t) - y(t).$$

Dividing by Δt you get

$$\frac{\Delta s}{\Delta t} = \sqrt{\left(\frac{\Delta x}{\Delta t}\right)^2 + \left(\frac{\Delta y}{\Delta t}\right)^2},$$

for the average velocity over the time interval $[t, t + \Delta t]$. Letting $\Delta t \to 0$ you find the velocity at time t to be

$$v = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}. \tag{9.7}$$

9.5.6 Example – the two motions on the circle from §9.5.4.

If a point moves along a circle according to $x(t) = \cos t$, $y(t) = \sin t$ (figure 9.8 on the left) then

$$\frac{dx}{dt} = -\sin t, \qquad , \frac{dy}{dt} = +\cos t$$

so

$$v(t) = \sqrt{(-\cos t)^2 + (\sin t)^2} = 1.$$

The velocity of this motion is therefore always the same; the point $(\cos t, \sin t)$ moves along the unit circle with constant velocity.
In the second example in §9.5.4 we had $x(t) = t$, $y(t) = \sqrt{1 - t^2}$, so

$$\frac{dx}{dt} = 1, \quad \frac{dy}{dt} = \frac{-t}{\sqrt{1-t^2}}$$

whence

$$v(t) = \sqrt{1^2 + \frac{t^2}{1-t^2}} = \frac{1}{\sqrt{1-t^2}}.$$

Therefore the point $(t, \sqrt{1-t^2})$ moves along the upper half of the unit circle from the left to the right, and its velocity changes according to $v = 1/\sqrt{1-t^2}$.

9.6 The length of a curve

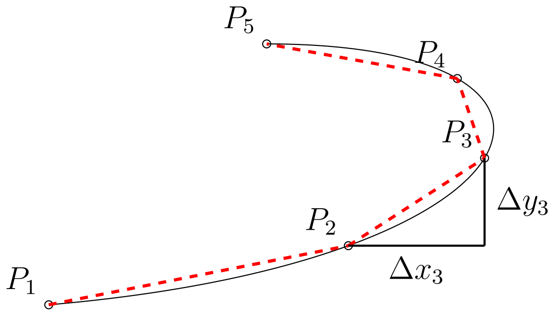
9.6.1 Length of a parametric curve.

Let $(x(t), y(t))$ be some parametric curve defined for $t_a \le t \le t_b$. To find the length of this curve you can reason as follows: The length of the curve should be the distance travelled by the point $(x(t), y(t))$ as t increases from t_a to t_b. At each moment in time the velocity $v(t)$ of the point is given by (9.7), and therefore the distance traveled should be

$$s = \int_{t_a}^{t_b} v(t)\,dt = \int_{t_a}^{t_b} \sqrt{x'(t)^2 + y'(t)^2}\,dt. \tag{9.8}$$

Alternatively, you could try to compute the distance travelled by means of Riemann sums.

def x(t): return 0.05+4*t*(1-t) def y(t): return 0.05+t+t*t*(1-t)



Choose a partition

$$t_a = t_0 < t_1 < \cdots < t_N = t_b$$

of the interval $[t_a, t_b]$. You then get a sequence of points $P_0(x(t_0), y(t_0))$, $P_1(x(t_1), y(t_1))$, $\ldots$, $P_N(x(t_N), y(t_N))$, and after "connecting the dots" you get a polygon. You could approximate the length of the curve by computing the length of this polygon. The distance between two consecutive points P_{k-1} and P_k is

$$\begin{aligned}\Delta s_k &= \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2}\\ &= \sqrt{\left(\frac{\Delta x_k}{\Delta t_k}\right)^2 + \left(\frac{\Delta y_k}{\Delta t_k}\right)^2}\ \Delta t_k\\ &\approx \sqrt{x'(c_k)^2 + y'(c_k)^2}\ \Delta t_k\end{aligned}$$

where we have approximated the difference quotients

$$\frac{\Delta x_k}{\Delta t_k} \text{ and } \frac{\Delta y_k}{\Delta t_k}$$

by the derivatives $x'(c_k)$ and $y'(c_k)$ for some c_k in the interval $[t_{k-1}, t_k]$.

The total length of the polygon is then

$$\sqrt{x'(c_1)^2 + y'(c_1)^2}\ \Delta t_1 + \cdots + \sqrt{x'(c_1)^2 + y'(c_1)^2}\ \Delta t_1$$

This is a Riemann sum for the integral $\int_{t_a}^{t_b} \sqrt{x'(t)^2 + y'(t)^2}\,dt$, and hence we find (once more) that the length of the curve is

$$s = \int_{t_a}^{t_b} \sqrt{x'(t)^2 + y'(t)^2}\,dt.$$

9.6.2 The length of the graph of a function.

The graph of a function ($y = f(x)$ with $a \le x \le b$) is also a curve in the plane, and you can ask what its length is. We will now find this length by *representing the graph as a parametric curve* and applying the formula (9.8) from the previous section.

The standard method of representing the graph of a function $y = f(x)$ by a parametric curve is to choose

$$x(t) = t, \text{ and } y(t) = f(t), \text{ for } a \le t \le b.$$

This parametric curve traces the graph of $y = f(x)$ from left to right as t increases from a to b.

Since $x'(t) = 1$ and $y'(t) = f(t)$ we find that the length of the graph is

$$L = \int_a^b \sqrt{1 + f'(t)^2}\, dt.$$

The variable t in this integral is a dummy variable and we can replace it with any other variable we like, for instance, x:

$$L = \int_a^b \sqrt{1 + f'(x)^2}\, dx. \tag{9.9}$$

In Leibniz' notation we have $y = f(x)$ and $f'(x) = dy/dx$ so that Leibniz would have written

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx.$$

9.7 Examples of length computations

9.7.1 Length of a circle.

In §9.6 we parametrized the unit circle by

$$x(t) = \cos t, \quad y(t) = \sin t, \quad (0 \le t \le 2\pi)$$

and computed $\sqrt{x'(t)^2 + y'(t)^2} = 1$. Therefore our formula tells us that the length of the unit circle is

$$L = \int_0^{2\pi} \sqrt{x'(t)^2 + y'(t)^2}\, dt = \int_0^{2\pi} 1\, dt = 2\pi.$$

This cannot be a PROOF that the unit circle has length 2π since we have already used that fact to define angles in radians, to define the trig functions Sine and Cosine, and to find their derivatives. But our computation shows that the length formula (9.9) is at least consistent with what we already knew.

9.7.2 Length of a parabola.

Consider our old friend, the parabola $y = x^2$, $0 \le x \le 1$. While the area under its graph was easy to compute ($\frac{1}{3}$), its length turns out to be much more complicated.

Our length formula (9.9) says that the length of the parabola is given by

$$L = \int_0^1 \sqrt{1 + \big(\frac{dx^2}{dx}\big)^2}\, dx = \int_0^1 \sqrt{1+4x^2}\, dx.$$

To find this integral you would have to use one of the following (not at all obvious) substitutions[1]

$$x = \tfrac{1}{4}\big(z - \frac{1}{z}\big) \qquad \text{(then } 1 + 4x^2 = \frac{1}{4}(z + 1/z)^2 \text{ so you can simplify the } \sqrt{\cdot}\text{)}$$

or (if you like hyperbolic functions)

$$x = \tfrac{1}{2}\sinh w \qquad \text{(in which case } \sqrt{1+4x^2} = \cosh w.\text{)}$$

9.7.3 Length of the graph of the Sine function.

To compute the length of the curve given by $y = \sin x$, $0 \le x \le \pi$ you would have to compute this integral:

$$L = \int_0^\pi \sqrt{1 + \big(\frac{d\sin x}{dx}\big)^2}\, dx = \int_0^\pi \sqrt{1+\cos^2 x}\, dx. \qquad (9.10)$$

Unfortunately this is not an integral which can be computed in terms of the functions we know in this course (it's an "elliptic integral of the second kind.") This happens very often with the integrals that you run into when you try to compute the length of a curve. In spite of the fact that we get stuck when we try to compute the integral in (9.10), the formula is not useless. For example, since $-1 \le< \cos x \le 1$ we know that

$$1 \le \sqrt{1+\cos^2 x} \le \sqrt{1+1} = \sqrt{2},$$

and therefore the length of the Sine graph is bounded by

$$\int_0^\pi 1dx \;\le\; \int_0^\pi \sqrt{1+\cos^2 x}\, dx \;\le\; \int_0^\pi \sqrt{2}\, dx,$$

i.e.

$$\pi \le L \le \pi\sqrt{2}.$$

[1] Many calculus textbooks will tell you to substitute $x = \tan\theta$, but the resulting integral is still not easy.

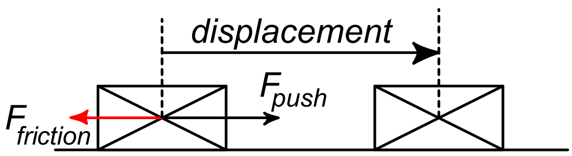
9.8 Work done by a force

9.8.1 Work as an integral.

In Newtonian mechanics a force which acts on an object in motion performs a certain amount of work, i.e. it spends a certain amount of energy. If the force which acts is constant, then the work done by this force is

$$\text{Work} = \text{Force} \times \text{Displacement}.$$

For example if you are pushing a box forward then there will be two forces acting on the box: the force you apply, and the friction force of the floor on the box.



The amount of work you do is the product of the force you exert and the length of the displacement. Both displacement and the force you apply are pointed towards the right, so both are positive, and the work you do (energy you provide to the box) is positive.

The amount of work done by the friction is similarly the product of the friction force and the displacement. Here the displacement is still to the right, but the friction force points to the left, so it is negative. The work done by the friction force is therefore negative. Friction extracts energy from the box.

Suppose now that the force $F(t)$ on the box is not constant, and that its motion is described by saying that its position at time t is $x(t)$. The basic formula work = force × displacement does not apply directly since it assumes that the force is constant. To compute the work done by the varying force $F(t)$ we choose a partition of the time interval $t_a \le t \le t_b$ into

$$t_a = t_0 < t_1 < \cdots < t_{N-1} < t_N = t_b$$

In each short time interval $t_{k-1} \le t \le t_k$ we assume the force is (almost) constant and we approximate it by $F(c_k)$ for some $t_{k-1} \le c_k \le t_k$. If we also assume that the velocity $v(t) = x'(t)$ is approximately constant between times t_{k-1} and t_k then the displacement during this time interval will be

$$x(t_k) - x(t_{k-1}) \approx v(c_k)\Delta t_k,$$

where $\Delta t_k = t_k - t_{k-1}$. Therefore the work done by the force F during the time interval $t_{k-1} \le t \le t_k$ is

$$\Delta W_k = F(c_k)v(c_k)\Delta t_k.$$

Adding the work done during each time interval we get the total work done by the force between time t_a and t_b:

$$W = F(c_1)v(c_1)\Delta t_1 + \cdots + F(c_N)v(c_N)\Delta t_N.$$

Again we have a Riemann sum for an integral. If we take the limit over finer and finer partitions we therefore find that the work done by the force $F(t)$ on an object whose motion is described by $x(t)$ is

$$W = \int_{t_a}^{t_b} F(t)v(t)dt, \tag{9.11}$$

in which $v(t) = x'(t)$ is the velocity of the object.

9.8.2 Kinetic energy.

Newton's famous law relating the force exerted on an object and its motion says $F = ma$, where a is the acceleration of the object, m is its mass, and F is the combination of all forces acting on the object. If the position of the object at time t is $x(t)$, then its velocity and acceleration are $v(t) = x'(t)$ and $a(t) = v'(t) = x''(t)$, and thus the total force acting on the object is

$$F(t) = ma(t) = m\frac{dv}{dt}.$$

The work done by the total force is therefore

$$W = \int_{t_a}^{t_b} F(t)v(t)dt = \int_{t_a}^{t_b} m\frac{dv(t)}{dt}\ v(t)\ dt. \tag{9.12}$$

Even though we have not assumed anything about the motion, so we don't know anything about the velocity $v(t)$, we can still do this integral. The key is to notice that, by the chain rule,

$$m\frac{dv(t)}{dt}\ v(t) = \frac{d\frac{1}{2}mv(t)^2}{dt}.$$

(Remember that m is a constant.) This says that the quantity

$$K(t) = \tfrac{1}{2}mv(t)^2$$

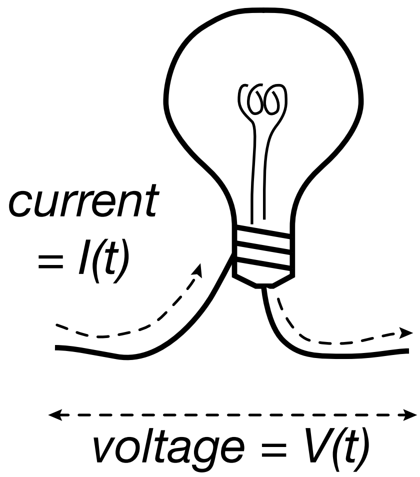
is the antiderivative we need to do the integral (9.12). We get

$$W = \int_{t_a}^{t_b} m\frac{dv(t)}{dt}\ v(t)\ dt = \int_{t_a}^{t_b} K'(t)\ dt = K(t_b) - K(t_a).$$

In Newtonian mechanics the quantity $K(t)$ is called ***the kinetic energy*** of the object, and our computation shows that *the amount by which the kinetic energy of an object increases is equal to the amount of work done on the object.*

9.9 Work done by an electric current

If at time t an electric current $I(t)$ (measured in Ampère) flows through an electric circuit, and if the voltage across this circuit is $V(t)$ (measured in Volts) then the energy supplied tot the circuit per second is $I(t)V(t)$.



Therefore the total energy supplied during a time interval $t_0 \le t \le t_1$ is the integral

$$\text{Energy supplied} = \int_{t_0}^{t_1} I(t)V(t)dt.$$

(measured in Joule; the energy consumption of a circuit is defined to be how much energy it consumes per time unit, and the power consumption of a circuit which consumes 1 Joule per second is said to be one Watt.)

If a certain voltage is applied to a simple circuit (like a light bulb) then the current flowing through that circuit is determined by the resistance R of that circuit by *Ohm's law*[2] which says

$$I = \frac{V}{R}.$$

9.9.1 Example.

If the resistance of a light bulb is $R = 200\Omega$, and if the voltage applied to it is

$$V(t) = 150 \sin 2\pi ft$$

where $f = 50\text{sec}^{-1}$ is the frequency, then how much energy does the current supply to the light bulb in one second?

To compute this we first find the current using Ohm's law,

$$I(t) = \frac{V(t)}{R} = \frac{150}{200} \sin 2\pi ft = 0.75 \sin 2\pi ft. \quad \text{(Amp)}$$

[2] http://en.wikipedia.org/wiki/Ohm's_law

The energy supplied in one second is then

$$\begin{aligned}
E &= \int_0^{1\text{ sec}} I(t)V(t)dt \\
&= \int_0^1 (150 \sin 2\pi f t) \times (0.75 \sin 2\pi f t)\ dt \\
&= 112.5 \int_0^1 \sin^2(2\pi f t)\ dt
\end{aligned}$$

You can do this last integral by using the double angle formula for the cosine, to rewrite

$$\sin^2(2\pi f t) = \tfrac{1}{2}\big\{1 - \cos 4\pi f t\big\} = \tfrac{1}{2} - \tfrac{1}{2}\cos 4\pi f t.$$

Keep in mind that $f = 50$, and you find that the integral is

$$\int_0^1 \sin^2(2\pi f t)\ dt = \Big[\frac{t}{2} - \frac{1}{4\pi f}\sin 4\pi f t\Big]_0^1 = \tfrac{1}{2},$$

and hence the energy supplied to the light bulb during one second is

$$E = 112.5 \times \tfrac{1}{2} = 56.25(\text{Joule}).$$

9.10 PROBLEMS

AREAS BETWEEN GRAPHS

486. Find the area of the region bounded by the parabola $y^2 = 4x$ and the line $y = 2x$.

487. Find the area bounded by the curve $y = x(2 - x)$ and the line $x = 2y$.

488. Find the area bounded by the curve $x^2 = 4y$ and the line $x = 4y - 2$.

489. Calculate the area of the region bounded by the parabolas $y = x^2$ and $x = y^2$.

490. Find the area of the region included between the parabola $y^2 = x$ and the line $x + y = 2$.

491. Find the area of the region bounded by the curves $y = \sqrt{x}$ and $y = x$.

492. Use integration to find the area of the triangular region bounded by the lines $y = 2x + 1$, $y = 3x + 1$ and $x = 4$.

493. Find the area bounded by the parabola $x^2 - 2 = y$ and the line $x + y = 0$.

494. Where do the graphs of $f(x) = x^2$ and $g(x) = 3/(2 + x^2)$ intersect? Find the area of the region which lies above the graph of g and below the graph of g. (Hint: if you need to integrate $1/(2 + x^2)$ you could substitute $x = u\sqrt{2}$.)

495. Graph the curve $y = (1/2)x^2 + 1$ and the straight line $y = x + 1$ and find the area between the curve and the line.

496. Find the area of the region between the parabolas $y^2 = x$ and $x^2 = 16y$.

497. Find the area of the region enclosed by the parabola $y^2 = 4ax$ and the line $y = mx$.

498. Find a so that the curves $y = x^2$ and $y = a\cos x$ intersect at the points $(x, y) = (\frac{\pi}{4}, \frac{\pi^2}{16})$. Then find the area between these curves.

499. Write a definite integral whose value is the area of the region between the two circles $x^2 + y^2 = 1$ and $(x - 1)^2 + y^2 = 1$. Find this area. If you cannot evaluate the integral by calculus you may use geometry to find the area. Hint: The part of a circle cut off by a line is a circular sector with a triangle removed.

VOLUMES OF REVOLUTION

Draw and describe the solids whose volume you are asked to compute in the following problems:

500. What do the dots in "lim..." in equation (9.2) stand for? (i.e. what approaches what in this limit?)

501. Find the volume enclosed by the paraboloid obtained by rotating the graph of $f(x) = R\sqrt{x/H}$ $(0 \le x \le H)$ around the x-axis. Here R and H are positive constants. Draw the solid whose volume you are asked to compute, and indicate what R and H are in your drawing.

502. Find the volume of the solids you get by rotating each of the following graphs around the x-axis:

(***i***) $f(x) = x$, $0 \le x \le 2$

(***ii***) $f(x) = \sqrt{2-x}$, $0 \le x \le 2$

(***iii***) $f(x) = \left(1+x^2\right)^{-1/2}$, $|x| \le 1$

(***iv***) $f(x) = \sin x$, $0 \le x \le \pi$

(***v***) $f(x) = 1 - x^2$, $|x| \le 1$

(***vi***) $f(x) = \cos x$, $0 \le x \le \pi$ (**!!**)

(***vii***) $f(x) = 1/\cos x$, $0 \le x \le \pi/4$

503. Find the volume that results by rotating the semicircle $y = \sqrt{R^2 - x^2}$ about the x-axis.

504. Let $\mathcal{T}$ be triangle $1 \le x \le 2$, $0 \le y \le 3x - 3$.

(***i***) Find the volume of the solid obtained by rotating the triangle $\mathcal{T}$ around the x-axis.

(***ii***) Find the volume that results by rotating the triangle $\mathcal{T}$ around the y axis.

(***iii***) Find the volume that results by rotating the triangle $\mathcal{T}$ around the line $x = -1$.

(***iv***) Find the volume that results by rotating the triangle $\mathcal{T}$ around the line $y = -1$.

505. A spherical bowl of radius a contains water to a depth $h < 2a$. Find the volume of the water in the bowl. (Which solid of revolution is implied in this problem?)

506. Water runs into a spherical bowl of radius 5 ft at the rate of 0.2 ft^3/sec. How fast is the water level rising when the water is 4 ft deep?

LENGTH OF CURVE

507. Find the length of the piece of the graph of $y = \sqrt{1-x^2}$ where $0 \le x \le \frac{1}{2}$.

The graph is a circle, so there are two ways of computing this length. One uses geometry (length of a circular arc = radius × angle), the other uses an integral.

Use both methods and check that you get the same answer. †387

508. Compute the length of the part of the *evolute of the circle*, given by

$$x(t) = \cos t - t\sin t, \quad y(t) = \sin t + t\cos t$$

where $0 < t < \pi$.

509. Show that the *Archimedean spiral*, given by

$$x(\theta) = \theta\cos\theta,\ y(\theta) = \theta\sin\theta,\ 0 \le \theta \le \pi$$

has the same length as the parabola given by

$$y = \tfrac{1}{2}x^2, \quad 0 \le x \le \pi.$$

Hint: you can set up integrals for both lengths. If you get the same integral in both cases, then you know the two curves have the same length (even if you don't try to compute the integrals).

Chapter 10

Methods of Integration

10.1 The indefinite integral

We recall some facts about integration from earlier chapters.

Definition 10.1.1. A function $y = F(x)$ is called an ***antiderivative*** of another function $y = f(x)$ if $F'(x) = f(x)$ for all x.

10.1.1 Example

$F_1(x) = x^2$ is an antiderivative of $f(x) = 2x$.
$F_2(x) = x^2 + 2004$ is also an antiderivative of $f(x) = 2x$.
$G(t) = \frac{1}{2}\sin(2t + 1)$ is an antiderivative of $g(t) = \cos(2t + 1)$.
The Fundamental Theorem of Calculus states that if a function $y = f(x)$ is continuous on an interval $a \le x \le b$, then there always exists an antiderivative $F(x)$ of f, and one has

$$\int_a^b f(x)\,\mathrm{d}x = F(b) - F(a). \tag{10.1}$$

The best way of computing an integral is often to find an antiderivative F of the given function f, and then to use the Fundamental Theorem (10.1). *How you go about finding an antiderivative F for some given function f is the subject of this chapter.*
The following notation is commonly used for antiderivates:

$$F(x) = \int f(x)\mathrm{d}x. \tag{10.2}$$

The integral which appears here does not have the integration bounds a and b. It is called an ***indefinite integral***, as opposed to the integral in (10.1) which is called a ***definite integral***. It's important to distinguish between the two kinds of integrals. Here is a list of differences:

Indefinite integral	Definite integral
$\int f(x)\mathrm{d}x$ is a function of x.	$\int_a^b f(x)\mathrm{d}x$ is a number.
By definition $\int f(x)\mathrm{d}x$ is *any function of x whose derivative is $f(x)$*.	$\int_a^b f(x)\mathrm{d}x$ was defined in terms of Riemann sums and can be interpreted as "area under the graph of $y = f(x)$", at least when $f(x) > 0$.
x is not a dummy variable, for example, $\int 2x\mathrm{d}x = x^2 + C$ and $\int 2t\mathrm{d}t = t^2 + C$ are functions of diffferent variables, so they are not equal.	x is a dummy variable, for example, $\int_0^1 2x\mathrm{d}x = 1$, and $\int_0^1 2t\mathrm{d}t = 1$, so $\int_0^1 2x\mathrm{d}x = \int_0^1 2t\mathrm{d}t$.

10.2 You can always check the answer

Suppose you want to find an antiderivative of a given function $f(x)$ and after a long and messy computation which you don't really trust you get an "answer", $F(x)$. You can then throw away the dubious computation and differentiate the $F(x)$ you had found. If $F'(x)$ turns out to be equal to $f(x)$, then your $F(x)$ is indeed an antiderivative and your computation isn't important anymore.

10.2.1 Example

Suppose we want to find $\int \ln x \, \mathrm{d}x$. Emily says it might be $F(x) = x \ln x - x$. Let's see if she's right:

$$\frac{\mathrm{d}}{\mathrm{d}x}(x \ln x - x) = x \cdot \frac{1}{x} + 1 \cdot \ln x - 1 = \ln x.$$

Who knows how Emily thought of this[1], but she's right! We now know that $\int \ln x\mathrm{d}x = x \ln x - x + C$.

10.3 About "$+C$"

Let $f(x)$ be a function defined on some interval $a \le x \le b$. If $F(x)$ is an antiderivative of $f(x)$ on this interval, then for any constant C the function $\tilde{F}(x) = F(x) + C$ will also be an antiderivative of $f(x)$. So one given function $f(x)$ has many different antiderivatives, obtained by adding different constants to one given antiderivative.

Theorem 10.3.1. If $F_1(x)$ and $F_2(x)$ are antiderivatives of the same function $f(x)$ on some interval $a \le x \le b$, then there is a constant C such that $F_1(x) = F_2(x) + C$.

[1] She integrated by parts.

Proof. Consider the difference $G(x) = F_1(x) - F_2(x)$. Then $G'(x) = F_1'(x) - F_2'(x) = f(x) - f(x) = 0$, so that $G(x)$ must be constant. Hence $F_1(x) - F_2(x) = C$ for some constant. □

It follows that there is some ambiguity in the notation $\int f(x)\,\mathrm{d}x$. Two functions $F_1(x)$ and $F_2(x)$ can both equal $\int f(x)\,\mathrm{d}x$ without equaling each other. When this happens, they (F_1 and F_2) differ by a constant. This can sometimes lead to confusing situations, e.g. you can check that

$$\int 2\sin x \cos x\,\mathrm{d}x = \sin^2 x$$
$$\int 2\sin x \cos x\,\mathrm{d}x = -\cos^2 x$$

are both correct. (Just differentiate the two functions $\sin^2 x$ and $-\cos^2 x$!) These two answers look different until you realize that because of the trig identity $\sin^2 x + \cos^2 x = 1$ they really only differ by a constant: $\sin^2 x = -\cos^2 x + 1$.

To avoid this kind of confusion we will from now on never forget to include the "arbitrary constant $+C$" in our answer when we compute an antiderivative.

10.4 Standard Integrals

Here is a list of the standard derivatives and hence the standard integrals everyone should know.

$$\int f(x)\,\mathrm{d}x = F(x) + C$$
$$\int x^n\,\mathrm{d}x = \frac{x^{n+1}}{n+1} + C \qquad \text{for all } n \neq -1$$
$$\int \frac{1}{x}\,\mathrm{d}x = \ln|x| + C$$
$$\int \sin x\,\mathrm{d}x = -\cos x + C$$
$$\int \cos x\,\mathrm{d}x = \sin x + C$$
$$\int \tan x\,\mathrm{d}x = -\ln \cos x + C$$
$$\int \frac{1}{1+x^2}\,\mathrm{d}x = \arctan x + C$$
$$\int \frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x = \arcsin x + C \qquad \left(= \frac{\pi}{2} - \arccos x + C\right)$$
$$\int \frac{\mathrm{d}x}{\cos x} = \frac{1}{2}\ln\frac{1+\sin x}{1-\sin x} + C \qquad \text{for } -\frac{\pi}{2} < x < \frac{\pi}{2}.$$

All of these integrals should be familiar except for the last one. You can check the last one by differentiation (using $\ln \frac{a}{b} = \ln a - \ln b$ simplifies things a bit).

10.5 Method of substitution

The chain rule says that

$$\frac{\mathrm{d}F(G(x))}{\mathrm{d}x} = F'(G(x)) \cdot G'(x),$$

so that

$$\int F'(G(x)) \cdot G'(x) \,\mathrm{d}x = F(G(x)) + C.$$

10.5.1 Example

Consider the function $f(x) = 2x\sin(x^2+3)$. It does not appear in the list of standard integrals we know by heart. But we do notice[2] that $2x = \frac{\mathrm{d}}{\mathrm{d}x}(x^2+3)$. So let's call $G(x) = x^2+3$, and $F(u) = -\cos u$, then

$$F(G(x)) = -\cos(x^2+3)$$

and

$$\frac{\mathrm{d}F(G(x))}{\mathrm{d}x} = \underbrace{\sin(x^2+3)}_{F'(G(x))} \cdot \underbrace{2x}_{G'(x)} = f(x),$$

so that

$$\int 2x\sin(x^2+3) \,\mathrm{d}x = -\cos(x^2+3) + C.$$

The most transparent way of computing an integral by substitution is by introducing new variables. Thus to do the integral

$$\int f(G(x))G'(x) \,\mathrm{d}x$$

where $f(u) = F'(u)$, we introduce the substitution $u = G(x)$, and agree to write $\mathrm{d}u = \mathrm{d}G(x) = G'(x)\,\mathrm{d}x$. Then we get

$$\int f(G(x))G'(x) \,\mathrm{d}x = \int f(u) \,\mathrm{d}u = F(u) + C.$$

At the end of the integration we must remember that u really stands for $G(x)$, so that

$$\int f(G(x))G'(x) \,\mathrm{d}x = F(u) + C = F(G(x)) + C.$$

For definite integrals this implies

$$\int_a^b f(G(x))G'(x) \,\mathrm{d}x = F(G(b)) - F(G(a)).$$

[2] You *will* start noticing things like this after doing several examples.

which you can also write as

$$\int_a^b f(G(x))G'(x)\,\mathrm{d}x = \int_{G(a)}^{G(b)} f(u)\,\mathrm{d}u. \tag{10.3}$$

To see this in action consider watching: YouTube by Michael Penn .

10.5.2 Example

[Substitution in a definite integral.] As an example we compute

$$\int_0^1 \frac{x}{1+x^2}\,\mathrm{d}x,$$

using the substitution $u = G(x) = 1 + x^2$. Since $\mathrm{d}u = 2x\,\mathrm{d}x$, the associated ***indefinite*** integral is

$$\int \underbrace{\frac{1}{1+x^2}}_{\frac{1}{u}} \underbrace{x\,\mathrm{d}x}_{\frac{1}{2}\mathrm{d}u} = \tfrac{1}{2}\int \frac{1}{u}\,\mathrm{d}u.$$

To find the definite integral you must compute the new integration bounds $G(0)$ and $G(1)$ (see equation (10.3).) If x runs between $x = 0$ and $x = 1$, then $u = G(x) = 1 + x^2$ runs between $u = 1 + 0^2 = 1$ and $u = 1 + 1^2 = 2$, so the definite integral we must compute is

$$\int_0^1 \frac{x}{1+x^2}\,\mathrm{d}x = \tfrac{1}{2}\int_1^2 \frac{1}{u}\,\mathrm{d}u,$$

which is in our list of memorable integrals. So we find

$$\int_0^1 \frac{x}{1+x^2}\,\mathrm{d}x = \tfrac{1}{2}\int_1^2 \frac{1}{u}\,\mathrm{d}u = \tfrac{1}{2}\big[\ln u\big]_1^2 = \tfrac{1}{2}\ln 2.$$

10.6 The double angle trick

If an integral contains $\sin^2 x$ or $\cos^2 x$, then you can remove the squares by using the double angle formulas from trigonometry.

Recall that

$$\cos^2\alpha - \sin^2\alpha = \cos 2\alpha \quad \text{and} \quad \cos^2\alpha + \sin^2\alpha = 1,$$

Adding these two equations gives

$$\cos^2\alpha = \frac{1}{2}\left(\cos 2\alpha + 1\right)$$

while substracting them gives

$$\sin^2\alpha = \frac{1}{2}\left(1 - \cos 2\alpha\right).$$

10.6.1 Example

The following integral shows up in many contexts, so it is worth knowing:

$$\begin{aligned}\int \cos^2 x \,\mathrm{d}x &= \frac{1}{2}\int (1+\cos 2x)\mathrm{d}x \\ &= \frac{1}{2}\left\{x + \frac{1}{2}\sin 2x\right\} + C \\ &= \frac{x}{2} + \frac{1}{4}\sin 2x + C.\end{aligned}$$

Since $\sin 2x = 2\sin x \cos x$ this result can also be written as

$$\int \cos^2 x \,\mathrm{d}x = \frac{x}{2} + \frac{1}{2}\sin x \cos x + C.$$

If you don't want to memorize the double angle formulas, then you can use "Complex Exponentials" to do these and many similar integrals. However, you will have to wait until we are in §12.6 where this is explained.

10.7 Integration by Parts

The product rule states

$$\frac{\mathrm{d}}{\mathrm{d}x}(F(x)G(x)) = \frac{\mathrm{d}F(x)}{\mathrm{d}x}G(x) + F(x)\frac{\mathrm{d}G(x)}{\mathrm{d}x}$$

and therefore, after rearranging terms,

$$F(x)\frac{\mathrm{d}G(x)}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}(F(x)G(x)) - \frac{\mathrm{d}F(x)}{\mathrm{d}x}G(x).$$

This implies the formula for ***integration by parts***

$$\int F(x)\frac{\mathrm{d}G(x)}{\mathrm{d}x}\,\mathrm{d}x = F(x)G(x) - \int \frac{\mathrm{d}F(x)}{\mathrm{d}x}G(x)\,\mathrm{d}x.$$

10.7.1 Example – Integrating by parts once

$$\int \underbrace{x}_{F(x)}\underbrace{e^x}_{G'(x)}\,\mathrm{d}x = \underbrace{x}_{F(x)}\underbrace{e^x}_{G(x)} - \int \underbrace{e^x}_{G(x)}\underbrace{1}_{F'(x)}\,\mathrm{d}x = xe^x - e^x + C.$$

Observe that in this example e^x was easy to integrate, while the factor x becomes an easier function when you differentiate it. This is the usual state of affairs when integration by parts works: differentiating one of the factors ($F(x)$) should simplify the integral, while integrating the other ($G'(x)$) should not complicate things (too much).

Another example: $\sin x = \frac{\mathrm{d}}{\mathrm{d}x}(-\cos x)$ so

$$\int x\sin x\,\mathrm{d}x = x(-\cos x) - \int(-\cos x)\cdot 1\,\mathrm{d}x == -x\cos x + \sin x + C.$$

The reader may want to watch this YouTube by Michael Penn to reinforce this technique.

10.7.2 Example – Repeated Integration by Parts

Sometimes one integration by parts is not enough: since $e^{2x} = \frac{\mathrm{d}}{\mathrm{d}x}(\frac{1}{2}e^{2x})$ one has

$$\begin{aligned}
\int \underbrace{x^2}_{F(x)} \underbrace{e^{2x}}_{G'(x)} \,\mathrm{d}x &= x^2 \frac{e^{2x}}{2} - \int \frac{e^{2x}}{2} 2x \,\mathrm{d}x \\
&= x^2 \frac{e^{2x}}{2} - \left\{ \frac{e^{2x}}{4} 2x - \int \frac{e^{2x}}{4} 2 \,\mathrm{d}x \right\} \\
&= x^2 \frac{e^{2x}}{2} - \left\{ \frac{e^{2x}}{4} 2x - \frac{e^{2x}}{8} 2 + C \right\} \\
&= \frac{1}{2} x^2 e^{2x} - \frac{1}{2} x e^{2x} + \frac{1}{4} e^{2x} - C
\end{aligned}$$

(Be careful with all the minus signs that appear when you integrate by parts.)
The same procedure will work whenever you have to integrate

$$\int P(x) e^{ax} \,\mathrm{d}x$$

where $P(x)$ is a polynomial, and a is a constant. Each time you integrate by parts, you get this

$$\begin{aligned}
\int P(x) e^{ax} \,\mathrm{d}x &= P(x) \frac{e^{ax}}{a} - \int \frac{e^{ax}}{a} P'(x) \,\mathrm{d}x \\
&= \frac{1}{a} P(x) e^{ax} - \frac{1}{a} \int P'(x) e^{ax} \,\mathrm{d}x.
\end{aligned}$$

You have replaced the integral $\int P(x)e^{ax}\,\mathrm{d}x$ with the integral $\int P'(x)e^{ax}\,\mathrm{d}x$. This is the same kind of integral, but it is a little easier since the degree of the derivative $P'(x)$ is less than the degree of $P(x)$.

10.7.3 Example – Emily's computation

Sometimes the factor $G'(x)$ is "invisible". Here is how you can get the antiderivative of $\ln x$ by integrating by parts:

$$\begin{aligned}
\int \ln x \,\mathrm{d}x &= \int \underbrace{\ln x}_{F(x)} \cdot \underbrace{1}_{G'(x)} \,\mathrm{d}x \\
&= \ln x \cdot x - \int \frac{1}{x} \cdot x \,\mathrm{d}x \\
&= x \ln x - \int 1 \,\mathrm{d}x \\
&= x \ln x - x + C.
\end{aligned}$$

You can do $\int P(x) \ln x \,\mathrm{d}x$ in the same way if $P(x)$ is a polynomial.

10.8 Reduction Formulas

Consider the integral

$$I_n = \int x^n e^{ax}\,\mathrm{d}x.$$

Integration by parts gives you

$$\begin{aligned} I_n &= x^n \frac{1}{a} e^{ax} - \int n x^{n-1} \frac{1}{a} e^{ax}\,\mathrm{d}x \\ &= \frac{1}{a} x^n e^{ax} - \frac{n}{a} \int x^{n-1} e^{ax}\,\mathrm{d}x. \end{aligned}$$

We haven't computed the integral, and in fact the integral that we still have to do is of the same kind as the one we started with (integral of $x^{n-1}e^{ax}$ instead of $x^n e^{ax}$). What we have derived is the following ***reduction formula***

$$I_n = \frac{1}{a} x^n e^{ax} - \frac{n}{a} I_{n-1} \qquad (R)$$

which holds for all n.

For $n = 0$ the reduction formula says

$$I_0 = \frac{1}{a} e^{ax}, \text{ i.e. } \int e^{ax}\,\mathrm{d}x = \frac{1}{a} e^{ax} + C.$$

When $n \neq 0$ the reduction formula tells us that we have to compute I_{n-1} if we want to find I_n. The point of a reduction formula is that the same formula also applies to I_{n-1}, and I_{n-2}, etc., so that after repeated application of the formula we end up with I_0, i.e., an integral we know.

10.8.1 Example

To compute $\int x^3 e^{ax}\,\mathrm{d}x$ we use the reduction formula three times:

$$\begin{aligned} I_3 &= \frac{1}{a} x^3 e^{ax} - \frac{3}{a} I_2 \\ &= \frac{1}{a} x^3 e^{ax} - \frac{3}{a} \left\{ \frac{1}{a} x^2 e^{ax} - \frac{2}{a} I_1 \right\} \\ &= \frac{1}{a} x^3 e^{ax} - \frac{3}{a} \left\{ \frac{1}{a} x^2 e^{ax} - \frac{2}{a} \left(\frac{1}{a} x e^{ax} - \frac{1}{a} I_0 \right) \right\} \end{aligned}$$

Insert the known integral $I_0 = \frac{1}{a} e^{ax} + C$ and simplify the other terms and you get

$$\int x^3 e^{ax}\,\mathrm{d}x = \frac{1}{a} x^3 e^{ax} - \frac{3}{a^2} x^2 e^{ax} + \frac{6}{a^3} x e^{ax} - \frac{6}{a^4} e^{ax} + C.$$

10.8.2 Reduction formula requiring two partial integrations

Consider

$$S_n = \int x^n \sin x \, \mathrm{d}x.$$

Then for $n \geq 2$ one has

$$\begin{aligned} S_n &= -x^n \cos x + n \int x^{n-1} \cos x \, \mathrm{d}x \\ &= -x^n \cos x + n x^{n-1} \sin x - n(n-1) \int x^{n-2} \sin x \, \mathrm{d}x. \end{aligned}$$

Thus we find the reduction formula

$$S_n = -x^n \cos x + n x^{n-1} \sin x - n(n-1) S_{n-2}.$$

Each time you use this reduction, the exponent n drops by 2, so in the end you get either S_1 or S_0, depending on whether you started with an odd or even n.

10.8.3 A reduction formula where you have to solve for I_n

We try to compute

$$I_n = \int (\sin x)^n \, \mathrm{d}x$$

by a reduction formula. Integrating by parts twice we get

$$\begin{aligned} I_n &= \int (\sin x)^{n-1} \sin x \, \mathrm{d}x \\ &= -(\sin x)^{n-1} \cos x - \int (-\cos x)(n-1)(\sin x)^{n-2} \cos x \, \mathrm{d}x \\ &= -(\sin x)^{n-1} \cos x + (n-1) \int (\sin x)^{n-2} \cos^2 x \, \mathrm{d}x. \end{aligned}$$

We now use $\cos^2 x = 1 - \sin^2 x$, which gives

$$\begin{aligned} I_n &= -(\sin x)^{n-1} \cos x + (n-1) \int \left\{ \sin^{n-2} x - \sin^n x \right\} \, \mathrm{d}x \\ &= -(\sin x)^{n-1} \cos x + (n-1) I_{n-2} - (n-1) I_n. \end{aligned}$$

You can think of this as an equation for I_n, which, when you solve it tells you

$$n I_n = -(\sin x)^{n-1} \cos x + (n-1) I_{n-2}$$

and thus implies

$$I_n = -\frac{1}{n} \sin^{n-1} x \cos x + \frac{n-1}{n} I_{n-2}. \qquad (\mathcal{S})$$

Since we know the integrals

$$I_0 = \int (\sin x)^0 \mathrm{d}x = \int \mathrm{d}x = x + C \text{ and } I_1 = \int \sin x \, \mathrm{d}x = -\cos x + C$$

the reduction formula ($\mathcal{S}$) allows us to calculate I_n for any $n \geq 0$.

10.8.4 A reduction formula which will be handy later

In the next section you will see how the integral of any "rational function" can be transformed into integrals of easier functions, the hardest of which turns out to be

$$I_n = \int \frac{\mathrm{d}x}{(1+x^2)^n}.$$

When $n = 1$ this is a standard integral, namely

$$I_1 = \int \frac{\mathrm{d}x}{1+x^2} = \arctan x + C.$$

When $n > 1$ integration by parts gives you a reduction formula. Here's the computation:

$$\begin{aligned}
I_n &= \int (1+x^2)^{-n}\,\mathrm{d}x \\
&= \frac{x}{(1+x^2)^n} - \int x\,(-n)\big(1+x^2\big)^{-n-1}\,2x\,\mathrm{d}x \\
&= \frac{x}{(1+x^2)^n} + 2n \int \frac{x^2}{(1+x^2)^{n+1}}\,\mathrm{d}x
\end{aligned}$$

Apply

$$\frac{x^2}{(1+x^2)^{n+1}} = \frac{(1+x^2)-1}{(1+x^2)^{n+1}} = \frac{1}{(1+x^2)^n} - \frac{1}{(1+x^2)^{n+1}}$$

to get

$$\int \frac{x^2}{(1+x^2)^{n+1}}\,\mathrm{d}x = \int \left\{ \frac{1}{(1+x^2)^n} - \frac{1}{(1+x^2)^{n+1}} \right\} \mathrm{d}x = I_n - I_{n+1}.$$

Our integration by parts therefore told us that

$$I_n = \frac{x}{(1+x^2)^n} + 2n\big(I_n - I_{n+1}\big),$$

which you can solve for I_{n+1}. You find the reduction formula

$$I_{n+1} = \frac{1}{2n}\frac{x}{(1+x^2)^n} + \frac{2n-1}{2n} I_n.$$

As an example of how you can use it, we start with $I_1 = \arctan x + C$, and conclude that

$$\begin{aligned}
\int \frac{\mathrm{d}x}{(1+x^2)^2} &= I_2 = I_{1+1} \\
&= \frac{1}{2\cdot 1}\frac{x}{(1+x^2)^1} + \frac{2\cdot 1 - 1}{2\cdot 1} I_1 \\
&= \tfrac{1}{2}\frac{x}{1+x^2} + \tfrac{1}{2}\arctan x + C.
\end{aligned}$$

Apply the reduction formula again, now with $n = 2$, and you get

$$\begin{aligned}\int \frac{dx}{(1+x^2)^3} &= I_3 = I_{2+1} \\ &= \frac{1}{2\cdot 2}\frac{x}{(1+x^2)^2} + \frac{2\cdot 2-1}{2\cdot 2} I_2 \\ &= \tfrac{1}{4}\frac{x}{(1+x^2)^2} + \tfrac{3}{4}\left\{\tfrac{1}{2}\frac{x}{1+x^2} + \tfrac{1}{2}\arctan x\right\} \\ &= \tfrac{1}{4}\frac{x}{(1+x^2)^2} + \tfrac{3}{8}\frac{x}{1+x^2} + \tfrac{3}{8}\arctan x + C.\end{aligned}$$

10.9 Partial Fraction Expansion

A ***rational function*** is one which is a ratio of polynomials,

$$f(x) = \frac{P(x)}{Q(x)} = \frac{p_n x^n + p_{n-1}x^{n-1} + \cdots + p_1 x + p_0}{q_d x^d + q_{d-1}x^{d-1} + \cdots + q_1 x + q_0}.$$

Such rational functions can always be integrated, and the trick which allows you to do this is called a ***partial fraction expansion.*** The whole procedure consists of several steps which are explained in this section. The procedure itself has nothing to do with integration: it's just a way of rewriting rational functions. It is in fact useful in other situations, such as finding Taylor series (see Part 11 of these notes) and computing "inverse Laplace transforms" (see MATH 319.)

10.9.1 Reduce to a proper rational function

A ***proper rational function*** is a rational function $P(x)/Q(x)$ where the degree of $P(x)$ is strictly less than the degree of $Q(x)$. the method of partial fractions only applies to proper rational functions. Fortunately there's an additional trick for dealing with rational functions that are not proper.

If P/Q isn't proper, i.e. if degree$(P) \geq$ degree(Q), then you divide P by Q, with result

$$\frac{P(x)}{Q(x)} = S(x) + \frac{R(x)}{Q(x)}$$

where $S(x)$ is the quotient, and $R(x)$ is the remainder after division. In practice you would do a long division to find $S(x)$ and $R(x)$.

10.9.2 Example

Consider the rational function

$$f(x) = \frac{x^3 - 2x + 2}{x^2 - 1}.$$

Here the numerator has degree 3 which is more than the degree of the denominator (which is 2). To apply the method of partial fractions we must first do a division with remainder. One has

$$\begin{array}{r|llll} & x & +1 & = S(x) & \\ \hline x^2-1 & x^3 & & -2x & +2 \\ & x^3 & & -x & \\ & & & -x & +2 \quad = R(x) \end{array}$$

so that

$$f(x) = \frac{x^3 - 2x + 2}{x^2 - 1} = x + \frac{-x + 2}{x^2 - 1}$$

When we integrate we get

$$\begin{aligned} \int \frac{x^3 - 2x + 2}{x^2 - 1}\,\mathrm{d}x &= \int \left\{ x + \frac{-x+2}{x^2-1} \right\} \mathrm{d}x \\ &= \frac{x^2}{2} + \int \frac{-x+2}{x^2-1}\,\mathrm{d}x. \end{aligned}$$

The rational function which still have to integrate, namely $\frac{-x+2}{x^2-1}$, is proper, i.e. its numerator has lower degree than its denominator.

10.9.3 Partial Fraction Expansion: The Easy Case

To compute the partial fraction expansion of a proper rational function $P(x)/Q(x)$ you must factor the denominator $Q(x)$. Factoring the denominator is a problem as difficult as finding all of its roots; we shall only do problems where the denominator is already factored into linear and quadratic factors, or where this factorization is easy to find.

In the easiest partial fractions problems, all the roots of $Q(x)$ are real numbers and distinct, so the denominator is factored into distinct linear factors, say

$$\frac{P(x)}{Q(x)} = \frac{P(x)}{(x - a_1)(x - a_2)\cdots(x - a_n)}.$$

To integrate this function we find constants $A_1, A_2, \ldots, A_n$ so that

$$\frac{P(x)}{Q(x)} = \frac{A_1}{x - a_1} + \frac{A_2}{x - a_2} + \cdots + \frac{A_n}{x - a_n}. \tag{\#}$$

Then the integral is

$$\int \frac{P(x)}{Q(x)}\,dx = A_1 \ln|x - a_1| + A_2 \ln|x - a_2| + \cdots + A_n \ln|x - a_n| + C.$$

One way to find the coefficients A_i in (#) is called the ***method of equating coefficients***. In this method we multiply both sides of (#) with $Q(x) = (x - a_1)\cdots(x - a_n)$. The result is a polynomial of degree n on both sides. Equating the coefficients of these polynomial gives a system of n linear equations for $A_1, \ldots, A_n$. You get the A_i by solving that system of equations.

Another much faster way to find the coefficients A_i is the ***Heaviside trick***[3]. Multiply equation (#) by $x - a_i$ and then plug in[4] $x = a_i$. On the right you are left with A_i so

$$A_i = \left.\frac{P(x)(x-a_i)}{Q(x)}\right|_{x=a_i} = \frac{P(a_i)}{(a_i - a_1)\cdots(a_i - a_{i-1})(a_i - a_{i+1})\cdots(a_i - a_n)}.$$

10.9.4 Previous Example continued

To integrate $\dfrac{-x+2}{x^2-1}$ we factor the denominator,

$$x^2 - 1 = (x-1)(x+1).$$

The partial fraction expansion of $\dfrac{-x+2}{x^2-1}$ then is

$$\frac{-x+2}{x^2-1} = \frac{-x+2}{(x-1)(x+1)} = \frac{A}{x-1} + \frac{B}{x+1}. \qquad (\dagger)$$

Multiply with $(x-1)(x+1)$ to get

$$-x + 2 = A(x+1) + B(x-1) = (A+B)x + (A-B).$$

The functions of x on the left and right are equal only if the coefficient of x and the constant term are equal. In other words we must have

$$A + B = -1 \text{ and } A - B = 2.$$

These are two linear equations for two unknowns A and B, which we now proceed to solve. Adding both equations gives $2A = 1$, so that $A = \frac{1}{2}$; from the first equation one then finds $B = -1 - A = -\frac{3}{2}$. So

$$\frac{-x+2}{x^2-1} = \frac{1/2}{x-1} - \frac{3/2}{x+1}.$$

Instead, we could also use the Heaviside trick: multiply $(\dagger)$ with $x-1$ to get

$$\frac{-x+2}{x+1} = A + B\frac{x-1}{x+1}$$

Take the limit $x \to 1$ and you find

$$\frac{-1+2}{1+1} = A, \text{ i.e. } A = \frac{1}{2}.$$

[3] Named after Oliver Heaviside, a physicist and electrical engineer in the late 19th and early 20ieth century.

[4] More properly, you should take the limit $x \to a_i$. The problem here is that equation (#) has $x - a_i$ in the denominator, so that it does not hold for $x = a_i$. Therefore you cannot set x equal to a_i in any equation derived from (#), but you can take the limit $x \to a_i$, which in practice is just as good.

Similarly, after multiplying (†) with $x+1$ one gets

$$\frac{-x+2}{x-1} = A\frac{x+1}{x-1} + B,$$

and letting $x \to -1$ you find

$$B = \frac{-(-1)+2}{(-1)-1} = -\frac{3}{2},$$

as before.

Either way, the integral is now easily found, namely,

$$\begin{aligned}\int \frac{x^3-2x+1}{x^2-1}\,\mathrm{d}x &= \frac{x^2}{2} + x + \int \frac{-x+2}{x^2-1}\,\mathrm{d}x \\ &= \frac{x^2}{2} + x + \int \left\{\frac{1/2}{x-1} - \frac{3/2}{x+1}\right\}\mathrm{d}x \\ &= \frac{x^2}{2} + x + \frac{1}{2}\ln|x-1| - \frac{3}{2}\ln|x+1| + C.\end{aligned}$$

10.9.5 Partial Fraction Expansion: The General Case

Buckle up.

When the denominator $Q(x)$ contains repeated factors or quadratic factors (or both) the partial fraction decomposition is more complicated. In the most general case the denominator $Q(x)$ can be factored in the form

$$Q(x) = (x-a_1)^{k_1}\cdots(x-a_n)^{k_n}(x^2+b_1x+c_1)^{\ell_1}\cdots(x^2+b_mx+c_m)^{\ell_m} \tag{10.4}$$

Here we assume that the factors $x-a_1, \ldots, x-a_n$ are all different, and we also assume that the factors $x^2+b_1x+c_1, \ldots, x^2+b_mx+c_m$ are all different.

It is a theorem from advanced algebra that you can always write the rational function $P(x)/Q(x)$ as a sum of terms like this

$$\frac{P(x)}{Q(x)} = \cdots + \frac{A}{(x-a_i)^k} + \cdots + \frac{Bx+C}{(x^2+b_jx+c_j)^\ell} + \cdots \tag{10.5}$$

How did this sum come about?

For each linear factor $(x-a)^k$ in the denominator (10.4) you get terms

$$\frac{A_1}{x-a} + \frac{A_2}{(x-a)^2} + \cdots + \frac{A_k}{(x-a)^k}$$

in the decomposition. There are as many terms as the exponent of the linear factor that generated them.

For each quadratic factor $(x^2+bx+c)^\ell$ you get terms

$$\frac{B_1x+C_1}{x^2+bx+c} + \frac{B_2x+C_2}{(x^2+bx+c)^2} + \cdots + \frac{B_mx+C_m}{(x^2+bx+c)^\ell}.$$

Again, there are as many terms as the exponent ℓ with which the quadratic factor appears in the denominator (10.4).

In general, you find the constants $A_{\ldots}$, $B_{\ldots}$ and $C_{\ldots}$ by the method of equating coefficients.

10.9.6 Example

To do the integral

$$\int \frac{x^2+3}{x^2(x+1)(x^2+1)^2}\,dx$$

apply the method of equating coefficients to the form

$$\frac{x^2+3}{x^2(x+1)(x^2+1)^2} = \frac{A_1}{x} + \frac{A_2}{x^2} + \frac{A_3}{x+1} + \frac{B_1x+C_1}{x^2+1} + \frac{B_2x+C_2}{(x^2+1)^2}. \qquad (\mathcal{EX})$$

Solving this last problem will require solving a system of seven linear equations in the seven unknowns $A_1, A_2, A_3, B_1, C_1, B_2, C_2$. A computer program like `Maple` can do this easily, but it is a lot of work to do it by hand. In general, the method of equating coefficients requires solving n linear equations in n unknowns where n is the degree of the denominator $Q(x)$.

See Problem 613 for a worked example where the coefficients are found.

!! Unfortunately, in the presence of quadratic factors or repeated linear factors the Heaviside trick does not give the whole answer; you must use the method of equating coefficients. **!!**

Once you have found the partial fraction decomposition $(\mathcal{EX})$ you still have to integrate the terms which appeared. The first three terms are of the form $\int A(x-a)^{-p}\,dx$ and they are easy to integrate:

$$\int \frac{A\,dx}{x-a} = A\ln|x-a| + C$$

and

$$\int \frac{A\,dx}{(x-a)^p} = \frac{A}{(1-p)(x-a)^{p-1}} + C$$

if $p>1$. The next, fourth term in $(\mathcal{EX})$ can be written as

$$\begin{aligned}\int \frac{B_1x+C_1}{x^2+1}\,\mathrm{d}x &= B_1\int \frac{x}{x^2+1}\,\mathrm{d}x + C_1\int \frac{\mathrm{d}x}{x^2+1}\\ &= \frac{B_1}{2}\ln(x^2+1) + C_1\arctan x + C_{\text{integration const.}}\end{aligned}$$

While these integrals are already not very simple, the integrals

$$\int \frac{Bx+C}{(x^2+bx+c)^p}\,\mathrm{d}x \qquad \text{with } p>1$$

which can appear are particularly unpleasant. If you really must compute one of these, then complete the square in the denominator so that the integral takes the form

$$\int \frac{Ax+B}{((x+b)^2+a^2)^p}\,dx.$$

After the change of variables $u = x + b$ and factoring out constants you have to do the integrals

$$\int \frac{du}{(u^2+a^2)^p} \qquad \text{and} \qquad \int \frac{u\,du}{(u^2+a^2)^p}.$$

Use the reduction formula we found in example 10.8.4 to compute this integral.

An alternative approach is to use complex numbers, which are on the menu later. If you allow complex numbers then the quadratic factors $x^2 + bx + c$ can be factored, and your partial fraction expansion only contains terms of the form $A/(x-a)^p$, although A and a can now be complex numbers. The integrals are then easy, but the answer has complex numbers in it, and rewriting the answer in terms of real numbers again can be quite involved.

Before attempting the problems in this section the reader might make use of tricks outlined in YouTube by Michael Penn .

10.10 PROBLEMS

DEFINITE VERSUS INDEFINITE INTEGRALS

510. Compute the following three integrals:

$$A = \int x^{-2}\,dx, \qquad B = \int t^{-2}\,dt, \qquad C = \int x^{-2}\,dt.$$

511. One of the following three integrals is not the same as the other two, which one?

$$A = \int_1^4 x^{-2}\,dx, \qquad B = \int_1^4 t^{-2}\,dt, \qquad C = \int_1^4 x^{-2}\,dt.$$

BASIC INTEGRALS

The following integrals are straightforward provided you know the list of standard antiderivatives. They can be done without using substitution or any other tricks.

512. $\displaystyle\int \{6x^5 - 2x^{-4} - 7x + 3/x - 5 + 4e^x + 7^x\}\, dx$

513. $\displaystyle\int (x/a + a/x + x^a + a^x + ax)\, dx$

514. $\displaystyle\int \left\{\sqrt{x} - \sqrt[3]{x^4} + \frac{7}{\sqrt[3]{x^2}} - 6e^x + 1\right\} dx$

515. $\displaystyle\int \left\{2^x + \left(\tfrac{1}{2}\right)^x\right\} dx$

516. $\displaystyle\int_{-3}^{0} (5y^4 - 6y^2 + 14)\, dy$

517. $\displaystyle\int_{1}^{3} \left(\frac{1}{t^2} - \frac{1}{t^4}\right) dt$

518. $\displaystyle\int_{1}^{2} \frac{t^6 - t^2}{t^4}\, dt$

519. $\displaystyle\int_{1}^{2} \frac{x^2 + 1}{\sqrt{x}}\, dx$

520. $\displaystyle\int_{0}^{2} (x^3 - 1)^2\, dx$

521. $\displaystyle\int_{1}^{2} (x + 1/x)^2\, dx$

522. $\displaystyle\int_{3}^{3} \sqrt{x^5 + 2}\, dx$

523. $\displaystyle\int_{1}^{-1} (x - 1)(3x + 2)\, dx$

524. $\displaystyle\int_{1}^{4} (\sqrt{t} - 2/\sqrt{t})\, dt$

525. $\displaystyle\int_{1}^{8} \left(\sqrt[3]{r} + \frac{1}{\sqrt[3]{r}}\right) dr$

526. $\displaystyle\int_{-1}^{0} (x + 1)^3\, dx$

527. $\displaystyle\int_{1}^{e} \frac{x^2 + x + 1}{x}\, dx$

528. $\displaystyle\int_{4}^{9} \left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)^2 dx$

529. $\displaystyle\int_{0}^{1} \left(\sqrt[4]{x^5} + \sqrt[5]{x^4}\right) dx$

530. $\displaystyle\int_{1}^{8} \frac{x - 1}{\sqrt[3]{x^2}}\, dx$

531. $\displaystyle\int_{\pi/4}^{\pi/3} \sin t\, dt$

532. $\displaystyle\int_0^{\pi/2} (\cos\theta + 2\sin\theta)\, d\theta$

533. $\displaystyle\int_0^{\pi/2} (\cos\theta + \sin 2\theta)\, d\theta$

534. $\displaystyle\int_{2\pi/3}^{\pi} \frac{\tan x}{\cos x}\, dx$

535. $\displaystyle\int_{\pi/3}^{\pi/2} \frac{\cot x}{\sin x}\, dx$

536. $\displaystyle\int_1^{\sqrt{3}} \frac{6}{1+x^2}\, dx$

537. $\displaystyle\int_0^{0.5} \frac{dx}{\sqrt{1-x^2}}$

538. $\displaystyle\int_4^8 (1/x)\, dx$

539. $\displaystyle\int_{\ln 3}^{\ln 6} 8e^x\, dx$

540. $\displaystyle\int_8^9 2^t\, dt$

541. $\displaystyle\int_{-e^2}^{-e} \frac{3}{x}\, dx$

542. $\displaystyle\int_{-2}^{3} |x^2-1|\, dx$

543. $\displaystyle\int_{-1}^{2} |x-x^2|\, dx$ †387

544. $\displaystyle\int_{-1}^{2} (x-2|x|)\, dx$

545. $\displaystyle\int_0^2 (x^2-|x-1|)\, dx$ †387

546. $\displaystyle\int_0^2 f(x)\, dx$ where

$$f(x) = \begin{cases} x^4 & \text{if } 0 \le x < 1, \\ x^5, & \text{if } 1 \le x \le 2. \end{cases}$$

547. $\displaystyle\int_{-\pi}^{\pi} f(x)\, dx$ where

$$f(x) = \begin{cases} x, & \text{if } -\pi \le x \le 0, \\ \sin x, & \text{if } 0 < x \le \pi. \end{cases}$$

548. Compute

$$I = \int_0^2 2x\big(1+x^2\big)^3\, dx$$

in two different ways:
(*i*) Expand $(1+x^2)^3$, multiply with $2x$, and integrate each term.
(*ii*) Use the substitution $u = 1 + x^2$.

549. Compute

$$I_n = \int 2x\big(1+x^2\big)^n\, dx.$$

550. If $f'(x) = x - 1/x^2$ and $f(1) = 1/2$ find $f(x)$.

551. Consider $\int_0^2 |x-1|\, dx$. Let $f(x) = |x-1|$ so that

$$f(x) = \begin{cases} x-1 & \text{if } x \ge 1 \\ 1-x & \text{if } x < 1 \end{cases}$$

Define

$$F(x) = \begin{cases} \frac{x^2}{2} - x & \text{if } x \ge 1 \\ x - \frac{x^2}{2} & \text{if } x < 1 \end{cases}$$

Then since F is an antiderivative of f we have by the Fundamental Theorem of Calculus:

$$\int_0^2 |x-1|\, dx = \int_0^2 f(x)\, dx = F(2) - F(0) = (\frac{2^2}{2} - 2) - (0 - \frac{0^2}{2}) = 0$$

But this integral cannot be zero, $f(x)$ is positive except at one point. How can this be?

BASIC SUBSTITUTIONS

Use a substitution to evaluate the following integrals.

552. $\displaystyle\int_1^2 \frac{u\,\mathrm{d}u}{1+u^2}$

553. $\displaystyle\int_1^2 \frac{x\,\mathrm{d}x}{1+x^2}$

554. $\displaystyle\int_{\pi/4}^{\pi/3} \sin^2\theta\cos\theta\,\mathrm{d}\theta$

555. $\displaystyle\int_2^3 \frac{1}{r\ln r}, \mathrm{d}r$

556. $\displaystyle\int \frac{\sin 2x}{1+\cos^2 x}\,dx$ †387

557. $\displaystyle\int \frac{\sin 2x}{1+\sin x}\,dx$

558. $\displaystyle\int_0^1 z\sqrt{1-z^2}\,\mathrm{d}z$

559. $\displaystyle\int_1^2 \frac{\ln 2x}{x}\,\mathrm{d}x$

560. $\displaystyle\int \frac{\ln(2x^2)}{x}\,dx$ †387

561. $\displaystyle\int_{\xi=0}^{\sqrt{2}} \xi(1+2\xi^2)^{10}\,\mathrm{d}\xi$

562. $\displaystyle\int_2^3 \sin\rho\big(\cos 2\rho\big)^4\,\mathrm{d}\rho$

563. $\displaystyle\int \alpha e^{-\alpha^2}\,\mathrm{d}\alpha$

564. $\displaystyle\int \frac{e^{\frac{1}{t}}}{t^2}\,\mathrm{d}t$

INVERSE TRIGONOMETRIC FUNCTIONS

565. The ***inverse sine function*** is the inverse function to the (restricted) sine function, i.e. when $-\pi/2 \le \theta \le \pi/2$ we have

$$\theta = \arcsin(y) \iff y = \sin\theta.$$

The inverse sine function is sometimes called ***Arc Sine function*** and denoted $\theta = \arcsin(y)$. We avoid the notation $\sin^{-1}(x)$ which is used by some as it is ambiguous (it could stand for either $\arcsin x$ or for $(\sin x)^{-1} = 1/(\sin x)$).

(***i***) If $y = \sin\theta$, express $\sin\theta$, $\cos\theta$, and $\tan\theta$ in terms of y when $0 \le \theta < \pi/2$.

(***ii***) If $y = \sin\theta$, express $\sin\theta$, $\cos\theta$, and $\tan\theta$ in terms of y when $\pi/2 < \theta \le \pi$.

(***iii***) If $y = \sin\theta$, express $\sin\theta$, $\cos\theta$, and $\tan\theta$ in terms of y when $-\pi/2 < \theta < 0$.

(***iv***) Evaluate $\displaystyle\int \frac{\mathrm{d}y}{\sqrt{1-y^2}}$ using the substitution $y = \sin\theta$, but give the final answer in terms of y.

566. Express in simplest form:

$$(\boldsymbol{i})\quad \cos(\arcsin^{-1}(x)); \qquad (\boldsymbol{ii})\quad \tan\left\{\arcsin\frac{\ln\frac{1}{4}}{\ln 16}\right\}; \qquad (\boldsymbol{iii})\quad \sin\big(2\arctan a\big)$$

567. Draw the graph of $y = f(x) = \arcsin\big(\sin(x)\big)$, for $-2\pi \le x \le +2\pi$. Make sure you get the same answer as your graphing calculator.

568. Use the change of variables formula to evaluate $\displaystyle\int_{1/2}^{\sqrt{3}/2} \frac{\mathrm{d}x}{\sqrt{1-x^2}}$ first using the substitution $x = \sin u$ and then using the substitution $x = \cos u$.

569. The ***inverse tangent function*** is the inverse function to the (restricted) tangent function, i.e. for $\pi/2 < \theta < \pi/2$ we have

$$\theta = \arctan(w) \iff w = \tan\theta.$$

The inverse tangent function is sometimes called ***Arc Tangent function*** and denoted $\theta = \arctan(y)$. We avoid the notation $\tan^{-1}(x)$ which is used by some as it is ambiguous (it could stand for either $\arctan x$ or for $(\tan x)^{-1} = 1/(\tan x)$).

(***i***) If $w = \tan\theta$, express $\sin\theta$ and $\cos\theta$ in terms of w when

$$(a)\ \ 0 \le \theta < \pi/2 \qquad (b)\ \ \pi/2 < \theta \le \pi \qquad (c)\ \ -\pi/2 < \theta < 0$$

(***ii***) Evaluate $\displaystyle\int \frac{\mathrm{d}w}{1+w^2}$ using the substitution $w = \tan\theta$, but give the final answer in terms of w.

570. Use the substitution $x = \tan(\theta)$ to find the following integrals. Give the final answer in terms of x.

(a) $\displaystyle\int \sqrt{1+x^2}\,\mathrm{d}x$ (b) $\displaystyle\int \frac{1}{(1+x^2)^2}\,\mathrm{d}x$ (c) $\displaystyle\int \frac{\mathrm{d}x}{\sqrt{1+x^2}}$

†387

571. $\displaystyle\int \frac{\mathrm{d}x}{\sqrt{1-x^2}}$

572. $\displaystyle\int \frac{\mathrm{d}x}{\sqrt{4-x^2}}$

573. $\displaystyle\int \frac{\mathrm{d}x}{\sqrt{2x-x^2}}$ †387

574. $\displaystyle\int \frac{x\,\mathrm{d}x}{\sqrt{1-4x^4}}$

575. $\displaystyle\int_{-1/2}^{1/2} \frac{\mathrm{d}x}{\sqrt{4-x^2}}$

576. $\displaystyle\int_{-1}^{1} \frac{\mathrm{d}x}{\sqrt{4-x^2}}$

577. $\displaystyle\int_{0}^{\sqrt{3}/2} \frac{\mathrm{d}x}{\sqrt{1-x^2}}$

578. $\displaystyle\int \frac{\mathrm{d}x}{x^2+1},$

579. $\displaystyle\int \frac{\mathrm{d}x}{x^2+a^2},$

580. $\displaystyle\int \frac{\mathrm{d}x}{7+3x^2},$

581. $\displaystyle\int \frac{\mathrm{d}x}{3x^2+6x+6}$ †387

582. $\displaystyle\int_{1}^{\sqrt{3}} \frac{\mathrm{d}x}{x^2+1},$

583. $\displaystyle\int_{a}^{a\sqrt{3}} \frac{\mathrm{d}x}{x^2+a^2}.$

INTEGRATION BY PARTS AND REDUCTION FORMULAE

584. Evaluate $\displaystyle\int x^n \ln x\,\mathrm{d}x$ where $n \ne -1$.

†387

585. Evaluate $\displaystyle\int e^{ax}\sin bx\,\mathrm{d}x$ where $a^2 + b^2 \ne 0$. [Hint: Integrate by parts twice.]

†387

586. Evaluate $\displaystyle\int e^{ax}\cos bx\,\mathrm{d}x$ where $a^2 + b^2 \ne 0$.

†387

587. Prove the formula

$$\int x^n e^x\,\mathrm{d}x = x^n e^x - n\int x^{n-1}e^x\,\mathrm{d}x$$

and use it to evaluate $\int x^2 e^x \, dx$.

588. Prove the formula

$$\int \sin^n x \, dx = -\frac{1}{n} \cos x \sin^{n-1} x + \frac{n-1}{n} \int \sin^{n-2} x \, dx, \qquad n \neq 0$$

589. Evaluate $\int \sin^2 x \, dx$. Show that the answer is the same as the answer you get using the half angle formula.

590. Evaluate $\int_0^{\pi} \sin^{14} x \, dx$. †387

591. Prove the formula

$$\int \cos^n x \, dx = \frac{1}{n} \sin x \cos^{n-1} x + \frac{n-1}{n} \int \cos^{n-2} x \, dx, \qquad n \neq 0$$

and use it to evaluate $\int_0^{\pi/4} \cos^4 x \, dx$. †387

592. Prove the formula

$$\int x^m (\ln x)^n \, dx = \frac{x^{m+1} (\ln x)^n}{m+1} - \frac{n}{m+1} \int x^m (\ln x)^{n-1} \, dx, \qquad m \neq -1,$$

and use it to evaluate the following integrals: †387

593. $\int \ln x \, dx$ †387

594. $\int (\ln x)^2 \, dx$ †387

595. $\int x^3 (\ln x)^2 \, dx$

596. Evaluate $\int x^{-1} \ln x \, dx$ by another method. [Hint: the solution is short!] †387

597. For an integer $n > 1$ derive the formula

$$\int \tan^n x \, dx = \frac{1}{n-1} \tan^{n-1} x - \int \tan^{n-2} x \, dx$$

Using this, find $\int_0^{\pi/4} \tan^5 x \, dx$ by doing just one explicit integration. †387

Use the reduction formula from example 10.8.4 to compute these integrals:

598. $\int \frac{dx}{(1+x^2)^3}$

599. $\int \frac{dx}{(1+x^2)^4}$

600. $\int \frac{x dx}{(1+x^2)^4}$ [Hint: $\int x/(1+x^2)^n dx$ is easy.]

601. $\int \frac{1+x}{(1+x^2)^2} \, dx$

602. The reduction formula from example 10.8.4 is valid for all $n \neq 0$. In particular, n does not have to be an integer, and it does not have to be positive.

Find a relation between $\int \sqrt{1+x^2}\,\mathrm{d}x$ and $\int \frac{\mathrm{d}x}{\sqrt{1+x^2}}$ by setting $n = -\frac{1}{2}$.

603. Apply integration by parts to

$$\int \frac{1}{x}\,dx$$

Let $u = \frac{1}{x}$ and $dv = dx$. This gives us, $du = \frac{-1}{x^2}\,dx$ and $v = x$.

$$\int \frac{1}{x}\,dx = (\frac{1}{x})(x) - \int x\,\frac{-1}{x^2}\,dx$$

Simplifying

$$\int \frac{1}{x}\,dx = 1 + \int \frac{1}{x}\,dx$$

and subtracting the integral from both sides gives us $0 = 1$. How can this be?

INTEGRATION OF RATIONAL FUNCTIONS

Express each of the following rational functions as a polynomial plus a proper rational function. (See §10.9.1 for definitions.)

604. $\dfrac{x^3}{x^3-4}$, †387

605. $\dfrac{x^3+2x}{x^3-4}$, †387

606. $\dfrac{x^3-x^2-x-5}{x^3-4}$. †388

607. $\dfrac{x^3-1}{x^2-1}$. †388

COMPLETING THE SQUARE

Write ax^2+bx+c in the form $a(x+p)^2+q$, i.e. find p and q in terms of a, b, and c (this procedure, which you might remember from high school algebra, is called "completing the square."). Then evaluate the integrals

608. $\displaystyle\int \frac{\mathrm{d}x}{x^2+6x+8}$, †388

609. $\displaystyle\int \frac{\mathrm{d}x}{x^2+6x+10}$, †388

610. $\displaystyle\int \frac{\mathrm{d}x}{5x^2+20x+25}$. †388

611. Use the method of equating coefficients to find numbers A, B, C such that

$$\frac{x^2+3}{x(x+1)(x-1)} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{x-1}$$

and then evaluate the integral $\displaystyle\int \frac{x^2+3}{x(x+1)(x-1)}\,\mathrm{d}x$.

†388

612. Do the previous problem using the Heaviside trick.

†388

613. Find the integral $\displaystyle\int \frac{x^2+3}{x^2(x-1)}\,\mathrm{d}x$.

†389

Evaluate the following integrals:

614. $\displaystyle\int_{-5}^{-2} \frac{x^4-1}{x^2+1}\,dx$

615. $\displaystyle\int \frac{x^3\,\mathrm{d}x}{x^4+1}$

616. $\displaystyle\int \frac{x^5\,dx}{x^2-1}$

617. $\displaystyle\int \frac{x^5\,dx}{x^4-1}$

618. $\displaystyle\int \frac{x^3}{x^2-1}\,dx$ †389

619. $\displaystyle\int \frac{e^{3x}\,dx}{e^{4x}-1}$ †389

620. $\displaystyle\int \frac{e^x\,dx}{\sqrt{1+e^{2x}}}$

621. $\displaystyle\int \frac{e^x\,dx}{e^{2x}+2e^x+2}$ †389

622. $\displaystyle\int \frac{dx}{1+e^x}$ †389

623. $\displaystyle\int \frac{dx}{x(x^2+1)}$

624. $\displaystyle\int \frac{dx}{x(x^2+1)^2}$

625. $\displaystyle\int \frac{dx}{x^2(x-1)}$ †389

626. $\displaystyle\int \frac{1}{(x-1)(x-2)(x-3)}\,dx$

627. $\displaystyle\int \frac{x^2+1}{(x-1)(x-2)(x-3)}\,dx$

628. $\displaystyle\int \frac{x^3+1}{(x-1)(x-2)(x-3)}\,dx$

629.

(a) Compute $\displaystyle\int_1^2 \frac{dx}{x(x-h)}$ where h is a positive number.

(b) What happens to your answer to (a) when $h \to 0^+$?

(c) Compute $\displaystyle\int_1^2 \frac{dx}{x^2}$.

MISCELLANEOUS AND MIXED INTEGRALS

630. Find the area of the region bounded by the curves

$$x=1, \qquad x=2, \qquad y=\frac{2}{x^2-4x+5}, \qquad y=\frac{x^2-8x+7}{x^2-8x+16}.$$

631. Let $\mathcal{P}$ be the piece of the parabola $y=x^2$ on which $0 \le x \le 1$.

(*i*) Find the area between $\mathcal{P}$, the x-axis and the line $x=1$.

(*ii*) Find the length of $\mathcal{P}$.

632. Let a be a positive constant and

$$F(x)=\int_0^x \sin(a\theta)\cos(\theta)\,d\theta.$$

[Hint: use a trig identity for $\sin A\cos B$, or wait until we have covered complex exponentials and then come back to do this problem.]

(*i*) Find $F(x)$ if $a \ne 1$.

(*ii*) Find $F(x)$ if $a=1$. (Don't divide by zero.)

Evaluate the following integrals:

633. $\displaystyle\int_0^a x\sin x\,dx$ †389

634. $\displaystyle\int_0^a x^2\cos x\,dx$ †389

635. $\displaystyle\int_3^4 \frac{x\,dx}{\sqrt{x^2-1}}$ †389

636. $\displaystyle\int_{1/4}^{1/3} \frac{x\,dx}{\sqrt{1-x^2}}$ †389

637. $\displaystyle\int_3^4 \frac{dx}{x\sqrt{x^2-1}}$ †389

638. $\displaystyle\int \frac{x\,\mathrm{d}x}{x^2+2x+17}$ †389

639. $\displaystyle\int \frac{x^4}{x^2-36}\,\mathrm{d}x$

640. $\displaystyle\int \frac{x^4}{36-x^2}\,\mathrm{d}x$

641. $\displaystyle\int \frac{(x^2+1)\,\mathrm{d}x}{x^4-x^2}$ †389

642. $\displaystyle\int \frac{(x^2+3)\,\mathrm{d}x}{x^4-2x^2}$

643. $\displaystyle\int e^x(x+\cos(x))\,\mathrm{d}x$

644. $\displaystyle\int (e^x+\ln(x))\,\mathrm{d}x$

645. $\displaystyle\int \frac{3x^2+2x-2}{x^3-1}\,\mathrm{d}x$

646. $\displaystyle\int \frac{x^4}{x^4-16}\,\mathrm{d}x$

647. $\displaystyle\int \frac{x}{(x-1)^3}\,\mathrm{d}x$

648. $\displaystyle\int \frac{4}{(x-1)^3(x+1)}\,\mathrm{d}x$

649. $\displaystyle\int \frac{1}{\sqrt{1-2x-x^2}}\,\mathrm{d}x$

650. $\displaystyle\int_1^e x\ln x\,\mathrm{d}x$

651. $\displaystyle\int 2x\ln(x+1)\,dx$ †389

652. $\displaystyle\int_{e^2}^{e^3} x^2\ln x\,\mathrm{d}x$

653. $\displaystyle\int_1^e x(\ln x)^3\,\mathrm{d}x$

654. $\displaystyle\int \arctan(\sqrt{x})\,\mathrm{d}x$ †389

655. $\displaystyle\int x(\cos x)^2\,\mathrm{d}x$

656. $\displaystyle\int_0^\pi \sqrt{1+\cos(6w)}\,\mathrm{d}w$

657. $\displaystyle\int \frac{1}{1+\sin(x)}\,\mathrm{d}x$ †389

658. Find

$$\int \frac{\mathrm{d}x}{x(x-1)(x-2)(x-3)}$$

and

$$\int \frac{(x^3+1)\,\mathrm{d}x}{x(x-1)(x-2)(x-3)}$$

659. Find

$$\int \frac{\mathrm{d}x}{x^3+x^2+x+1}$$

†389

660. You don't always have to find the antiderivative to find a definite integral. This problem gives you two examples of how you can avoid finding the antiderivative.

(***i***) To find

$$I = \int_0^{\pi/2} \frac{\sin x\,\mathrm{d}x}{\sin x+\cos x}$$

you use the substitution $u=\pi/2-x$. The new integral you get must of course be equal to the integral I you started with, so if you *add the old and new integrals* you get $2I$. If you actually do this you will see that the sum of the old and new integrals is *very* easy to compute.

(***ii***) Use the same trick to find $\displaystyle\int_0^{\pi/2} \sin^2 x\,\mathrm{d}x$

661. Graph the equation $x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}}$. Compute the area bounded by this curve.

662. THE BOW-TIE GRAPH. Graph the equation $y^2 = x^4 - x^6$. Compute the area bounded by this curve.

663. THE FAN-TAILED FISH. Graph the equation

$$y^2 = x^2 \left(\frac{1-x}{1+x}\right).$$

Find the area enclosed by the loop. (HINT: Rationalize the denominator of the integrand.)

664. Find the area of the region bounded by the curves

$$x = 2, \qquad y = 0, \qquad y = x \ln \frac{x}{2}$$

665. Find the volume of the solid of revolution obtained by rotating around the $x-$axis the region bounded by the lines $x = 5$, $x = 10$, $y = 0$, and the curve

$$y = \frac{x}{\sqrt{x^2 + 25}}.$$

666.

How to find the integral of $f(x) = \dfrac{1}{\cos x}$

(***i***) Verify the answer given in the table in the lecture notes.

(***ii***) Note that

$$\frac{1}{\cos x} = \frac{\cos x}{\cos^2 x} = \frac{\cos x}{1 - \sin^2 x},$$

and apply the substitution $s = \sin x$ followed by a partial fraction decomposition to compute $\int \frac{\mathrm{d}x}{\cos x}$.

RATIONALIZING SUBSTITUTIONS

Recall that a rational function is the ratio of two polynomials.

667. Prove that the family of rational functions is closed under taking sums, products, quotients (except do not divide by the zero polynomial), and compositions.

To integrate rational functions of x and $\sqrt{1 + x^2}$ one may do a trigonometric substitution, e.g., $x = \tan(\theta)$ and $1 + \tan^2(\theta) = \sec^2(\theta)$. This turns the problem into a trig integral. Or one could use $1 + \sinh^2(t) = \cosh^2(t)$ and convert the problem into a rational function of e^t.

Another technique which works is to use the parameterization of the hyperbola by rational functions:

$$x = \frac{1}{2}(t - \frac{1}{t}) \qquad y = \frac{1}{2}(t + \frac{1}{t})$$

668. Show that $y^2 - x^2 = 1$ and hence $y = \sqrt{1 + x^2}$.

Use this to rationalize the integrals, i.e, make them into an integral of a rational function of t. You do not need to integrate the rational function.

669. $\int \sqrt{1+x^2}\,\mathrm{d}x$ †389

670. $\int \frac{x^4}{\sqrt{1+x^2}}\,\mathrm{d}x$

671. $\int \frac{\mathrm{d}s}{\sqrt{s^2+2s+3}}$

672. Show that $t = x + y = x + \sqrt{1+x^2}$.
Hence if

$$\int g(x)\ dx = \int f(t)\ dt = F(t) + C$$

then

$$\int g(x)\ dx = F(x + \sqrt{1-x^2}) + C.$$

673. Note that $x = \sqrt{y^2-1}$. Show that t is a function of y.

†389

Express these integrals as integrals of rational functions of t.

674. $\int \frac{\mathrm{d}y}{(y^2-1)^{1/2}}$ †389

675. $\int \frac{y^4}{(y^2-1)^{1/2}}\,\mathrm{d}y$

676. $\int \frac{s^4}{(s^2-36)^{3/2}}\,\mathrm{d}s$

677. $\int \frac{\mathrm{d}s}{(s^2+2s)^{1/2}}$

678. Note that $1 = (\frac{x}{y})^2 + (\frac{1}{y})^2$. What substitution would rationalize integrands which have $\sqrt{1-z^2}$ in them? Show how to write t as a function of z.

†390

Express these integrals as integrals of rational functions of t.

679. $\int \sqrt{1-z^2}\,\mathrm{d}z$ †390

680. $\int \frac{\mathrm{d}z}{\sqrt{1-z^2}}$ †390

681. $\int \frac{z^2}{\sqrt{1-z^2}}\,\mathrm{d}z$

682. $\int \frac{s^4}{(36-s^2)^{3/2}}\,\mathrm{d}s$

683. $\int \frac{\mathrm{d}s}{(s+5)\sqrt{s^2+5s}}$

RATIONAL FUNCTIONS OF sin AND cos

Examples of such integrals are:

$$\int \frac{(\cos\theta)^2 - (\cos\theta)(\sin\theta) + 1}{(\cos\theta)^2 + (\sin\theta)^3 + (\cos\theta) + 1}\,d\theta$$

or

$$\int \frac{(\sin\theta)^3(\cos\theta) + (\cos\theta) + (\sin\theta) + 1}{(\cos\theta)^2(\sin\theta)^3 - (\cos\theta)}\,d\theta$$

The goal of the following problems is to show that such integrals can be rationalized, not to integrat the rational function.

684. Substitute $z = \sin(\theta)$ and express $\int r(sin(\theta), cos(\theta))\ d\theta$ as a rational function of z and $\sqrt{1-z^2}$.

†390

685. Express it as rational function of t.

†390

686. Express t as a function of θ. †390

Chapter 11

Taylor's Formula and Infinite Series

All continuous functions which vanish at $x = a$
are approximately equal at $x = a$,
but some are more approximately equal than others.

11.1 Taylor Polynomials

Suppose you need to do some computation with a complicated function $y = f(x)$, and suppose that the only values of x you care about are close to some constant $x = a$. Since polynomials are simpler than most other functions, you could then look for a polynomial $y = P(x)$ which somehow "matches" your function $y = f(x)$ for values of x close to a. And you could then replace your function f with the polynomial P, hoping that the error you make isn't too big. *Which* polynomial you will choose depends on when you think a polynomial "matches" a function. In this chapter we will say that a polynomial P of degree n matches a function f at $x = a$ ***if** P **has the same value and the same derivatives of order** $1, 2, \ldots, n$ **at** $x = a$ **as the function** f**.*** The polynomial which matches a given function at some point $x = a$ is the Taylor polynomial of f. It is given by the following formula.

Definition 11.1.1. The Taylor polynomial of a function $y = f(x)$ of degree n at a point a is the polynomial

$$T_n^a f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n. \qquad (11.1)$$

(Recall that $n! = 1 \cdot 2 \cdot 3 \cdots n$, and by definition $0! = 1$.

Theorem 11.1.1. The Taylor polynomial has the following property: it is the only polynomial $P(x)$ of degree n whose value and whose derivatives of orders 1, 2, ..., and n are the same as those of f, i.e. it's the only polynomial of degree n for which

$$P(a) = f(a), \quad P'(a) = f'(a), \quad P''(a) = f''(a), \quad \ldots, \quad P^{(n)}(a) = f^{(n)}(a)$$

holds.

Proof. We do the case $a = 0$, for simplicity. Let n be given, consider a polynomial $P(x)$ of degree n, say,

$$P(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_n x^n,$$

and let's see what its derivatives look like. They are:

$$\begin{array}{rclllllllll}
P(x) & = & a_0 & + & a_1 x & + & a_2 x^2 & + & a_3 x^3 & + & a_4 x^4 & + \cdots \\
P'(x) & = & & & a_1 & + & 2a_2 x & + & 3a_3 x^2 & + & 4a_4 x^3 & + \cdots \\
P^{(2)}(x) & = & & & & & 1\cdot 2a_2 & + & 2\cdot 3a_3 x & + & 3\cdot 4a_4 x^2 & + \cdots \\
P^{(3)}(x) & = & & & & & & & 1\cdot 2\cdot 3a_3 & + & 2\cdot 3\cdot 4a_4 x & + \cdots \\
P^{(4)}(x) & = & & & & & & & & & 1\cdot 2\cdot 3\cdot 4a_4 & + \cdots
\end{array}$$

When you set $x = 0$ all the terms which have a positive power of x vanish, and you are left with the first entry on each line, i.e.

$$P(0) = a_0, \quad P'(0) = a_1, \quad P^{(2)}(0) = 2a_2, \quad P^{(3)}(0) = 2\cdot 3a_3, \text{ etc.}$$

and in general

$$P^{(k)}(0) = k!a_k \text{ for } 0 \le k \le n.$$

For $k \ge n+1$ the derivatives $p^{(k)}(x)$ all vanish of course, since $P(x)$ is a polynomial of degree n.

Therefore, if we want P to have the same values and derivatives at $x = 0$ of orders $1,, \dots, n$ as the function f, then we must have $k!a_k = P^{(k)}(0) = f^{(k)}(0)$ for all $k \le n$. Thus

$$a_k = \frac{f^{(k)}(0)}{k!} \qquad \text{for } 0 \le k \le n.$$

□

11.2 Examples

Note that the zeroth order Taylor polynomial is just a constant,

$$T_0^a f(x) = f(a),$$

while the first order Taylor polynomial is

$$T_1^a f(x) = f(a) + f'(a)(x-a).$$

This is exactly the *linear approximation of* $f(x)$ for x close to a which was derived earlier in this text.

The Taylor polynomial generalizes this first order approximation by providing "higher order approximations" to f.

Most of the time we will take $a = 0$ in which case we write $T_n f(x)$ instead of $T_n^a f(x)$, and we get a slightly simpler formula

$$T_n f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n. \tag{11.2}$$

You will see below that for many functions $f(x)$ the Taylor polynomials $T_n f(x)$ give better and better approximations as you add more terms (i.e. as you increase n). For this reason the limit when $n \to \infty$ is often considered, which leads to the *infinite sum*

$$T_\infty f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots$$

At this point we will not try to make sense of the "sum of infinitely many numbers".

11.2.1 Example: Compute the Taylor polynomials of degree 0, 1 and 2 of $f(x) = e^x$ at $a = 0$, and plot them

One has

$$f(x) = e^x \implies f'(x) = e^x \implies f''(x) = e^x,$$

so that

$$f(0) = 1, \quad f'(0) = 1, \quad f''(0) = 1.$$

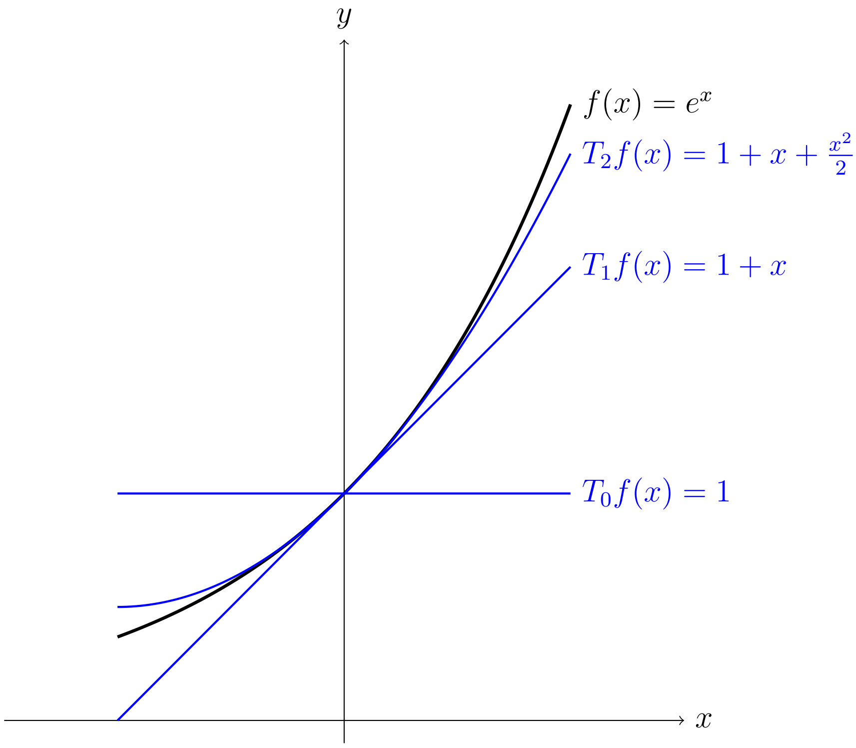


Figure 11.1: The Taylor polynomials of degree 0, 1 and 2 of $f(x) = e^x$ at $a = 0$. The zeroth order Taylor polynomial has the right value at $x = 0$ but it doesn't know whether or not the function f is increasing at $x = 0$. The first order Taylor polynomial has the right slope at $x = 0$, but it doesn't see if the graph of f is curved up or down at $x = 0$. The second order Taylor polynomial also has the right curvature at $x = 0$.

Therefore the first three Taylor polynomials of e^x at $a = 0$ are

$$T_0 f(x) = 1$$
$$T_1 f(x) = 1 + x$$
$$T_2 f(x) = 1 + x + \frac{1}{2}x^2.$$

The graphs are found in Figure 11.1. As you can see from the graphs, the Taylor polynomial $T_0 f(x)$ of degree 0 is close to e^x for small x, by virtue of the continuity of e^x

The Taylor polynomial of degree 0, i.e. $T_0 f(x) = 1$ captures the fact that e^x by virtue of its continuity does not change very much if x stays close to $x = 0$.

The Taylor polynomial of degree 1, i.e. $T_1 f(x) = 1 + x$ corresponds to the tangent line to the graph of $f(x) = e^x$, and so it also captures the fact that the function $f(x)$ is increasing near $x = 0$.

Clearly $T_1 f(x)$ is a better approximation to e^x than $T_0 f(x)$.

The graphs of both $y = T_0 f(x)$ and $y = T_1 f(x)$ are straight lines, while the graph of $y = e^x$ is curved (in fact, convex). The second order Taylor polynomial captures this convexity. In fact, the graph of $y = T_2 f(x)$ is a parabola, and since it has the same first and second derivative at $x = 0$, its curvature is the same as the curvature of the graph of $y = e^x$ at $x = 0$.

So it seems that $y = T_2 f(x) = 1 + x + x^2/2$ is an approximation to $y = e^x$ which beats both $T_0 f(x)$ and $T_1 f(x)$.

11.2.2 Example: Find the Taylor polynomials of $f(x) = \sin x$

When you start computing the derivatives of $\sin x$ you find

$$f(x) = \sin x, \quad f'(x) = \cos x, \quad f''(x) = -\sin x, \quad f^{(3)}(x) = -\cos x,$$

and thus

$$f^{(4)}(x) = \sin x.$$

So after four derivatives you're back to where you started, and the sequence of derivatives of $\sin x$ cycles through the pattern

$$\sin x,\ \cos x,\ -\sin x,\ -\cos x,\ \sin x,\ \cos x,\ -\sin x,\ -\cos x,\ \sin x,\ \dots$$

on and on. At $x = 0$ you then get the following values for the derivatives $f^{(j)}(0)$,

j	1	2	3	4	5	6	7	8	$\cdots$
$f^{(j)}(0)$	0	1	0	-1	0	1	0	-1	$\cdots$

This gives the following Taylor polynomials

$$\begin{aligned}
T_0 f(x) &= 0 \\
T_1 f(x) &= x \\
T_2 f(x) &= x \\
T_3 f(x) &= x - \frac{x^3}{3!} \\
T_4 f(x) &= x - \frac{x^3}{3!} \\
T_5 f(x) &= x - \frac{x^3}{3!} + \frac{x^5}{5!}
\end{aligned}$$

Note that since $f^{(2)}(0) = 0$ the Taylor polynomials $T_1 f(x)$ and $T_2 f(x)$ are the same! The second order Taylor polynomial in this example is really only a polynomial of degree 1. In general the Taylor polynomial $T_n f(x)$ of any function is a polynomial of degree at most n, and this example shows that the degree can sometimes be strictly less.

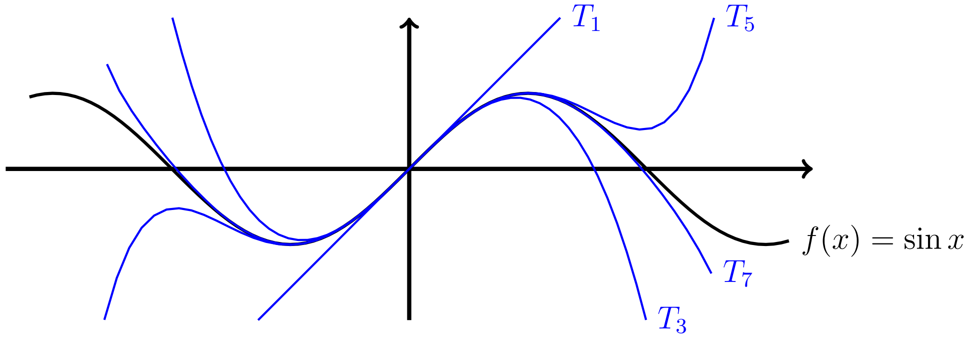


Figure 11.2: Taylor polynomials of $f(x) = \sin x$

11.2.3 Example – Compute the Taylor polynomials of degree two and three of $f(x) = 1 + x + x^2 + x^3$ at $a = 3$

Solution: Remember that our notation for the n^{th} degree Taylor polynomial of a function f at a is $T_n^a f(x)$, and that it is defined by (11.1).

We have

$$f'(x) = 1 + 2x + 3x^2, \quad f''(x) = 2 + 6x, \quad f'''(x) = 6$$

Therefore $f(3) = 40$, $f'(3) = 34$, $f''(3) = 20$, $f'''(3) = 6$, and thus

$$T_2^s f(x) = 40 + 34(x-3) + \frac{20}{2!}(x-3)^2 = 40 + 34(x-3) + 10(x-3)^2. \tag{11.3}$$

Why don't we expand the answer? You could do this (i.e. replace $(x-3)^2$ by $x^2 - 6x + 9$ throughout and sort the powers of x), but as we will see in this chapter, the Taylor polynomial $T_n^a f(x)$ is used as an approximation for $f(x)$ when x is close to a. In this example $T_2^3 f(x)$ is to be used when x is close to 3. If $x - 3$ is a small number then the successive powers $x - 3$, $(x-3)^2$, $(x-3)^3$, ... decrease rapidly, and so the terms in (11.3) are arranged in decreasing order.

We can also compute the third degree Taylor polynomial. It is

$$\begin{aligned} T_3^3 f(x) &= 40 + 34(x-3) + \frac{20}{2!}(x-3)^2 + \frac{6}{3!}(x-3)^3 \\ &= 40 + 34(x-3) + 10(x-3)^2 + (x-3)^3. \end{aligned}$$

If you expand this (this takes a little work) you find that

$$40 + 34(x-3) + 10(x-3)^2 + (x-3)^3 = 1 + x + x^2 + x^3.$$

So the third degree Taylor polynomial is the function f itself! Why is this so? Because of Theorem 11.1.1! Both sides in the above equation are third degree polynomials, and their derivatives of order 0, 1, 2 and 3 are the same at $x = 3$, so they must be the same polynomial.

11.3 Some special Taylor polynomials

Here is a list of functions whose Taylor polynomials are sufficiently regular that you can write a formula for the nth term.

$$
\begin{aligned}
T_n e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} \\
T_{2n+1}\{\sin x\} &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} \\
T_{2n}\{\cos x\} &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + (-1)^n \frac{x^{2n}}{(2n)!} \\
T_n\left\{\frac{1}{1-x}\right\} &= 1 + x + x^2 + x^3 + x^4 + \cdots + x^n \qquad \text{(Geometric Series)} \\
T_n\{\ln(1+x)\} &= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + (-1)^{n+1}\frac{x^n}{n}
\end{aligned}
$$

All of these Taylor polynomials can be computed directly from the definition, by repeatedly differentiating $f(x)$. To see this in action consider viewing YouTube by 3Blue1Brown .

Another function whose Taylor polynomial you should know is $f(x) = (1+x)^a$, where a is a constant. You can compute $T_n f(x)$ directly from the definition, and when you do this you find

$$
T_n\{(1+x)^a\} = 1 + ax + \frac{a(a-1)}{1\cdot 2}x^2 + \frac{a(a-1)(a-2)}{1\cdot 2\cdot 3}x^3 + \cdots + \frac{a(a-1)\cdots(a-n+1)}{1\cdot 2\cdots n}x^n. \quad (11.4)
$$

This formula is called ***Newton's binomial formula***. The coefficient of x^n is called a ***binomial coefficient***, and it is written

$$
\binom{a}{n} = \frac{a(a-1)\cdots(a-n+1)}{n!}. \quad (11.5)
$$

When a is an integer $\binom{a}{n}$ is also called "a choose n."

Note that you already knew special cases of the binomial formula: when a is a positive integer the binomial coefficients are just the numbers in ***Pascal's triangle***. When $a = -1$ the binomial formula is the Geometric series.

11.4 The Remainder Term

The Taylor polynomial $T_n f(x)$ is almost never exactly equal to $f(x)$, but often it is a good approximation, especially if x is small. To see how good the approximation is we define the "error term" or, "remainder term".

Definition 11.4.1. If f is an n times differentiable function on some interval containing a, then

$$
R_n^a f(x) = f(x) - T_n^a f(x)
$$

is called the n^{th} order remainder (or error) term in the Taylor polynomial of f.

If $a = 0$, as will be the case in most examples we do, then we write

$$
R_n f(x) = f(x) - T_n f(x).
$$

11.4.1 Example

If $f(x) = \sin x$ then we have found that $T_3 f(x) = x - \frac{1}{6}x^3$, so that

$$R_3\{\sin x\} = \sin x - x + \tfrac{1}{6}x^3.$$

This is a completely correct formula for the remainder term, but it's rather useless: there's nothing about this expression that suggests that $x - \frac{1}{6}x^3$ is a much better approximation to $\sin x$ than, say, $x + \frac{1}{6}x^3$.

The usual situation is that there is no simple formula for the remainder term.

11.4.2 An unusual example, in which there *is* a simple formula for $R_n f(x)$

Consider $f(x) = 1 - x + 3\,x^2 - 15\,x^3$.

Then you find

$$T_2 f(x) = 1 - x + 3\,x^2, \text{ so that } R_2 f(x) = f(x) - T_2 f(x) = -15\,x^3.$$

The moral of this example is this: *Given a polynomial $f(x)$ you find its n^{th} degree Taylor polynomial by taking all terms of degree $\le n$ in $f(x)$; the remainder $R_n f(x)$ then consists of the remaining terms.*

11.4.3 Another unusual, but important example where you can compute $R_n f(x)$

Consider the function

$$f(x) = \frac{1}{1-x}.$$

Then repeated differentiation gives

$$f'(x) = \frac{1}{(1-x)^2}, \quad f^{(2)}(x) = \frac{1\cdot 2}{(1-x)^3}, \quad f^{(3)}(x) = \frac{1\cdot 2\cdot 3}{(1-x)^4}, \quad \ldots$$

and thus

$$f^{(n)}(x) = \frac{1\cdot 2\cdot 3\cdots n}{(1-x)^{n+1}}.$$

Consequently,

$$f^{(n)}(0) = n! \implies \frac{1}{n!}f^{(n)}(0) = 1,$$

and you see that the Taylor polynomials of this function are really simple, namely

$$T_n f(x) = 1 + x + x^2 + x^3 + x^4 + \cdots + x^n.$$

But this sum should be really familiar: it is just the ***Geometric Sum*** (each term is x times the previous term). Its sum is given by[1]

$$T_n f(x) = 1 + x + x^2 + x^3 + x^4 + \cdots + x^n = \frac{1-x^{n+1}}{1-x},$$

[1] Multiply both sides with $1-x$ to verify this, in case you had forgotten the formula!

which we can rewrite as

$$T_n f(x) = \frac{1}{1-x} - \frac{x^{n+1}}{1-x} = f(x) - \frac{x^{n+1}}{1-x}.$$

The remainder term therefore is

$$R_n f(x) = f(x) - T_n f(x) = \frac{x^{n+1}}{1-x}.$$

11.5 Lagrange's Formula for the Remainder Term

Theorem 11.5.1. Let f be an $n+1$ times differentiable function on some interval I containing $x = 0$. Then for every x in the interval I there is a ξ between 0 and x such that

$$R_n f(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!} x^{n+1}.$$

(ξ between 0 and x means either $0 < \xi < x$ or $x < \xi < 0$, depending on the sign of x.)

This theorem (including the proof) is similar to the Mean Value Theorem. The proof is a bit involved, and I've put it at the end of this chapter.

There are calculus textbooks which, after presenting this remainder formula, give a whole bunch of problems which ask you to find ξ for given f and x. Such problems completely miss the point of Lagrange's formula. The point is that *even though you usually can't compute the mystery point ξ precisely, Lagrange's formula for the remainder term allows you to **estimate** it.* Here is the most common way to estimate the remainder:

Theorem 11.5.2 (Estimate of remainder term)**.** If f is an $n+1$ times differentiable function on an interval containing $x = 0$, and if you have a constant M such that

$$\left|f^{(n+1)}(t)\right| \leq M \text{ for all } t \text{ between 0 and } x, \tag{†}$$

then

$$|R_n f(x)| \leq \frac{M|x|^{n+1}}{(n+1)!}.$$

Proof. We don't know what ξ is in Lagrange's formula, but it doesn't matter, for wherever it is, it must lie between 0 and x so that our assumption (†) implies $|f^{(n+1)}(\xi)| \leq M$. Put that in Lagrange's formula and you get the stated inequality. □

11.5.1 How to compute e in a few decimal places

Consider $f(x) = e^x$. We computed the Taylor polynomials before. If you set $x = 1$, then you get $e = f(1) = T_n f(1) + R_n f(1)$, and thus, taking $n = 8$,

$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \frac{1}{6!} + \frac{1}{7!} + \frac{1}{8!} + R_8(1).$$

By Lagrange's formula there is a ξ between 0 and 1 such that

$$R_8(1) = \frac{f^{(9)}(\xi)}{9!}\, 1^9 = \frac{e^\xi}{9!}.$$

(remember: $f(x) = e^x$, so all its derivatives are also e^x.) We don't really know where ξ is, but since it lies between 0 and 1 we know that $1 < e^\xi < e$. So the remainder term $R_8(1)$ is positive and no more than $e/9!$. Estimating $e < 3$, we find

$$\frac{1}{9!} < R_8(1) < \frac{3}{9!}.$$

Thus we see that

$$1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{7!} + \frac{1}{8!} + \frac{1}{9!} < e < 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{7!} + \frac{1}{8!} + \frac{3}{9!}$$

or, in decimals,

$$2.718\,281\ldots < e < 2.718\,287\ldots$$

11.5.2 Error in the approximation $\sin x \approx x$

In many calculations involving $\sin x$ for small values of x one makes the simplifying approximation $\sin x \approx x$, justified by the known limit

$$\lim_{x\to 0} \frac{\sin x}{x} = 1.$$

Question: How big is the error in this approximation?

To answer this question, we use Lagrange's formula for the remainder term again.

Let $f(x) = \sin x$. Then the first degree Taylor polynomial of f is

$$T_1 f(x) = x.$$

The approximation $\sin x \approx x$ is therefore exactly what you get if you approximate $f(x) = \sin x$ by its first degree Taylor polynomial. Lagrange tells us that

$$f(x) = T_1 f(x) + R_1 f(x), \text{ i.e. } \sin x = x + R_1 f(x),$$

where, since $f''(x) = -\sin x$,

$$R_1 f(x) = \frac{f''(\xi)}{2!} x^2 = -\tfrac{1}{2} \sin \xi \cdot x^2$$

for some ξ between 0 and x.

As always with Lagrange's remainder term, we don't know where ξ is precisely, so we have to estimate the remainder term. The easiest way to do this (but not the best: see below) is to say that no matter what ξ is, $\sin \xi$ will always be between -1 and 1. Hence the remainder term is bounded by

$$(\P) \qquad |R_1 f(x)| \le \tfrac{1}{2} x^2,$$

and we find that

$$x - \tfrac{1}{2}x^2 \le \sin x \le x + \tfrac{1}{2}x^2.$$

Question: How small must we choose x to be sure that the approximation $\sin x \approx x$ isn't off by more than 1% ?

If we want the error to be less than 1% of the estimate, then we should require $\frac{1}{2}x^2$ to be less than 1% of $|x|$, i.e.

$$\tfrac{1}{2}x^2 < 0.01 \cdot |x| \Leftrightarrow |x| < 0.02$$

So we have shown that, if you choose $|x| < 0.02$, then the error you make in approximating $\sin x$ by just x is no more than 1%.

A final comment about this example: the estimate for the error we got here can be improved quite a bit in two different ways:

(1) You could notice that one has $|\sin x| \leq x$ for all x, so if ξ is between 0 and x, then $|\sin \xi| \leq |\xi| \leq |x|$, which gives you the estimate

$$|R_1 f(x)| \leq \tfrac{1}{2}|x|^3 \quad \text{instead of } \tfrac{1}{2}x^2 \text{ as in (¶)}.$$

(2) For this particular function the two Taylor polynomials $T_1 f(x)$ and $T_2 f(x)$ are the same (because $f''(0) = 0$). So $T_2 f(x) = x$, and we can write

$$\sin x = f(x) = x + R_2 f(x),$$

In other words, the error in the approximation $\sin x \approx x$ is also given by the *second* order remainder term, which according to Lagrange is given by

$$R_2 f(x) = \frac{-\cos \xi}{3!} x^3 \quad \overset{|\cos \xi| \leq 1}{\Longrightarrow} \quad |R_2 f(x)| \leq \tfrac{1}{6}|x|^3,$$

which is the best estimate for the error in $\sin x \approx x$ we have so far.

11.6 The limit as $x \to 0$, keeping n fixed

11.6.1 Little-oh

Lagrange's formula for the remainder term lets us write a function $y = f(x)$, which is defined on some interval containing $x = 0$, in the following way

$$f(x) = f(0) + f'(0)x + \frac{f^{(2)}(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \frac{f^{(n+1)}(\xi)}{(n+1)!}x^{n+1} \tag{11.6}$$

The last term contains the ξ from Lagrange's theorem, which depends on x, and of which you only know that it lies between 0 and x. For many purposes it is not necessary to know the last term in this much detail – often it is enough to know that "in some sense" the last term is the smallest term, in particular, as $x \to 0$ it is much smaller than x, or x^2, or, …, or x^n:

Theorem 11.6.1. If the $n+1$st derivative $f^{(n+1)}(x)$ is continuous at $x = 0$ then the remainder term $R_n f(x) = f^{(n+1)}(\xi)x^{n+1}/(n+1)!$ satisfies

$$\lim_{x \to 0} \frac{R_n f(x)}{x^k} = 0$$

for any $k = 0, 1, 2, \ldots, n$.

Proof. Since ξ lies between 0 and x, one has $\lim_{x \to 0} f^{(n+1)}(\xi) = f^{(n+1)}(0)$, and therefore

$$\lim_{x \to 0} \frac{R_n f(x)}{x^k} = \lim_{x \to 0} f^{(n+1)}(\xi) \frac{x^{n+1}}{x^k} = \lim_{x \to 0} f^{(n+1)}(\xi) \cdot x^{n+1-k} = f^{(n+1)}(0) \cdot 0 = 0.$$

□

So we can rephrase (11.6) by saying

$$f(x) = f(0) + f'(0)x + \frac{f^{(2)}(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \text{ remainder}$$

where the remainder is much smaller than x^n, x^{n-1}, ..., x^2, x or 1. In order to express the condition that some function is "much smaller than x^n," at least for very small x, Landau introduced the following notation which many people find useful.

Definition 11.6.1. "$o(x^n)$" is an abbreviation for any function $h(x)$ which satisfies

$$\lim_{x\to 0} \frac{h(x)}{x^n} = 0.$$

So you can rewrite (11.6) as

$$f(x) = f(0) + f'(0)x + \frac{f^{(2)}(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + o(x^n).$$

The nice thing about Landau's little-oh is that you can compute with it, as long as you obey the following (at first sight rather strange) rules which will be proved in class

$$\begin{aligned}
x^n \cdot o(x^m) &= o(x^{n+m}) \\
o(x^n) \cdot o(x^m) &= o(x^{n+m}) \\
x^m &= o(x^n) && \text{if } n < m \\
o(x^n) + o(x^m) &= o(x^n) && \text{if } n < m \\
o(Cx^n) &= o(x^n) && \text{for any constant } C
\end{aligned}$$

11.6.2 Example: prove one of these little-oh rules

Let's do the first one, i.e. let's show that $x^n \cdot o(x^m)$ is $o(x^{n+m})$ as $x \to 0$.

Remember, if someone writes $x^n \cdot o(x^m)$, then the $o(x^m)$ is an abbreviation for some function $h(x)$ which satisfies $\lim_{x\to 0} h(x)/x^m = 0$. So the $x^n \cdot o(x^m)$ we are given here really is an abbreviation for $x^n h(x)$. We then have

$$\lim_{x\to 0} \frac{x^n h(x)}{x^{n+m}} = \lim_{x\to 0} \frac{h(x)}{x^m} = 0, \text{ since } h(x) = o(x^m).$$

11.6.3 Can you see that $x^3 = o(x^2)$ by looking at the graphs of these functions?

A picture is of course never a proof, but have a look at figure 11.3 which shows you the graphs of $y = x$, x^2, x^3, x^4, x^5 and x^{10}. As you see, when x approaches 0, the graphs of higher powers of x approach the x-axis (much?) faster than do the graphs of lower powers.

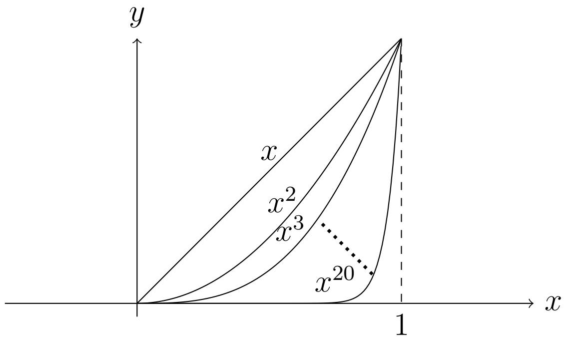


Figure 11.3: How the powers stack up. All graphs of $y = x^n$ $(n > 1)$ are tangent to the x-axis at the origin. But the larger the exponent n the "flatter" the graph of $y = x^n$ is.

You should also have a look at figure 11.4 which exhibits the graphs of $y = x^2$, as well as several linear functions $y = Cx$ (with $C = 1, \frac{1}{2}, \frac{1}{5}$ and $\frac{1}{10}$.) For each of these linear functions one has $x^2 < Cx$ if x is small enough; *how* small is actually small enough depends on C. The smaller the constant C, the closer you have to keep x to 0 to be sure that x^2 is smaller than Cx. Nevertheless, no matter how small C is, the parabola will eventually always reach the region below the line $y = Cx$.

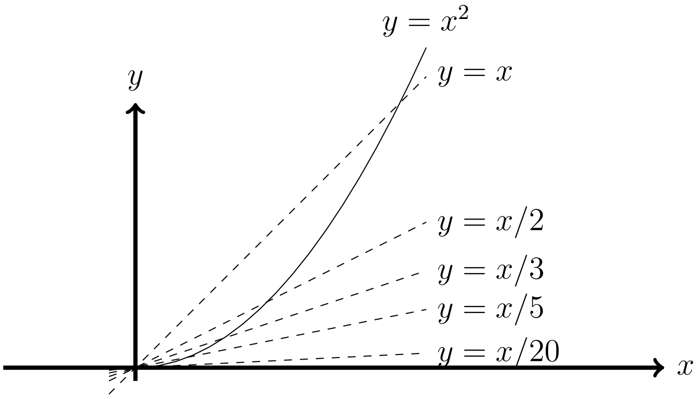


Figure 11.4: x^2 is smaller than any multiple of x, if x is small enough. Compare the quadratic function $y = x^2$ with a linear function $y = Cx$. Their graphs are a parabola and a straight line. Parts of the parabola may lie above the line, but as $x \searrow 0$ the parabola will always duck underneath the line.

11.6.4 Example: Little-oh arithmetic is a little funny

Both x^2 and x^3 are functions which are $o(x)$, i.e.

$$x^2 = o(x) \quad \text{and} \quad x^3 = o(x)$$

Nevertheless $x^2 \neq x^3$. So in working with little-oh we are giving up on the principle that says that two things which both equal a third object must themselves be equal; in other words, $a = b$ and $b = c$ implies $a = c$, but not when you're using little-ohs! You can also put it like this: just because two quantities both are much smaller than x, they don't have to be equal. In particular,

you can never cancel little-ohs!!!

In other words, the following is pretty wrong

$$o(x^2) - o(x^2) = 0.$$

Why? The two $o(x^2)$'s both refer to functions $h(x)$ which satisfy $\lim_{x\to 0} h(x)/x^2 = 0$, but there are many such functions, and the two $o(x^2)$'s could be abbreviations for different functions $h(x)$.

Contrast this with the following computation, which at first sight looks wrong even though it is actually right:

$$o(x^2) - o(x^2) = o(x^2).$$

In words: if you subtract two quantities both of which are negligible compared to x^2 for small x then the result will also be negligible compared to x^2 for small x.

11.6.5 Computations with Taylor polynomials

The following theorem is very useful because it lets you compute Taylor polynomials of a function without differentiating it.

Theorem 11.6.2. If $f(x)$ and $g(x)$ are $n+1$ times differentiable functions then

$$T_n f(x) = T_n g(x) \iff f(x) = g(x) + o(x^n). \tag{11.7}$$

In other words, if two functions have the same nth degree Taylor polynomial, then their difference is much smaller than x^n, at least, if x is small.

In principle the definition of $T_n f(x)$ lets you compute as many terms of the Taylor polynomial as you want, but in many (most) examples the computations quickly get out of hand. To see what can happen go though the following example:

11.6.6 How *NOT* to compute the Taylor polynomial of degree 12 of $f(x) = 1/(1+x^2)$

Diligently computing derivatives one by one you find

$$
\begin{aligned}
f(x) &= \frac{1}{1+x^2} && \text{so } f(0) = 1\\
f'(x) &= \frac{-2x}{(1+x^2)^2} && \text{so } f'(0) = 0\\
f''(x) &= \frac{6x^2-2}{(1+x^2)^3} && \text{so } f''(0) = -2\\
f^{(3)}(x) &= 24\frac{x-x^3}{(1+x^2)^4} && \text{so } f^{(3)}(0) = 0\\
f^{(4)}(x) &= 24\frac{1-10x^2+5x^4}{(1+x^2)^5} && \text{so } f^{(4)}(0) = 24 = 4!\\
f^{(5)}(x) &= 240\frac{-3x+10x^3-3x^5}{(1+x^2)^6} && \text{so } f^{(4)}(0) = 0\\
f^{(6)}(x) &= -720\frac{-1+21x^2-35x^4+7x^6}{(1+x^2)^7} && \text{so } f^{(4)}(0) = 720 = 6!\\
&\ \vdots
\end{aligned}
$$

I'm getting tired of differentiating – can you find $f^{(12)}(x)$? After a lot of work we give up at the sixth derivative, and all we have found is

$$T_6\left\{\frac{1}{1+x^2}\right\} = 1 - x^2 + x^4 - x^6.$$

By the way,

$$f^{(12)}(x) = 479001600\frac{1 - 78\,x^2 + 715\,x^4 - 1716\,x^6 + 1287\,x^8 - 286\,x^{10} + 13\,x^{12}}{(1+x^2)^{13}}$$

and $479001600 = 12!$.

11.6.7 The right approach to finding the Taylor polynomial of any degree of $f(x) = 1/(1+x^2)$

Start with the Geometric Series: if $g(t) = 1/(1-t)$ then

$$g(t) = 1 + t + t^2 + t^3 + t^4 + \cdots + t^n + o(t^n).$$

Now substitute $t = -x^2$ in this limit,

$$g(-x^2) = 1 - x^2 + x^4 - x^6 + \cdots + (-1)^n x^{2n} + o\left((-x^2)^n\right)$$

Since $o\left((-x^2)^n\right) = o(x^{2n})$ and

$$g(-x^2) = \frac{1}{1-(-x^2)} = \frac{1}{1+x^2},$$

we have found

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \cdots + (-1)^n x^{2n} + o(x^{2n})$$

By Theorem (11.6.2) this implies

$$T_{2n}\left\{\frac{1}{1+x^2}\right\} = 1 - x^2 + x^4 - x^6 + \cdots + (-1)^n x^{2n}.$$

11.6.8 Example of multiplication of Taylor series

Finding the Taylor series of $e^{2x}/(1+x)$ directly from the definition is another recipe for headaches. Instead, you should exploit your knowledge of the Taylor series of both factors e^{2x} and $1/(1+x)$:

$$\begin{aligned} e^{2x} &= 1 + 2x + \frac{2^2x^2}{2!} + \frac{2^3x^3}{3!} + \frac{2^4x^4}{4!} + o(x^4) \\ &= 1 + 2x + 2x^2 + \frac{4}{3}x^3 + \frac{2}{3}x^4 + o(x^4) \\ \frac{1}{1+x} &= 1 - x + x^2 - x^3 + x^4 + o(x^4). \end{aligned}$$

Then multiply these two

$$\begin{aligned}
e^{2x}\cdot\frac{1}{1+x} &= \left(1+2x+2x^2+\frac{4}{3}x^3+\frac{2}{3}x^4+o(x^4)\right)\cdot\left(1-x+x^2-x^3+x^4+o(x^4)\right)\\
&= \begin{array}{ccccccccccc}
1 & - & x & + & x^2 & - & x^3 & + & x^4 & + & o(x^4)\\
 & + & 2x & - & 2x^2 & + & 2x^3 & - & 2x^4 & + & o(x^4)\\
 & & & + & 2x^2 & - & 2x^3 & + & 2x^4 & + & o(x^4)\\
 & & & & & + & \frac{4}{3}x^3 & - & \frac{4}{3}x^4 & + & o(x^4)\\
 & & & & & & & + & \frac{2}{3}x^4 & + & o(x^4)
\end{array}\\
&= 1+x+x^2+\frac{1}{3}x^3+\frac{1}{3}x^4+o(x^4)\quad (x\to 0)
\end{aligned}$$

11.6.9 Taylor's formula and Fibonacci numbers

The Fibonacci numbers are defined as follows: the first two are $f_0 = 1$ and $f_1 = 1$, and the others are defined by the equation

$$\text{(Fib)}\qquad \boxed{f_n = f_{n-1} + f_{n-2}}$$

So

$$\begin{aligned}
f_2 &= f_1 + f_0 = 1 + 1 = 2,\\
f_3 &= f_2 + f_1 = 2 + 1 = 3,\\
f_4 &= f_3 + f_2 = 3 + 2 = 5,\\
&\text{etc.}
\end{aligned}$$

The equation (Fib) lets you compute the whole sequence of numbers, one by one, when you are given only the first few numbers of the sequence (f_0 and f_1 in this case). Such an equation for the elements of a sequence is called a ***recursion relation***.

Now consider the function

$$f(x) = \frac{1}{1-x-x^2}.$$

Let

$$T_\infty f(x) = c_0 + c_1x + c_2x^2 + c_3x^3 + \cdots$$

be its Taylor series.

Due to Lagrange's remainder theorem you have, for any n,

$$\frac{1}{1-x-x^2} = c_0 + c_1x + c_2x^2 + c_3x^3 + \cdots + c_nx^n + o(x^n)\quad (x\to 0).$$

Multiply both sides with $1-x-x^2$ and you get

$$\begin{aligned}
1 &= (1-x-x^2)\cdot(c_0 + c_1x + c_2x^2 + \cdots + c_n + o(x^n))\quad (x\to 0)\\
&= \begin{array}{ccccccccccc}
c_0 & + & c_1x & + & c_2x^2 & + & \cdots & + & c_nx^n & + & o(x^n)\\
 & - & c_0x & - & c_1x^2 & - & \cdots & - & c_{n-1}x^n & + & o(x^n)\\
 & & & - & c_0x^2 & - & \cdots & - & c_{n-2}x^n & - & o(x^n)
\end{array}\qquad (x\to 0)\\
&= c_0 + (c_1-c_0)x + (c_2-c_1-c_0)x^2 + (c_3-c_2-c_1)x^3 + \cdots\\
&\qquad \cdots + (c_n - c_{n-1} - c_{n-2})x^n + o(x^n)\quad (x\to 0)
\end{aligned}$$

Compare the coefficients of powers x^k on both sides for $k = 0, 1, \ldots, n$ and you find

$$c_0 = 1, \quad c_1 - c_0 = 0 \implies c_1 = c_0 = 1, \quad c_2 - c_1 - c_0 = 0 \implies c_2 = c_1 + c_0 = 2$$

and in general

$$c_n - c_{n-1} - c_{n-2} = 0 \implies c_n = c_{n-1} + c_{n-2}$$

Therefore the coefficients of the Taylor series $T_\infty f(x)$ are exactly the Fibonacci numbers:

$$c_n = f_n \text{ for } n = 0, 1, 2, 3, \ldots$$

Since it is much easier to compute the Fibonacci numbers one by one than it is to compute the derivatives of $f(x) = 1/(1 - x - x^2)$, this is a better way to compute the Taylor series of $f(x)$ than just directly from the definition.

11.6.10 More about the Fibonacci numbers

In this example you'll see a trick that lets you compute the Taylor series of ***any rational function***. You already know the trick: find the partial fraction decomposition of the given rational function. Ignoring the case that you have quadratic expressions in the denominator, this lets you represent your rational function as a sum of terms of the form

$$\frac{A}{(x-a)^p}.$$

These are easy to differentiate any number of times, and thus they allow you to write their Taylor series.

Let's apply this to the function $f(x) = 1/(1 - x - x^2)$ from the example 11.6.9. First we factor the denominator.

$$1 - x - x^2 = 0 \iff x^2 + x - 1 = 0 \iff x = \frac{-1 \pm \sqrt{5}}{2}.$$

The number

$$\phi = \frac{1+\sqrt{5}}{2} \approx 1.618\,033\,988\,749\,89\ldots$$

is called the ***Golden Ratio***. It satisfies[2]

$$\phi + \frac{1}{\phi} = \sqrt{5}.$$

The roots of our polynomial $x^2 + x - 1$ are therefore

$$x_- = \frac{-1-\sqrt{5}}{2} = -\phi, \qquad x_+ = \frac{-1+\sqrt{5}}{2} = \frac{1}{\phi}.$$

and we can factor $1 - x - x^2$ as follows

$$1 - x - x^2 = -(x^2 + x - 1) = -(x - x_-)(x - x_+) = -(x - \frac{1}{\phi})(x + \phi).$$

[2] To prove this, use $\frac{1}{\phi} = \frac{2}{1+\sqrt{5}} = \frac{2}{1+\sqrt{5}}\frac{1-\sqrt{5}}{1-\sqrt{5}} = \frac{-1+\sqrt{5}}{2}$.

So $f(x)$ can be written as

$$f(x) = \frac{1}{1-x-x^2} = \frac{-1}{(x-\frac{1}{\phi})(x+\phi)} = \frac{A}{x-\frac{1}{\phi}} + \frac{B}{x+\phi}$$

The Heaviside trick will tell you what A and B are, namely,

$$A = \frac{-1}{\frac{1}{\phi}+\phi} = \frac{-1}{\sqrt{5}}, \qquad B = \frac{1}{\frac{1}{\phi}+\phi} = \frac{1}{\sqrt{5}}$$

The nth derivative of $f(x)$ is

$$f^{(n)}(x) = \frac{A(-1)^n n!}{\left(x-\frac{1}{\phi}\right)^{n+1}} + \frac{B(-1)^n n!}{(x+\phi)^{n+1}}$$

Setting $x = 0$ and dividing by $n!$ finally gives you the coefficient of x^n in the Taylor series of $f(x)$. The result is the following formula for the nth Fibonacci number

$$c_n = \frac{f^{(n)}(0)}{n!} = \frac{1}{n!}\frac{A(-1)^n n!}{\left(-\frac{1}{\phi}\right)^{n+1}} + \frac{1}{n!}\frac{B(-1)^n n!}{(\phi)^{n+1}} = -A\phi^{n+1} - B\left(\frac{1}{\phi}\right)^{n+1}$$

Using the values for A and B you find

$$\boxed{f_n = c_n = \frac{1}{\sqrt{5}}\left\{\phi^{n+1} - \frac{1}{\phi^{n+1}}\right\}} \tag{11.8}$$

11.6.11 Differentiating Taylor polynomials

If

$$T_n f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$$

is the Taylor polynomial of a function $y = f(x)$, then what is the Taylor polynomial of its derivative $f'(x)$?

Theorem 11.6.3. The Taylor polynomial of degree $n-1$ of $f'(x)$ is given by

$$T_{n-1}\{f'(x)\} = a_1 + 2a_2 x + \cdots + n a_n x^{n-1}.$$

In other words, "the Taylor polynomial of the derivative is the derivative of the Taylor polynomial."

Proof. Let $g(x) = f'(x)$. Then $g^{(k)}(0) = f^{(k+1)}(0)$, so that

$$\begin{aligned} T_{n-1}g(x) &= g(0) + g'(0)x + g^{(2)}(0)\frac{x^2}{2!} + \cdots + g^{(n-1)}(0)\frac{x^{n-1}}{(n-1)!} \\ &= f'(0) + f^{(2)}(0)x + f^{(3)}(0)\frac{x^2}{2!} + \cdots + f^{(n)}(0)\frac{x^{n-1}}{(n-1)!} \qquad (\$) \end{aligned}$$

On the other hand, if $T_n f(x) = a_0 + a_1 x + \cdots + a_n x^n$, then $a_k = f^{(k)}(0)/k!$, so that

$$k a_k = \frac{k}{k!} f^{(k)}(0) = \frac{f^{(k)}(0)}{(k-1)!}.$$

In other words,

$$1 \cdot a_1 = f'(0),\ 2a_2 = f^{(2)}(0),\ 3a_3 = \frac{f^{(3)}(0)}{2!},\ \text{etc.}$$

So, continuing from ($) you find that

$$T_{n-1}\{f'(x)\} = T_{n-1}g(x) = a_1 + 2a_2x + \cdots + na_nx^{n-1}$$

as claimed. □

11.6.12 Example

We compute the Taylor polynomial of $f(x) = 1/(1-x)^2$ by noting that

$$f(x) = F'(x),\ \text{where } F(x) = \frac{1}{1-x}.$$

Since

$$T_{n+1}F(x) = 1 + x + x^2 + x^3 + \cdots + x^{n+1},$$

theorem 11.6.3 implies that

$$T_n\left\{\frac{1}{(1-x)^2}\right\} = 1 + 2x + 3x^2 + 4x^3 + \cdots + (n+1)x^n$$

11.6.13 Example

[Example: Taylor polynomials of $\arctan x$.] Let $f(x) = \arctan x$. Then know that

$$f'(x) = \frac{1}{1+x^2}.$$

By substitution of $t = -x^2$ in the Taylor polynomial of $1/(1-t)$ we had found

$$T_{2n}\{f'(x)\} = T_{2n}\left\{\frac{1}{1+x^2}\right\} = 1 - x^2 + x^4 - x^6 + \cdots + (-1)^n x^{2n}.$$

This Taylor polynomial must be the derivative of $T_{2n+1}f(x)$, so we have

$$T_{2n+1}\{\arctan x\} = x - \frac{x^3}{3} + \frac{x^5}{5} + \cdots + (-1)^n \frac{x^{2n+1}}{2n+1}.$$

11.7 The limit $n \to \infty$, keeping x fixed

11.7.1 Sequences and their limits

We shall call a ***sequence*** any ordered sequence of numbers $a_1, a_2, a_3, \ldots$: for each positive integer n we have to specify a number a_n.

11.7.2 Examples of sequences

definition	**first few number in the sequence**
$\downarrow$	$\downarrow$
$a_n = n$	$1, 2, 3, 4, \ldots$
$b_n = 0$	$0, 0, 0, 0, \ldots$
$c_n = \dfrac{1}{n}$	$\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$
$d_n = \left(-\frac{1}{3}\right)^n$	$-\frac{1}{3}, \frac{1}{9}, -\frac{1}{27}, \frac{1}{81}, \ldots$
$E_n = 1 + \dfrac{1}{1!} + \dfrac{1}{2!} + \dfrac{1}{3!} + \cdots + \dfrac{1}{n!}$	$1, 2, 2\frac{1}{2}, 2\frac{2}{3}, 2\frac{17}{24}, 2\frac{43}{60}, \ldots$
$S_n = T_{2n+1}\{\sin x\} = x - \dfrac{x^3}{3!} + \cdots + (-1)^n \dfrac{x^{2n+1}}{(2n+1)!}$	$x, x - \dfrac{x^3}{3!}, x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!}, \ldots$

The last two sequences are derived from the Taylor polynomials of e^x (at $x = 1$) and $\sin x$ (at any x). The last example S_n really is a sequence of functions, i.e. for every choice of x you get a different sequence.

Definition 11.7.1. A sequence of numbers $(a_n)_{n=1}^{\infty}$ converges to a limit L, if for every $\epsilon > 0$ there is a number N_ϵ such that for all $n > N_\epsilon$ one has

$$|a_n - L| < \epsilon.$$

One writes

$$\lim_{n\to\infty} a_n = L$$

11.7.3 Example: $\lim_{n\to\infty} \dfrac{1}{n} = 0$

The sequence $c_n = 1/n$ converges to 0. To prove this let $\epsilon > 0$ be given. We have to find an N_ϵ such that

$$|c_n| < \epsilon \text{ for all } n > N_\epsilon.$$

The c_n are all positive, so $|c_n| = c_n$, and hence

$$|c_n| < \epsilon \iff \frac{1}{n} < \epsilon \iff n > \frac{1}{\epsilon},$$

which prompts us to choose $N_\epsilon = 1/\epsilon$. The calculation we just did shows that if $n > \frac{1}{\epsilon} = N_\epsilon$, then $|c_n| < \epsilon$. That means that $\lim_{n\to\infty} c_n = 0$.

11.7.4 Example: $\lim_{n\to\infty} a^n = 0$ if $|a| < 1$

As in the previous example one can show that $\lim_{n\to\infty} 2^{-n} = 0$, and more generally, that for any constant a with $-1 < a < 1$ one has

$$\lim_{n\to\infty} a^n = 0.$$

Indeed,

$$|a^n| = |a|^n = e^{n \ln |a|} < \epsilon$$

holds if and only if

$$n \ln |a| < \ln \epsilon.$$

Since $|a| < 1$ we have $\ln |a| < 0$ so that dividing by $\ln |a|$ reverses the inequality, with result

$$|a^n| < \epsilon \iff n > \frac{\ln \epsilon}{\ln |a|}$$

The choice $N_\epsilon = (\ln \epsilon)/(\ln |a|)$ therefore guarantees that $|a^n| < \epsilon$ whenever $n > N_\epsilon$.

One can show that the operation of taking limits of sequences obeys the same rules as taking limits of functions.

Theorem 11.7.1. If

$$\lim_{n\to\infty} a_n = A \text{ and } \lim_{n\to\infty} b_n = B,$$

then one has

$$\begin{aligned}
\lim_{n\to\infty} a_n \pm b_n &= A \pm B \\
\lim_{n\to\infty} a_n b_n &= AB \\
\lim_{n\to\infty} \frac{a_n}{b_n} &= \frac{A}{B} \qquad \text{(assuming } B \neq 0\text{)}.
\end{aligned}$$

The so-called "sandwich theorem" for ordinary limits also applies to limits of sequences. Namely, one has

Theorem 11.7.2 ("Sandwich theorem")**.** If a_n is a sequence which satisfies $b_n < a_n < c_n$ for all n, and if $\lim_{n\to\infty} b_n = \lim_{n\to\infty} c_n = 0$, then $\lim_{n\to\infty} a_n = 0$.

Finally, one can show this:

Theorem 11.7.3. If $f(x)$ is a function which is continuous at $x = A$, and a_n is a sequence which converges to A, then

$$\lim_{n\to\infty} f(a_n) = f\left(\lim_{n\to\infty} a_n\right) = f(A).$$

11.7.5 Example

Since $\lim_{n\to\infty} 1/n = 0$ and since $f(x) = \cos x$ is continuous at $x = 0$ we have

$$\lim_{n\to\infty} \cos \frac{1}{n} = \cos 0 = 1.$$

11.7.6 Example

You can compute the limit of any rational function of n by dividing numerator and denominator by the highest occurring power of n. Here is an example:

$$\lim_{n\to\infty} \frac{2n^2 - 1}{n^2 + 3n} = \lim_{n\to\infty} \frac{2 - \left(\frac{1}{n}\right)^2}{1 + 3 \cdot \frac{1}{n}} = \frac{2 - 0^2}{1 + 3 \cdot 0^2} = 2$$

11.7.7 Example

[Application of the Sandwich theorem.] We show that $\lim_{n\to\infty} \frac{1}{\sqrt{n^2+1}} = 0$ in two different ways.

Method 1: Since $\sqrt{n^2+1} > \sqrt{n^2} = n$ we have

$$0 < \frac{1}{\sqrt{n^2+1}} < \frac{1}{n}.$$

The sequences "0" and $\frac{1}{n}$ both go to zero, so the Sandwich theorem implies that $1/\sqrt{n^2+1}$ also goes to zero.

Method 2: Divide numerator and denominator both by n to get

$$a_n = \frac{1/n}{\sqrt{1+(1/n)^2}} = f\left(\frac{1}{n}\right), \qquad \text{where } f(x) = \frac{x}{\sqrt{1+x^2}}.$$

Since $f(x)$ is continuous at $x = 0$, and since $\frac{1}{n} \to 0$ as $n \to \infty$, we conclude that a_n converges to 0.

11.7.8 Example: $\lim_{n\to\infty} \frac{x^n}{n!} = 0$ for any real number x

If $|x| \le 1$ then this is easy, for we would have $|x^n| \le 1$ for all $n \ge 0$ and thus

$$\left|\frac{x^n}{n!}\right| \le \frac{1}{n!} = \frac{1}{1\cdot \underbrace{2\cdot 3\cdots(n-1)\cdot n}_{n-1 \text{ factors}}} \le \frac{1}{1\cdot \underbrace{2\cdot 2\cdots 2\cdot 2}_{n-1 \text{ factors}}} = \frac{1}{2^{n-1}}$$

which shows that $\lim_{n\to\infty} \frac{x^n}{n!} = 0$, by the Sandwich Theorem.

For arbitrary x you first choose an integer $N \ge 2x$. Then for all $n \ge N$ one has

$$\begin{aligned} \frac{x^n}{n!} &\le \frac{|x|\cdot|x|\cdots|x|\cdot|x|}{1\cdot 2\cdot 3\cdots n} && \text{use } |x| \le \frac{N}{2} \\ &\le \frac{N\cdot N\cdot N\cdots N\cdot N}{1\cdot 2\cdot 3\cdots n}\left(\frac{1}{2}\right)^n \end{aligned}$$

Split fraction into two parts, one containing the first N factors from both numerator and denominator, the other the remaining factors:

$$\underbrace{\frac{N}{1}\cdot\frac{N}{2}\cdot\frac{N}{3}\cdots\frac{N}{N}}_{=N^N/N!}\cdot\frac{N}{N+1}\cdots\frac{N}{n} = \frac{N^N}{N!}\cdot\underbrace{\frac{N}{N+1}}_{<1}\cdot\underbrace{\frac{N}{N+2}}_{<1}\cdots\underbrace{\frac{N}{n}}_{<1} \le \frac{N^N}{N!}$$

Hence we have

$$\left|\frac{x^n}{n!}\right| \le \frac{N^N}{N!}\left(\frac{1}{2}\right)^n$$

if $2|x| \le N$ and $n \ge N$.

Here everything is independent of n, except for the last factor $(\frac{1}{2})^n$ which causes the whole thing to converge to zero as $n \to \infty$.

11.8 Convergence of Taylor Series

Definition 11.8.1. Let $y = f(x)$ be some function defined on an interval $a < x < b$ containing 0. We say the Taylor series $T_\infty f(x)$ converges to $f(x)$ for a given x if

$$\lim_{n\to\infty} T_n f(x) = f(x).$$

The most common notations which express this condition are

$$f(x) = \sum_{k=0}^{\infty} f^{(k)}(0)\frac{x^k}{k!}$$

or

$$f(x) = f(0) + f'(0)x + f''(0)\frac{x^2}{2!} + f^{(3)}(0)\frac{x^3}{3!} + \cdots$$

In both cases convergence justifies the idea that you can add infinitely many terms, as suggested by both notations.

There is no easy and general criterion which you could apply to a given function $f(x)$ that would tell you if its Taylor series converges for any particular x (except $x = 0$ – what does the Taylor series look like when you set $x = 0$?). On the other hand, it turns out that for many functions the Taylor series does converge to $f(x)$ for all x in some interval $-\rho < x < \rho$. In this section we will check this for two examples: the "geometric series" and the exponential function.

Before we do the examples I want to make this point about how we're going to prove that the Taylor series converges: Instead of taking the limit of the $T_n f(x)$ as $n \to \infty$, you are usually better off looking at the remainder term. Since $T_n f(x) = f(x) - R_n f(x)$ you have

$$\lim_{n\to\infty} T_n f(x) = f(x) \iff \lim_{n\to\infty} R_n f(x) = 0$$

So: to check that the Taylor series of $f(x)$ converges to $f(x)$ we must show that the remainder term $R_n f(x)$ goes to zero as $n \to \infty$.

11.8.1 Example: The Geometric series converges for $-1 < x < 1$

If $f(x) = 1/(1-x)$ then by the formula for the Geometric Sum you have

$$\begin{aligned} f(x) &= \frac{1}{1-x} \\ &= \frac{1 - x^{n+1} + x^{n+1}}{1-x} \\ &= 1 + x + x^2 + \cdots + x^n + \frac{x^{n+1}}{1-x} \\ &= T_n f(x) + \frac{x^{n+1}}{1-x}. \end{aligned}$$

We are not dividing by zero since $|x| < 1$ so that $1 - x \neq 0$. The remainder term is

$$R_n f(x) = \frac{x^{n+1}}{1-x}.$$

Since $|x| < 1$ we have

$$\lim_{n\to\infty} |R_n f(x)| = \lim_{n\to\infty} \frac{|x|^{n+1}}{|1-x|} = \frac{\lim_{n\to\infty} |x|^{n+1}}{|1-x|} = \frac{0}{|1-x|} = 0.$$

Thus we have shown that the series converges for all $-1 < x < 1$, i.e.

$$\boxed{\frac{1}{1-x} = \lim_{n\to\infty} \left\{1 + x + x^2 + \cdots + x^n\right\} = 1 + x + x^2 + x^3 + \cdots}$$

11.8.2 Convergence of the exponential Taylor series

Let $f(x) = e^x$. It turns out the Taylor series of e^x converges to e^x for every value of x. Here's why: we had found that

$$T_n e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!},$$

and by Lagrange's formula the remainder is given by

$$R_n e^x = e^{\xi} \frac{x^{n+1}}{(n+1)!},$$

where ξ is some number between 0 and x.

If $x > 0$ then $0 < \xi < x$ so that $e^{\xi} \le e^x$; if $x < 0$ then $x < \xi < 0$ implies that $e^{\xi} < e^0 = 1$. Either way one has $e^{\xi} \le e^{|x|}$, and thus

$$|R_n e^x| \le e^{|x|} \frac{|x|^{n+1}}{(n+1)!}.$$

We have shown before that $\lim_{n\to\infty} x^{n+1}/(n+1)! = 0$, so the Sandwich theorem again implies that $\lim_{n\to\infty} |R_n e^x| = 0$.

Conclusion:

$$\boxed{e^x = \lim_{n\to\infty} \left\{1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!}\right\} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots}$$

Do Taylor series always converge? And if the series of some function $y = f(x)$ converges, must it then converge to $f(x)$? Although the Taylor series of most function we run into converge to the functions itself, the following example shows that it doesn't have to be so.

11.8.3 The day that all Chemistry stood still

The rate at which a chemical reaction "A→B" proceeds depends among other things on the temperature at which the reaction is taking place. This dependence is described by the ***Arrhenius law*** which states that the rate at which a reaction takes place is proportional to

$$f(T) = e^{-\frac{\Delta E}{kT}}$$

where ΔE is the amount of energy involved in each reaction, k is Boltzmann's constant, and T is the temperature in degrees Kelvin. If you ignore the constants ΔE and k (i.e. if you set them equal to one by choosing the right units) then the reaction rate is proportional to

$$f(T) = e^{-1/T}.$$

If you have to deal with reactions at low temperatures you might be inclined to replace this function with its Taylor series at $T = 0$, or at least the first non-zero term in this series. If you were to do this you'd be in for a surprise. To see what happens, let's look at the following function,

$$f(x) = \begin{cases} e^{-1/x} & x > 0 \\ 0 & x \le 0 \end{cases}$$

This function goes to zero ***very*** quickly as $x \to 0$. In fact one has

$$\lim_{x \searrow 0} \frac{f(x)}{x^n} = \lim_{x \searrow 0} \frac{e^{-1/x}}{x^n} = \lim_{t \to \infty} t^n e^{-t} = 0. \qquad \text{(set } t = 1/x\text{)}$$

This implies

$$f(x) = o(x^n) \quad (x \to 0)$$

for any $n = 1, 2, 3 \ldots$. As $x \to 0$, this function vanishes faster than any power of x.

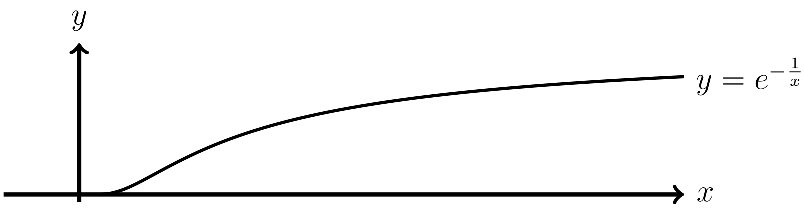


Figure 11.5: An innocent looking function with an unexpected Taylor series. The Taylor series at $x = 0$ does not converge to $f(0)$. See example 11.8.3 which shows that even when a Taylor series of some function f converges you can't be sure that it converges to f – it could converge to a different function.

If you try to compute the Taylor series of f you need its derivatives at $x = 0$ of all orders. These can be computed (not easily), and the result turns out to be that ***all derivatives of*** f ***vanish at*** $x = 0$***,***

$$f(0) = f'(0) = f''(0) = f^{(3)}(0) = \cdots = 0.$$

The Taylor series of f is therefore

$$T_\infty f(x) = 0 + 0 \cdot x + 0 \cdot \frac{x^2}{2!} + 0 \cdot \frac{x^3}{3!} + \cdots = 0.$$

Clearly this series converges (all terms are zero, after all), but instead of converging to the function $f(x)$ we started with, it converges to the function $g(x) = 0$.

What does this mean for the chemical reaction rates and Arrhenius' law? We wanted to "simplify" the Arrhenius law by computing the Taylor series of $f(T)$ at $T = 0$, but we have just seen that all terms in this series are zero. Therefore replacing the Arrhenius reaction rate by its Taylor series at $T = 0$ has the effect of setting all reaction rates equal to zero.

11.9 Leibniz' formulas for $\ln 2$ and $\pi/4$

Leibniz showed that

$$\frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots = \ln 2$$

and

$$\frac{1}{1}-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\cdots=\frac{\pi}{4}$$

Both formulas arise by setting $x=1$ in the Taylor series for

$$\begin{aligned}\ln(1+x)&=x-\frac{x^2}{2}+\frac{x^3}{3}+\frac{x^4}{4}-\cdots\\ \arctan x&=x-\frac{x^3}{3}+\frac{x^5}{5}+\frac{x^7}{7}-\cdots\end{aligned}$$

This is only justified if you show that the series actually converge, which we'll do here, at least for the first of these two formulas. The proof of the second is similar. The following is not Leibniz' original proof.

You begin with the geometric sum

$$1-x+x^2-x^3+\cdots+(-1)^n x^n=\frac{1}{1+x}+\frac{(-1)^{n+1}x^{n+1}}{1+x}$$

Then you integrate both sides from $x=0$ to $x=1$ and get

$$\begin{aligned}\frac{1}{1}-\frac{1}{2}+\frac{1}{3}-\cdots+(-1)^n\frac{1}{n+1}&=\int_0^1\frac{\mathrm{d}x}{1+x}+(-1)^{n+1}\int_0^1\frac{x^{n+1}\mathrm{d}x}{1+x}\\ &=\ln 2+(-1)^{n+1}\int_0^1\frac{x^{n+1}\mathrm{d}x}{1+x}\end{aligned}$$

(Use $\int_0^1 x^k\mathrm{d}x=\frac{1}{k+1}$.) Instead of computing the last integral you estimate it by saying

$$0\le\frac{x^{n+1}}{1+x}\le x^{n+1}\implies 0\le\int_0^1\frac{x^{n+1}\mathrm{d}x}{1+x}\le\int_0^1 x^{n+1}\mathrm{d}x=\frac{1}{n+2}$$

Hence

$$\lim_{n\to\infty}(-1)^{n+1}\int_0^1\frac{x^{n+1}\mathrm{d}x}{1+x}=0,$$

and we get

$$\begin{aligned}\lim_{n\to\infty}\frac{1}{1}-\frac{1}{2}+\frac{1}{3}-\cdots+(-1)^n\frac{1}{n+1}&=\ln 2+\lim_{n\to\infty}(-1)^{n+1}\int_0^1\frac{x^{n+1}\mathrm{d}x}{1+x}\\ &=\ln 2.\end{aligned}$$

Euler proved that

$$\frac{\pi^2}{6}=1+\frac{1}{4}+\frac{1}{9}+\cdots\frac{1}{n^2}+\cdots$$

11.10 Proof of Lagrange's formula

For simplicity assume $x>0$. Consider the function

$$g(t)=f(0)+f'(0)t+\frac{f''(0)}{2}t^2+\cdots+\frac{f^{(n)}(0)}{n!}t^n+Kt^{n+1}-f(t),$$

where

$$K\stackrel{\text{def}}{=}-\frac{f(0)+f'(0)x+\cdots+\frac{f^{(n)}(0)}{n!}x^n-f(x)}{x^{n+1}}\tag{11.9}$$

We have chosen this particular K to be sure that

$$g(x) = 0.$$

Just by computing the derivatives you also find that

$$g(0) = g'(0) = g''(0) = \cdots = g^{(n)}(0) = 0,$$

while

$$g^{(n+1)}(t) = (n+1)!K - f^{(n+1)}(t). \tag{11.10}$$

We now apply *Rolle's Theorem* n times:

- since $g(t)$ vanishes at $t = 0$ and at $t = x$ there exists an x_1 with $0 < x_1 < x$ such that $g'(x_1) = 0$
- since $g'(t)$ vanishes at $t = 0$ and at $t = x_1$ there exists an x_2 with $0 < x_2 < x_1$ such that $g'(x_2) = 0$
- since $g''(t)$ vanishes at $t = 0$ and at $t = x_2$ there exists an x_3 with $0 < x_3 < x_2$ such that $g''(x_3) = 0$

  $\vdots$
- since $g^{(n)}(t)$ vanishes at $t = 0$ and at $t = x_n$ there exists an x_{n+1} with $0 < x_{n+1} < x_n$ such that $g^{(n)}(x_{n+1}) = 0$.

We now set $\xi = x_{n+1}$, and observe that we have shown that $g^{(n+1)}(\xi) = 0$, so by (11.10) we get

$$K = \frac{f^{(n+1)}(\xi)}{(n+1)!}.$$

Apply that to (11.9) and you finally get

$$f(x) = f(0) + f'(0)x + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \frac{f^{(n+1)}(\xi)}{(n+1)!}x^{n+1}.$$

11.11 Proof of Theorem 11.6.2

Lemma. If $h(x)$ is a k times differentiable function on some interval containing 0, and if for some integer $k < n$ one has $h(0) = h'(0) = \cdots = h^{(k-1)}(0) = 0$, then

$$\lim_{x\to 0} \frac{h(x)}{x^k} = \frac{h^{(k)}(0)}{k!}. \tag{11.11}$$

Proof. Just apply l'Hopital's rule k times. You get

$$\lim_{x\to 0} \frac{h(x)}{x^k} \overset{=\frac{0}{0}}{=} \lim_{x\to 0} \frac{h'(x)}{kx^{k-1}} \overset{=\frac{0}{0}}{=} \lim_{x\to 0} \frac{h^{(2)}(x)}{k(k-1)x^{k-2}} \overset{=\frac{0}{0}}{=} \cdots$$

$$\cdots = \lim_{x\to 0} \frac{h^{(k-1)}(x)}{k(k-1)\cdots 2x^1} \overset{=\frac{0}{0}}{=} \frac{h^{(k)}(0)}{k(k-1)\cdots 2\cdot 1}$$

□

First define the function $h(x) = f(x) - g(x)$. If $f(x)$ and $g(x)$ are n times differentiable, then so is $h(x)$.

The condition $T_n f(x) = T_n g(x)$ means that

$$f(0) = g(0), \quad f'(0) = g'(0), \quad \ldots, \quad f^{(n)}(0) = g^{(n)}(0),$$

which says, in terms of $h(x)$,

$$(\dagger) \qquad h(0) = h'(0) = h''(0) = \cdots = h^{(n)}(0) = 0,$$

i.e.

$$T_n h(x) = 0.$$

We now prove the first pat of the theorem: suppose $f(x)$ and $g(x)$ have the same nth degree Taylor polynomial. Then we have just argued that $T_n h(x) = 0$, and Lemma 11.11 (with $k = n$) says that $\lim_{x\to 0} h(x)/x^n = 0$, as claimed.

To conclude we show the converse also holds. So suppose that $\lim_{x\to 0} h(x)/x^n = 0$. We'll show that ($\dagger$) follows. If ($\dagger$) were not true then there would be a smallest integer $k \le n$ such that

$$h(0) = h'(0) = h''(0) = \cdots = h^{(k-1)}(0) = 0, \text{ but } h^{(k)}(0) \neq 0.$$

This runs into the following contradiction with Lemma 11.11

$$0 \neq \frac{h^{(k)}(0)}{k!} = \lim_{x\to 0} \frac{h(x)}{x^k} = \lim_{x\to 0} \frac{h(x)}{x^n} \cdot \frac{x^n}{x^k} = 0 \cdot \underbrace{\lim_{x\to 0} x^{n-k}}_{(*)} = 0.$$

Here the limit $(*)$ exists because $n \ge k$.

11.12 PROBLEMS

TAYLOR'S FORMULA

687. Find a second order polynomial (i.e. a quadratic function) $Q(x)$ such that $Q(7) = 43, Q'(7) = 19, Q''(7) = 11$.

†390

688. Find a second order polynomial $p(x)$ such that $p(2) = 3$, $p'(2) = 8$, and $p''(2) = -1$.

†390

689. A Third order polynomial $P(x)$ satisfies $P(0) = 1, P'(0) = -3, P''(0) = -8, P'''(0) = 24$. Find $P(x)$.

690. Let $f(x) = \sqrt{x + 25}$. Find the polynomial $P(x)$ of degree three such that $P^{(k)}(0) = f^{(k)}(0)$ for $k = 0, 1, 2, 3$.

691. Let $f(x) = 1 + x - x^2 - x^3$. Compute and graph $T_0 f(x)$, $T_1 f(x)$, $T_2 f(x)$, $T_3 f(x)$, and $T_4 f(x)$, as well as $f(x)$ itself (so, for each of these functions find where they are positive or negative, where they are increasing/decreasing, and find the inflection points on their graph.)

692. Find $T_3 \sin x$ and $T_5 \sin x$.

Graph $T_3 \sin x$ and $T_5 \sin x$ as well as $y = \sin x$ in one picture. (As before, find where these functions are positive or negative, where they are increasing/decreasing, and find the inflection points on their graph. This problem can&should be done without a graphing calculator.)

Compute $T_0^a f(x)$, $T_1^a f(x)$ and $T_2^a f(x)$ for the following functions.

693. $f(x) = x^3$, $a = 0$; then for $a = 1$ and $a = 2$.

694. $f(x) = \dfrac{1}{x}$, $a = 1$. Also do $a = 2$.

695. $f(x) = \sqrt{x}$, $a = 1$.

696. $f(x) = \ln x$, $a = 1$. Also $a = e^2$.

697. $f(x) = \ln \sqrt{x}$, $a = 1$.

698. $f(x) = \sin(2x)$, $a = 0$, also $a = \pi/4$.

699. $f(x) = \cos(x)$, $a = \pi$.

700. $f(x) = (x-1)^2$, $a = 0$, and also $a = 1$.

701. $f(x) = \dfrac{1}{e^x}$, $a = 0$.

702. Find the nth degree Taylor polynomial $T_n^a f(x)$ of the following functions $f(x)$

n	a	$f(x)$
2	0	$1 + x - x^3$
3	0	$1 + x - x^3$
25	0	$1 + x - x^3$
25	2	$1 + x - x^3$
2	1	$1 + x - x^3$
1	1	x^2
2	1	x^2
5	1	$1/x$
5	0	$1/(1+x)$
3	0	$1/(1-3x+2x^2)$

For which of these combinations $(n, a, f(x))$ is $T_n^a f(x)$ the same as $f(x)$?

* * *

Compute the Taylor series $T_\infty f(t)$ for the following functions (α is a constant). Give a formula for the coefficient of x^n in $T_\infty f(t)$. (*Be smart. Remember properties of the logarithm, definitions of the hyperbolic functions, partial fraction decomposition.*)

703. e^t †390

704. $e^{\alpha t}$ †390

705. $\sin(3t)$ †390

706. $\sinh t$ †390

707. $\cosh t$ †390

708. $\dfrac{1}{1+2t}$ †390

709. $\dfrac{3}{(2-t)^2}$ †391

710. $\ln(1+t)$ †391

711. $\ln(2+2t)$ †391

712. $\ln \sqrt{1+t}$ †391

713. $\ln(1+2t)$ †391

714. $\ln \sqrt{\dfrac{1+t}{1-t}}$ †391

715. $\dfrac{1}{1-t^2}$ [hint:PFD!] †391

716. $\dfrac{t}{1-t^2}$ †391

717. $\sin t + \cos t$ †391

718. $2 \sin t \cos t$ †391

719. $\tan t$ (3 terms only) †391

720. $1 + t^2 - \dfrac{2}{3}t^4$ †391

721. $(1+t)^5$ †391

722. $\sqrt[3]{1+t}$ †391

723. $f(x) = \frac{x^4}{1+4x^2}$, what is $f^{(10)}(0)$? †391

724. Compute the Taylor series of the following two functions

$$f(x) = \sin a \cos x + \cos a \sin x$$

and

$$g(x) = \sin(a + x)$$

where a is a constant. †391

725. Compute the Taylor series of the following two functions

$$h(x) = \cos a \cos x - \sin a \sin x$$

and

$$k(x) = \cos(a + x)$$

where a is a constant.

726. The following questions ask you to rediscover ***Newton's Binomial Formula***, which is just the Taylor series for $(1+x)^n$. Newton's formula generalizes the formulas for $(a+b)^2$, $(a+b)^3$, etc that you get using

Pascal's triangle. It allows non integer exponents which are allowed to be either positive and negative. Reread section 11.3 before doing this problem.

(a) Find the Taylor series of $f(x) = \sqrt{1+x}$ $(= (1+x)^{1/2})$

(b) Find the coefficient of x^4 in the Taylor series of $f(x) = (1+x)^\pi$ (don't do the arithmetic!)

(c) Let p be any real number. Compute the terms of degree 0, 1, 2 and 3 of the Taylor series of

$$f(x) = (1+x)^p$$

(d) Compute the Taylor polynomial of degree n of $f(x) = (1+x)^p$.

(e) Write the result of (d) for the exponents $p = 2, 3$ and also, for $p = -1, -2, -3$ and finally for $p = \frac{1}{2}$. The ***Binomial Theorem*** states that this series converges when $|x| < 1$.

LAGRANGE'S FORMULA FOR THE REMAINDER

727. Find the fourth degree Taylor polynomial $T_4\{\cos x\}$ for the function $f(x) = \cos x$ and estimate the error $|\cos x - P_4(x)|$ for $|x| < 1$.

†392

728. Find the 4th degree Taylor polynomial $T_4\{\sin x\}$ for the function $f(x) = \sin x$. Estimate the error $|\sin x - T_4\{\sin x\}|$ for $|x| < 1$.

729. (*Computing the cube root of* 9) The cube root of $8 = 2\times2\times2$ is easy, and 9 is only one more than 8. So you could try to compute $\sqrt[3]{9}$ by viewing it as $\sqrt[3]{8+1}$.

(a) Let $f(x) = \sqrt[3]{8+x}$. Find $T_2f(x)$, and estimate the error $|\sqrt[3]{9} - T_2f(1)|$.

(b) Repeat part (***i***) for "$n = 3$", i.e. compute $T_3f(x)$ and estimate $|\sqrt[3]{9} - T_3f(1)|$.

†392

730. Follow the method of problem 729 to compute $\sqrt{10}$:

(a) Use Taylor's formula with $f(x) = \sqrt{9+x}$, $n = 1$, to calculate $\sqrt{10}$ approximately. Show that the error is less than 1/216.

(b) Repeat with $n = 2$. Show that the error is less than 0.0003.

731. Find the eighth degree Taylor polynomial $T_8f(x)$ about the point 0 for the function $f(x) = \cos x$ and estimate the error $|\cos x - T_8f(x)|$ for $|x| < 1$.

Now find the ninth degree Taylor polynomial, and estimate $|\cos x - T_9f(x)|$ for $|x| \leq 1$.

LITTLE-OH AND MANIPULATING TAYLOR POLYNOMIALS

Are the following statements *True or False?* In mathematics this means that you should either *show that the statement always holds* or else *give at least one counterexample,* thereby showing that the statement is not always true.

732. $(1+x^2)^2 - 1 = o(x)$?

733. $(1+x^2)^2 - 1 = o(x^2)$?

734. $\sqrt{1+x} - \sqrt{1-x} = o(x)$?

735. $o(x) + o(x) = o(x)$?

736. $o(x) - o(x) = o(x)$?

737. $o(x) \cdot o(x) = o(x)$?

738. $o(x^2) + o(x) = o(x^2)$?

739. $o(x^2) - o(x^2) = o(x^3)$?

740. $o(2x) = o(x)$?

741. $o(x) + o(x^2) = o(x)$?

742. $o(x) + o(x^2) = o(x^2)$?

743. $1 - \cos x = o(x)$?

744. Define

$$f(x) = \begin{cases} e^{-1/x^2} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

This function goes to zero ***very*** quickly as $x \to 0$ but is 0 only at 0. Prove that $f(x) = o(x^n)$ for every n.

745. For which value(s) of k is $\sqrt{1+x^2} = 1 + o(x^k)$ (as $x \to 0$)?

For which value(s) of k is $\sqrt[3]{1+x^2} = 1 + o(x^k)$ (as $x \to 0$)?

For which value(s) of k is $1 - \cos x^2 = o(x^k)$ (as $x \to 0$)?

746. Let g_n be the coefficient of x^n in the Taylor series of the function

$$g(x) = \frac{1}{2 - 3\,x + x^2}$$

(a) Compute g_0 and g_1 directly from the definition of the Taylor series.

(b) Show that the recursion relation $g_n = 3g_{n-1} - 2g_{n-2}$ holds for all $n \geq 2$.

(c) Compute g_2, g_3, g_4, g_5.

(d) Using a partial fraction decomposition of $g(x)$ find a formula for $g^{(n)}(0)$, and hence for g_n. †393

747. Answer the same questions as in the previous problem, for the functions

$$h(x) = \frac{x}{2 - 3\,x + x^2}$$

and

$$k(x) = \frac{2 - x}{2 - 3\,x + x^2}.$$

†393

748. Let h_n be the coefficient of x^n in the Taylor series of

$$h(x) = \frac{1 + x}{2 - 5x + 2x^2}.$$

(a) Find a recursion relation for the h_n.

(b) Compute h_0, h_1, $\ldots$, h_8.

(c) Derive a formula for h_n valid for all n, by using a partial fraction expansion.

(d) Is h_{2009} more or less than a million? A billion?

Find the Taylor series for the following functions, by substituting, adding, multiplying, applying long division and/or differentiating known series for $\frac{1}{1+x}$, e^x, $\sin x$, $\cos x$ *and* $\ln x$.

749. e^{at} †393

750. e^{1+t} †393

751. e^{-t^2} †393

752. $\dfrac{1+t}{1-t}$ †393

753. $\dfrac{1}{1+2t}$ †393

754.

$$f(x) = \begin{cases} \frac{\sin(x)}{x} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$$

†393

755. $\dfrac{\ln(1+x)}{x}$ †394

756. $\dfrac{e^t}{1-t}$ †394

757. $\dfrac{1}{\sqrt{1-t}}$ †394

758. $\dfrac{1}{\sqrt{1-t^2}}$ (recommendation: use the answer to problem 757) †394

759. $\arcsin t$
(use problem 757 again) †394

760. Compute $T_4[e^{-t}\cos t]$ (See example 11.6.8.) †394

761. $T_4[e^{-t}\sin 2t]$ †394

762. $\dfrac{1}{2 - t - t^2}$ †394

763. $\sqrt[3]{1 + 2t + t^2}$ †394

764. $\ln(1 - t^2)$

765. $\sin t \cos t$

LIMITS OF SEQUENCES

Compute the following limits:

766. $\lim_{n\to\infty} \frac{n}{2n-3}$ †394

767. $\lim_{n\to\infty} \frac{n^2}{2n-3}$ †394

768. $\lim_{n\to\infty} \frac{n^2}{2n^2+n-3}$ †394

769. $\lim_{n\to\infty} \frac{2^n+1}{1-2^n}$ †394

770. $\lim_{n\to\infty} \frac{2^n+1}{1-3^n}$ †394

771. $\lim_{n\to\infty} \frac{e^n+1}{1-2^n}$ †394

772. $\lim_{n\to\infty} \frac{n^2}{(1.01)^n}$ †395

773. $\lim_{n\to\infty} \frac{1000^n}{n!}$ †395

774. $\lim_{n\to\infty} \frac{n!+1}{(n+1)!}$ †395

775. Compute $\lim_{n\to\infty} \frac{(n!)^2}{(2n)!}$ [Hint: write out all the factors in numerator and denominator.]

776. Let f_n be the nth Fibonacci number. Compute

$$\lim_{n\to\infty} \frac{f_n}{f_{n-1}}$$

†395

CONVERGENCE OF TAYLOR SERIES

777. Prove that the Taylor series for $f(x) = \cos x$ converges to $f(x)$ for all real numbers x (by showing that the remainder term goes to zero as $n \to \infty$). †395

778. Prove that the Taylor series for $g(x) = \sin(2x)$ converges to $g(x)$ for all real numbers x . †395

779. Prove that the Taylor series for $h(x) = \cosh(x)$ converges to $h(x)$ for all real numbers x .

780. Prove that the Taylor series for $k(x) = e^{2x+3}$ converges to $k(x)$ for all real numbers x .

781. Prove that the Taylor series for $\ell(x) = \cos\left(x - \frac{\pi}{7}\right)$ converges to $\ell(x)$ for all real numbers x.

782. If the Taylor series of a function $y = f(x)$ converges for all x, does it have to converge to $f(x)$, or could it converge to some other function? †395

783. For which real numbers x does the Taylor series of $f(x) = \frac{1}{1-x}$ converge to $f(x)$? †395

784. For which real numbers x does the Taylor series of $f(x) = \frac{1}{1-x^2}$ converge to $f(x)$? (hint: a substitution may help.) †395

785. For which real numbers x does the Taylor series of $f(x) = \frac{1}{1+x^2}$ converge to $f(x)$? †395

786. For which real numbers x does the Taylor series of $f(x) = \frac{1}{3+2x}$ converge to $f(x)$? †395

787. For which real numbers x does the Taylor series of $f(x) = \frac{1}{2-5x}$ converge to $f(x)$? †395

788. For which real numbers x does the Taylor series of $f(x) = \frac{1}{2-x-x^2}$ converge to $f(x)$? (hint: use PFD and the Geometric Series to find the remainder term.)

789. Show that the Taylor series for $f(x) = \ln(1+x)$ converges when $-1 < x < 1$ by integrating the Geometric Series

$$\frac{1}{1+t} = 1 - t + t^2 - t^3 + \cdots + (-1)^n t^n + (-1)^{n+1}\frac{t^{n+1}}{1+t}$$

from $t = 0$ to $t = x$. (See §11.9.)

790. Show that the Taylor series for $f(x) = e^{-x^2}$ converges for all real numbers x. (Set $t = -x^2$ in the Taylor series with remainder for e^t.)

791. Show that the Taylor series for $f(x) = \sin(x^4)$ converges for all real numbers x. (Set $t = x^4$ in the Taylor series with remainder for $\sin t$.)

792. Show that the Taylor series for $f(x) = 1/(1 + x^3)$ converges whenever $-1 < x < 1$ (Use the GEOMETRIC SERIES.)

793. For which x does the Taylor series of $f(x) = 2/(1 + 4x^2)$ converge? (Again, use the GEOMETRIC SERIES.)

794. The error function from statistics is defined by

$$\operatorname{erf}(x) = \frac{1}{\sqrt{\pi}}\int_0^x e^{-t^2/2}\,\mathrm{d}t$$

(a) Find the Taylor series of the error function from the Taylor series of $f(r) = e^r$ (set $r = -t^2/2$ and integrate).

(b) Estimate the error term and show that the Taylor series of the error function converges for all real x.

795. Prove Leibniz' formula for $\frac{\pi}{4}$ by mimicking the proof in section 11.9. Specifically, find a formula for the remainder in :

$$\frac{1}{1+t^2} = 1 - t^2 + \cdots + (-1)^n t^{2n} + R_{2n}(t)$$

and integrate this from $t = 0$ to $t = 1$.

APPROXIMATING INTEGRALS

796. (a) Compute $T_2\{\sin t\}$ and give an upper bound for $R_2\{\sin t\}$ for $0 \le t \le 0.5$

(b) Use part (a) to approximate $\int_0^{0.5} \sin(x^2)\,dx$, and give an upper bound for the error in your approximation.

†395

797. (a) Find the second degree Taylor polynomial for the function e^t.

(b) Use it to give an estimate for the integral

$$\int_0^1 e^{x^2}\,dx$$

(c) Suppose instead we used the 5th degree Taylor polynomial $p(t)$ for e^t to give an estimate for the integral:

$$\int_0^1 e^{x^2}\,dx$$

Give an upper bound for the error:

$$\left|\int_0^1 e^{x^2}\,dx - \int_0^1 p(x^2)\,dx\right|$$

Note: You need not find $p(t)$ or the integral $\int_0^1 p(x^2)\,dx$.

†396

798. Approximate $\int_0^{0.1} \arctan x\,dx$ and estimate the error in your approximation by analyzing $T_2 f(t)$ and $R_2 f(t)$ where $f(t) = \arctan t$.

799. Approximate $\int_0^{0.1} x^2 e^{-x^2}\,dx$ and estimate the error in your approximation by analyzing $T_3 f(t)$ and $R_3 f(t)$ where $f(t) = te^{-t}$.

800. Estimate $\int_0^{0.5} \sqrt{1 + x^4}\,dx$ with an error of less than 10^{-4}.

801. Estimate $\int_0^{0.1} \arctan x\,dx$ with an error of less than 0.001.

Chapter 12

Complex Numbers

12.1 Complex numbers

The equation $x^2 + 1 = 0$ has no solutions, because for any real number x the square x^2 is nonnegative, and so $x^2 + 1$ can never be less than 1. In spite of this it turns out to be very useful to ***assume*** that there is a number i for which one has

$$i^2 = -1. \tag{12.1}$$

Any ***complex number*** is then an expression of the form $a + bi$, where a and b are old-fashioned real numbers. The number a is called the ***real part*** of $a + bi$, and b is called its ***imaginary part.***

Traditionally the letters z and w are used to stand for complex numbers.

Since any complex number is specified by two real numbers one can visualize them by plotting a point with coordinates (a, b) in the plane for a complex number $a + bi$. The plane in which one plot these complex numbers is called the Complex plane, or Argand plane.

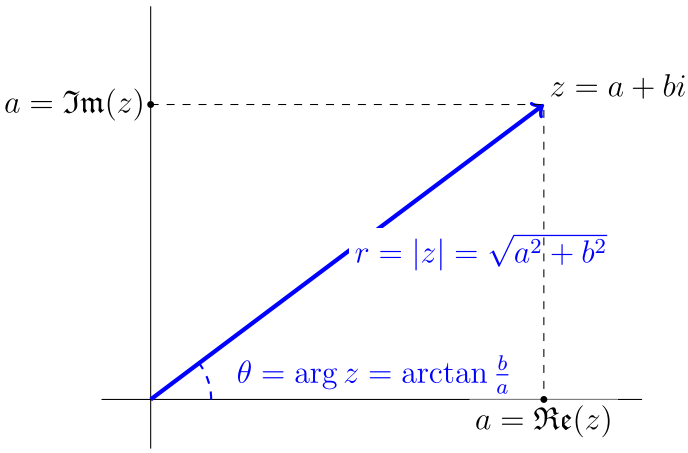


Figure 12.1: A complex number with cartesian representation (a, b) in black and polar representation (r, θ) in blue.

You can add, multiply and divide complex numbers. Here's how:

To add (subtract) $z = a + bi$ and $w = c + di$

$$z + w = (a + bi) + (c + di) = (a + c) + (b + d)i,$$
$$z - w = (a + bi) - (c + di) = (a - c) + (b - d)i.$$

To multiply z and w proceed as follows:

$$\begin{aligned} zw &= (a + bi)(c + di) \\ &= a(c + di) + bi(c + di) \\ &= ac + adi + bci + bdi^2 \\ &= (ac - bd) + (ad + bc)i \end{aligned}$$

where we have use the defining property $i^2 = -1$ to get rid of i^2.

To divide two complex numbers one always uses the following trick.

$$\begin{aligned} \frac{a + bi}{c + di} &= \frac{a + bi}{c + di} \cdot \frac{c - di}{c - di} \\ &= \frac{(a + bi)(c - di)}{(c + di)(c - di)} \end{aligned}$$

Now

$$(c + di)(c - di) = c^2 - (di)^2 = c^2 - d^2 i^2 = c^2 + d^2,$$

so

$$\begin{aligned} \frac{a + bi}{c + di} &= \frac{(ac + bd) + (bc - ad)i}{c^2 + d^2} \\ &= \frac{ac + bd}{c^2 + d^2} + \frac{bc - ad}{c^2 + d^2}\, i \end{aligned}$$

Obviously you do not want to memorize this formula: instead you remember the trick, i.e. to divide $c + di$ into $a + bi$ you multiply numerator and denominator with $c - di$.

For any complex number $w = c + di$ the number $c - di$ is called its ***complex conjugate.*** Notation:

$$w = c + di, \quad \bar{w} = c - di.$$

A frequently used property of the complex conjugate is the following formula

$$w\bar{w} = (c + di)(c - di) = c^2 - (di)^2 = c^2 + d^2. \tag{12.2}$$

The following notation is used for the ***real and imaginary parts*** of a complex number z. If $z = a + bi$ then

$$a = \text{the Real Part of } z = \mathfrak{Re}(z), \qquad b = \text{the Imaginary Part of } z = \mathfrak{Im}(z).$$

Note that both $\mathfrak{Re} z$ and $\mathfrak{Im} z$ are real numbers. A common mistake is to say that $\mathfrak{Im} z = bi$. The "i" should **not** be there.

12.2 Argument and Absolute Value

For any given complex number $z = a + bi$ one defines the ***absolute value*** or ***modulus*** to be

$$|z| = \sqrt{a^2 + b^2},$$

so $|z|$ is the distance from the origin to the point z in the complex plane (see figure 12.1).
The angle θ is called the ***argument*** of the complex number z. Notation:

$$\arg z = \theta.$$

The argument is defined in an ambiguous way: it is only defined up to a multiple of 2π. E.g. the argument of -1 could be π, or $-\pi$, or 3π, or, etc. In general one says $\arg(-1) = \pi + 2k\pi$, where k may be any integer.
From trigonometry one sees that for any complex number $z = a + bi$ one has

$$a = |z| \cos\theta, \text{ and } b = |z| \sin\theta,$$

so that

$$|z| = |z| \cos\theta + i|z| \sin\theta = |z|\big(\cos\theta + i \sin\theta\big).$$

and

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{b}{a}.$$

12.2.1 Example: Find argument and absolute value of $z = 2 + i$

Solution: $|z| = \sqrt{2^2 + 1^2} = \sqrt{5}$. z lies in the first quadrant so its argument θ is an angle between 0 and $\pi/2$. From $\tan\theta = \frac{1}{2}$ we then conclude $\arg(2 + i) = \theta = \arctan \dfrac{1}{2}$.

12.3 Geometry of Arithmetic

Since we can picture complex numbers as points in the complex plane, we can also try to visualize the arithmetic operations "addition" and "multiplication."

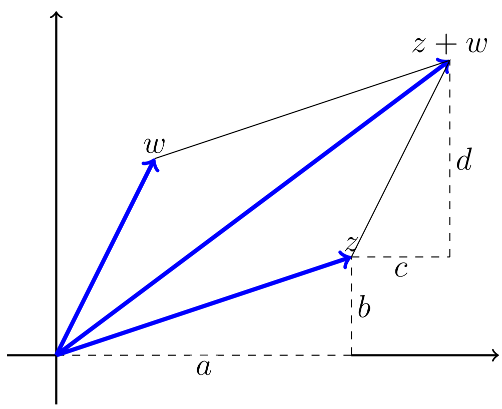


Figure 12.2: Addition of $z = a + bi$ and $w = c + di$

To add z and w one forms the parallelogram with the origin, z and w as vertices. The fourth vertex then is $z + w$. See figure 12.2.

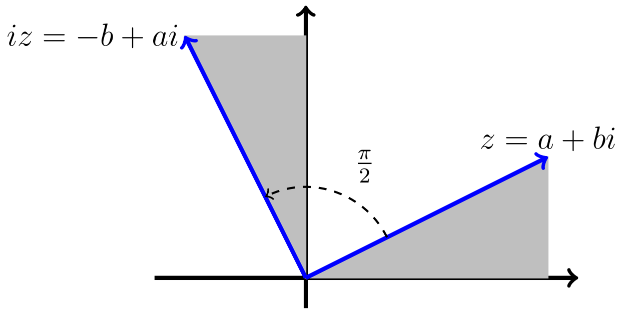


Figure 12.3: Multiplication of $a + bi$ by i.

To understand multiplication we first look at multiplication with i. If $z = a + bi$ then

$$iz = i(a + bi) = ia + bi^2 = ai - b = -b + ai.$$

Thus, to form iz from the complex number z one rotates z counterclockwise by 90 degrees. See figure 12.3.

If a is any real number, then multiplication of $w = c + di$ by a gives

$$aw = ac + adi,$$

so aw points in the same direction, but is a times as far away from the origin. If $a < 0$ then aw points in the opposite direction. See figure 12.4.

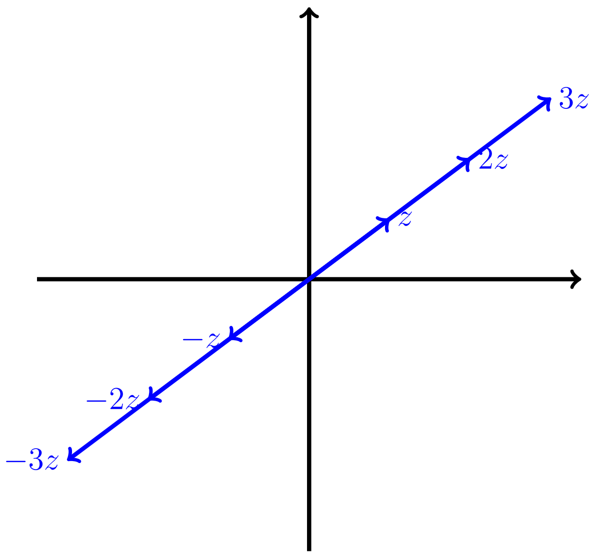


Figure 12.4: Multiplication of a real and a complex number

Next, to multiply $z = a + bi$ and $w = c + di$ we write the product as

$$zw = (a + bi)w = aw + biw.$$

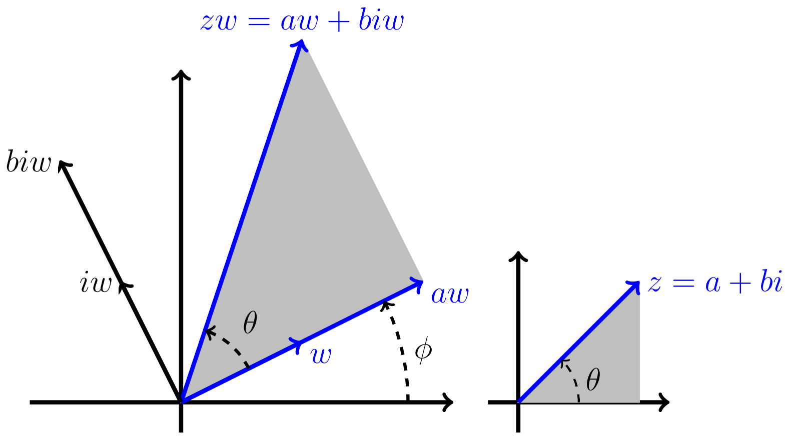


Figure 12.5: Multiplication of two complex numbers

Figure 12.5 shows $a+bi$ on the right. On the left, the complex number w was first drawn, then aw was drawn. Subsequently iw and biw were constructed, and finally $zw = aw + biw$ was drawn by adding aw and biw.

One sees from figure 12.5 that since iw is perpendicular to w, the line segment from 0 to biw is perpendicular to the segment from 0 to aw. Therefore the larger shaded triangle on the left is a right triangle. The length of the adjacent side is $a|w|$, and the length of the opposite side is $b|w|$. The ratio of these two lengths is $a : b$, which is the same as for the shaded right triangle on the right, so we conclude that these two triangles are similar.

The triangle on the left is $|w|$ times as large as the triangle on the right. The two angles marked θ are equal.

Since $|zw|$ is the length of the hypothenuse of the shaded triangle on the left, it is $|w|$ times the hypothenuse of the triangle on the right, i.e. $|zw| = |w| \cdot |z|$.

The argument of zw is the angle $\theta + \varphi$; since $\theta = \arg z$ and $\varphi = \arg w$ we get the following two formulas

$$|zw| = |z| \cdot |w| \tag{12.3}$$

$$\arg(zw) = \arg z + \arg w, \tag{12.4}$$

in other words,

when you multiply complex numbers, their lengths get multiplied and their arguments get added.

12.4 Applications in Trigonometry

12.4.1 Unit length complex numbers

For any θ the number $z = \cos\theta + i\sin\theta$ has length 1: it lies on the unit circle. Its argument is $\arg z = \theta$. Conversely, any complex number on the unit circle is of the form $\cos\phi + i\sin\phi$, where ϕ is its argument.

12.4.2 The Addition Formulas for Sine & Cosine

For any two angles θ and ϕ one can multiply $z = \cos\theta + i\sin\theta$ and $w = \cos\phi + i\sin\phi$. The product zw is a complex number of absolute value $|zw| = |z| \cdot |w| = 1 \cdot 1$, and with argument $\arg(zw) = \arg z + \arg w = \theta + \phi$. So zw lies on the unit circle and must be $\cos(\theta+\phi) + i\sin(\theta+\phi)$. Thus we have

$$(\cos\theta + i\sin\theta)(\cos\phi + i\sin\phi) = \cos(\theta+\phi) + i\sin(\theta+\phi). \tag{12.5}$$

By multiplying out the Left Hand Side we get

$$\begin{aligned}(\cos\theta + i\sin\theta)(\cos\phi + i\sin\phi) = \cos\theta\cos\phi - \sin\theta\sin\phi \\ + i(\sin\theta\cos\phi + \cos\theta\sin\phi).\end{aligned} \tag{12.6}$$

Compare the Right Hand Sides of (12.5) and (12.6), and you get the addition formulas for Sine and Cosine:

$$\begin{aligned}\cos(\theta+\phi) &= \cos\theta\cos\phi - \sin\theta\sin\phi \\ \sin(\theta+\phi) &= \sin\theta\cos\phi + \cos\theta\sin\phi\end{aligned}$$

12.4.3 De Moivre's formula

For any complex number z the argument of its square z^2 is $\arg(z^2) = \arg(z \cdot z) = \arg z + \arg z = 2\arg z$. The argument of its cube is $\arg z^3 = \arg(z \cdot z^2) = \arg(z) + \arg z^2 = \arg z + 2\arg z = 3\arg z$. Continuing like this one finds that

$$\arg z^n = n\ \arg z \tag{12.7}$$

for any integer n.

Applying this to $z = \cos\theta + i\sin\theta$ you find that z^n is a number with absolute value $|z^n| = |z|^n = 1^n = 1$, and argument $n\arg z = n\theta$. Hence $z^n = \cos n\theta + i\sin n\theta$. So we have found

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta. \tag{12.8}$$

This is ***de Moivre's formula.***

For instance, for $n = 2$ this tells us that

$$\cos 2\theta + i\sin 2\theta = (\cos\theta + i\sin\theta)^2 = \cos^2\theta - \sin^2\theta + 2i\cos\theta\sin\theta.$$

Comparing real and imaginary parts on left and right hand sides this gives you the double angle formulas $\cos 2\theta = \cos^2\theta - \sin^2\theta$ and $\sin 2\theta = 2\sin\theta\cos\theta$.

For $n = 3$ you get, using the *Binomial Theorem,* or Pascal's triangle,

$$\begin{aligned}(\cos\theta + i\sin\theta)^3 &= \cos^3\theta + 3i\cos^2\theta\sin\theta + 3i^2\cos\theta\sin^2\theta + i^3\sin^3\theta \\ &= \cos^3\theta - 3\cos\theta\sin^2\theta + i(3\cos^2\theta\sin\theta - \sin^3\theta)\end{aligned}$$

so that

$$\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$$

and

$$\sin 3\theta = \cos^2\theta\sin\theta - \sin^3\theta.$$

In this way it is fairly easy to write down similar formulas for $\sin 4\theta$, $\sin 5\theta$, etc....

12.5 Calculus of complex valued functions

A ***complex valued function*** on some interval $I = (a, b) \subseteq \mathbb{R}$ is a function $f : I \to \mathbb{C}$. Such a function can be written as in terms of its real and imaginary parts,

$$f(x) = u(x) + iv(x), \tag{12.9}$$

in which $u, v : I \to \mathbb{R}$ are two real valued functions.

One defines limits of complex valued functions in terms of limits of their real and imaginary parts. Thus we say that

$$\lim_{x \to x_0} f(x) = L$$

if $f(x) = u(x) + iv(x)$, $L = A + iB$, and both

$$\lim_{x \to x_0} u(x) = A \text{ and } \lim_{x \to x_0} v(x) = B$$

hold. From this definition one can prove that the usual limit theorems also apply to complex valued functions.

Theorem 12.5.1. If $\lim_{x \to x_0} f(x) = L$ and $\lim_{x \to x_0} g(x) = M$, then one has

$$\begin{aligned} \lim_{x \to x_0} f(x) \pm g(x) &= L \pm M, \\ \lim_{x \to x_0} f(x)g(x) &= LM, \\ \lim_{x \to x_0} \frac{f(x)}{g(x)} &= \frac{L}{M}, \text{ provided } M \neq 0. \end{aligned}$$

The ***derivative*** of a complex valued function $f(x) = u(x) + iv(x)$ is defined by simply differentiating its real and imaginary parts:

$$f'(x) = u'(x) + iv'(x). \tag{12.10}$$

Again, one finds that the sum,product and quotient rules also hold for complex valued functions.

Theorem 12.5.2. If $f, g : I \to \mathbb{C}$ are complex valued functions which are differentiable at some $x_0 \in I$, then the functions $f \pm g$, fg and f/g are differentiable (assuming $g(x_0) \neq 0$ in the case of the quotient.) One has

$$\begin{aligned} (f \pm g)'(x_0) &= f'(x_0) \pm g'(x_0) \\ (fg)'(x_0) &= f'(x_0)g(x_0) + f(x_0)g'(x_0) \\ \left(\frac{f}{g}\right)'(x_0) &= \frac{f'(x_0)g(x_0) - f(x_0)g'(x_0)}{g(x_0)^2} \end{aligned}$$

Note that the chain rule does not appear in this list! See problem 832 for more about the chain rule.

12.6 The Complex Exponential Function

We finally give a definition of e^{a+bi}. First we consider the case $a = 0$:

Definition 12.6.1. For any real number t we set $e^{it} = \cos t + i \sin t$.. See Figure 12.6.

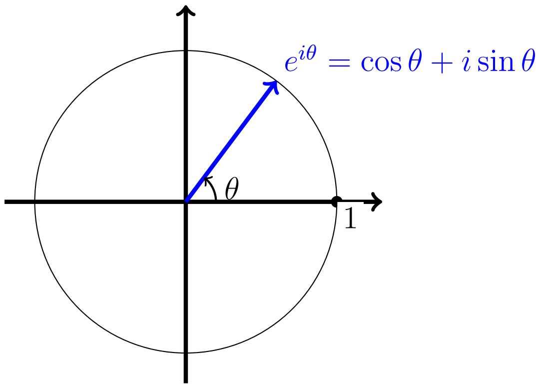


Figure 12.6: Euler's definition of $e^{i\theta}$

12.6.1 Example

$e^{\pi i} = \cos\pi + i\sin\pi = -1$. This leads to Euler's famous formula

$$e^{\pi i} + 1 = 0,$$

which combines the five most basic quantities in mathematics: e, π, i, 1, and 0.

For a brilliant lecture on this identity consider watching YouTube by Mathologer

Reasons why the definition 12.6.1 seems a good definition.

Reason 1. We haven't defined e^{it} before and we can do anything we like.

Reason 2. Substitute it in the Taylor series for e^x:

$$\begin{aligned} e^{it} &= 1 + it + \frac{(it)^2}{2!} + \frac{(it)^3}{3!} + \frac{(it)^4}{4!} + \cdots \\ &= 1 + it - \frac{t^2}{2!} - i\frac{t^3}{3!} + \frac{t^4}{4!} + i\frac{t^5}{5!} - \cdots \\ &= 1 - t^2/2! + t^4/4! - \cdots \\ &\qquad + i\big(t - t^3/3! + t^5/5! - \cdots\big) \\ &= \cos t + i\sin t. \end{aligned}$$

This is not a proof, because before we had only proved the convergence of the Taylor series for e^x if x was a real number, and here we have pretended that the series is also good if you substitute $x = it$.

Reason 3. As a function of t the definition 12.6.1 gives us the correct derivative. Namely, using the chain rule (i.e. pretending it still applies for complex functions) we would get

$$\frac{de^{it}}{dt} = ie^{it}.$$

Indeed, this is correct. To see this proceed from our definition 12.6.1:

$$\begin{aligned}\frac{de^{it}}{dt} &= \frac{d\cos t + i\sin t}{dt}\\ &= \frac{d\cos t}{dt} + i\frac{d\sin t}{dt}\\ &= -\sin t + i\cos t\\ &= i(\cos t + i\sin t)\end{aligned}$$

Reason 4. The formula $e^x \cdot e^y = e^{x+y}$ still holds. Rather, we have $e^{it+is} = e^{it}e^{is}$. To check this replace the exponentials by their definition:

$$e^{it}e^{is} = (\cos t + i\sin t)(\cos s + i\sin s) = \cos(t+s) + i\sin(t+s) = e^{i(t+s)}.$$

Requiring $e^x \cdot e^y = e^{x+y}$ to be true for all complex numbers helps us decide what e^{a+bi} shoud be for arbitrary complex numbers $a+bi$.

Definition 12.6.2. For any complex number $a+bi$ we set

$$e^{a+bi} = e^a \cdot e^{ib} = e^a(\cos b + i\sin b).$$

One verifies as above in "reason 3" that this gives us the right behaviour under differentiation. Thus, for any complex number $r = a+bi$ the function

$$y(t) = e^{rt} = e^{at}(\cos bt + i\sin bt)$$

satisfies

$$y'(t) = \frac{de^{rt}}{dt} = re^{rt}.$$

12.7 Complex solutions of polynomial equations

12.7.1 Quadratic equations

The well-known quadratic formula tells you that the equation

$$ax^2 + bx + c = 0 \tag{12.11}$$

has two solutions, given by

$$x_\pm = \frac{-b \pm \sqrt{D}}{2a}, \qquad D = b^2 - 4ac. \tag{12.12}$$

If the coefficients a, b, c are real numbers and if the ***discriminant*** D is positive, then this formula does indeed give two real solutions x_+ and x_-. However, if $D < 0$, then there are no real solutions, but there are two complex solutions, namely

$$x_\pm = \frac{-b}{2a} \pm i\frac{\sqrt{-D}}{2a}$$

12.7.2 Example: solve $x^2 + 2x + 5 = 0$

Solution: Use the quadratic formula, or complete the square:

$$\begin{aligned} & x^2 + 2x + 5 = 0 \\ \iff & x^2 + 2x + 1 = -4 \\ \iff & (x+1)^2 = -4 \\ \iff & x + 1 = \pm 2i \\ \iff & x = -1 \pm 2i. \end{aligned}$$

So, if you allow complex solutions then every quadratic equation has two solutions, unless the two solutions coincide (the case $D = 0$, in which there is only one solution.)

12.7.3 Complex roots of a number

For any given complex number w there is a method of finding all complex solutions of the equation

$$z^n = w \tag{12.13}$$

if $n = 2, 3, 4, \cdots$ is a given integer.

To find these solutions you write w in polar form, i.e. you find $r > 0$ and θ such that $w = re^{i\theta}$. Then

$$z = r^{1/n} e^{i\theta/n}$$

is a solution to (12.13). But it isn't the only solution, because the angle θ for which $w = re^{i\theta}$ isn't unique – it is only determined up to a multiple of 2π. Thus if we have found one angle θ for which $w = r^{i\theta}$, then we can also write

$$w = re^{i(\theta + 2k\pi)}, \qquad k = 0, \pm 1, \pm 2, \cdots$$

The n^{th} roots of w are then

$$z_k = r^{1/n} e^{i\left(\frac{\theta}{n} + 2\frac{k}{n}\pi\right)}$$

Here k can be any integer, so it looks as if there are infinitely many solutions. However, if you increase k by n, then the exponent above increases by $2\pi i$, and hence z_k does not change. In a formula:

$$z_n = z_0, \quad z_{n+1} = z_1, \quad z_{n+2} = z_2, \quad \ldots \quad z_{k+n} = z_k$$

So if you take $k = 0, 1, 2, \cdots, n-1$ then you have had all the solutions.

The solutions z_k always form a regular polygon with n sides.

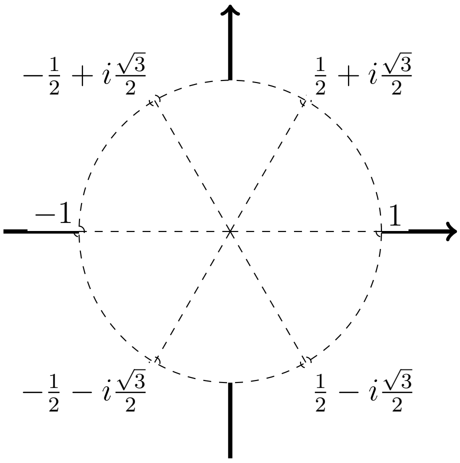


Figure 12.7: The sixth roots of 1. There are six of them, and they re arranged in a regular hexagon.

12.7.4 Example: find all sixth roots of $w = 1$

We are to solve $z^6 = 1$. First write 1 in polar form,

$$1 = 1 \cdot e^{0i} = 1 \cdot e^{2k\pi i}, \qquad (k = 0, \pm 1, \pm 2, \ldots).$$

Then we take the 6^{th} root and find

$$z_k = 1^{1/6} e^{2k\pi i/6} = e^{k\pi i/3}, \qquad (k = 0, \pm 1, \pm 2, \ldots).$$

The six roots are

$$z_0 = 1 \qquad z_1 = e^{\pi i/3} = \tfrac{1}{2} + \tfrac{i}{2}\sqrt{3} \qquad z_2 = e^{2\pi i/3} = -\tfrac{1}{2} + \tfrac{i}{2}\sqrt{3}$$

$$z_3 = -1 \qquad z_4 = e^{4\pi i/3} = -\tfrac{1}{2} - \tfrac{i}{2}\sqrt{3} \qquad z_5 = e^{5\pi i/3} = \tfrac{1}{2} - \tfrac{i}{2}\sqrt{3}$$

12.8 Other handy things you can do with complex numbers

12.8.1 Partial fractions

Consider the partial fraction decomposition

$$\frac{x^2 + 3x - 4}{(x-2)(x^2+4)} = \frac{A}{x-2} + \frac{Bx + C}{x^2 + 4}$$

The coefficient A is easy to find: multiply with $x - 2$ and set $x = 2$ (or rather, take the limit $x \to 2$) to get

$$A = \frac{2^2 + 3 \cdot 2 - 4}{2^2 + 4} = \cdots .$$

Before we had no similar way of finding B and C quickly, but now we can apply the same trick: multiply with $x^2 + 4$,

$$\frac{x^2 + 3x - 4}{(x-2)} = Bx + C + (x^2 + 4)\frac{A}{x-2},$$

and substitute $x = 2i$. This make $x^2 + 4 = 0$, with result

$$\frac{(2i)^2 + 3 \cdot 2i - 4}{(2i - 2)} = 2iB + C.$$

Simplify the complex number on the left:

$$\begin{aligned}\frac{(2i)^2 + 3 \cdot 2i - 4}{(2i - 2)} &= \frac{-4 + 6i - 4}{-2 + 2i} \\ &= \frac{-8 + 6i}{-2 + 2i} \\ &= \frac{(-8 + 6i)(-2 - 2i)}{(-2)^2 + 2^2} \\ &= \frac{28 + 4i}{8} \\ &= \frac{7}{2} + \frac{i}{2}\end{aligned}$$

So we get $2iB + C = \frac{7}{2} + \frac{i}{2}$; since B and C are real numbers this implies

$$B = \frac{1}{4}, \qquad C = \frac{7}{2}.$$

12.8.2 Certain trigonometric and exponential integrals

You can compute

$$I = \int e^{3x} \cos 2x \mathrm{d}x$$

by integrating by parts twice. You can also use that $\cos 2x$ is the real part of e^{2ix}. Instead of computing the real integral I, we look at the following related complex integral

$$J = \int e^{3x} e^{2ix} \mathrm{d}x$$

which we get from I by replacing $\cos 2x$ with e^{2ix}. Since $e^{2ix} = \cos 2x + i \sin 2x$ we have

$$J = \int e^{3x}(\cos 2x + i \sin 2x)\mathrm{d}x = \int e^{3x} \cos 2x \mathrm{d}x + i \int e^{3x} \sin 2x \mathrm{d}x$$

i.e.,

$$J = I + \text{something imaginary}.$$

The point of all this is that J is easier to compute than I:

$$J = \int e^{3x} e^{2ix} \mathrm{d}x = \int e^{3x+2ix} \mathrm{d}x = \int e^{(3+2i)x} \mathrm{d}x = \frac{e^{(3+2i)x}}{3 + 2i} + C$$

where we have used that

$$\int e^{ax} \mathrm{d}x = \frac{1}{a} e^{ax} + C$$

holds even if a is complex is a complex number such as $a = 3 + 2i$.

To find I you have to compute the real part of J, which you do as follows:

$$\begin{aligned}\frac{e^{(3+2i)x}}{3+2i} &= e^{3x}\frac{\cos 2x + i\sin 2x}{3+2i}\\ &= e^{3x}\frac{(\cos 2x + i\sin 2x)(3-2i)}{(3+2i)(3-2i)}\\ &= e^{3x}\frac{3\cos 2x + 2\sin 2x + i(\cdots)}{13}\end{aligned}$$

so

$$\int e^{3x}\cos 2x\mathrm{d}x = e^{3x}\big(\tfrac{3}{13}\cos 2x + \tfrac{2}{13}\sin 2x\big) + C.$$

12.8.3 Complex amplitudes

A harmonic oscillation is given by

$$y(t) = A\cos(\omega t - \phi),$$

where A is the ***amplitude***, ω is the ***frequency***, and ϕ is the ***phase*** of the oscillation. If you add two harmonic oscillations with the same frequency ω, then you get another harmonic oscillation with frequency ω. You can prove this using the addition formulas for cosines, but there's another way using complex exponentials. It goes like this.

Let $y(t) = A\cos(\omega t - \phi)$ and $z(t) = B\cos(\omega t - \theta)$ be the two harmonic oscillations we wish to add. They are the real parts of

$$\begin{aligned}Y(t) &= A\left\{\cos(\omega t - \phi) + i\sin(\omega t - \phi)\right\} = Ae^{i\omega t - i\phi} = Ae^{-i\phi}e^{i\omega t}\\ Z(t) &= B\left\{\cos(\omega t - \theta) + i\sin(\omega t - \theta)\right\} = Be^{i\omega t - i\theta} = Be^{-i\theta}e^{i\omega t}\end{aligned}$$

Therefore $y(t) + z(t)$ is the real part of $Y(t) + Z(t)$, i.e.

$$y(t) + z(t) = \mathfrak{Re}\big(Y(t)\big) + \mathfrak{Re}\big(Z(t)\big) = \mathfrak{Re}\big(Y(t) + Z(t)\big).$$

The quantity $Y(t) + Z(t)$ is easy to compute:

$$Y(t) + Z(t) = Ae^{-i\phi}e^{i\omega t} + Be^{-i\theta}e^{i\omega t} = \left(Ae^{-i\phi} + Be^{-i\theta}\right)e^{i\omega t}.$$

If you now do the complex addition

$$Ae^{-i\phi} + Be^{-i\theta} = Ce^{-i\psi},$$

i.e. you add the numbers on the right, and compute the absolute value C and argument $-\psi$ of the sum, then we see that $Y(t) + Z(t) = Ce^{i(\omega t - \psi)}$. Since we were looking for the real part of $Y(t) + Z(t)$, we get

$$y(t) + z(t) = A\cos(\omega t - \phi) + B\cos(\omega t - \theta) = C\cos(\omega t - \psi).$$

The complex numbers $Ae^{-i\phi}$, $Be^{-i\theta}$ and $Ce^{-i\psi}$ are called the complex amplitudes for the harmonic oscillations $y(t)$, $z(t)$ and $y(t) + z(t)$.

The recipe for adding harmonic oscillations can therefore be summarized as follows: ***Add the complex amplitudes.***

12.9 PROBLEMS

COMPUTING AND DRAWING COMPLEX NUMBERS

802. Compute the following complex numbers by hand.

Draw ***all*** numbers in the complex (or "Argand") plane (use graph paper or quad paper if necessary).

Compute absolute value and argument of all numbers involved.

i^2; i^3; i^4; $1/i$;

$(1+2i)(2-i)$;

$(1+i)(1+2i)(1+3i)$;

$(\frac{1}{2}\sqrt{2}+\frac{i}{2}\sqrt{2})^2$; $(\frac{1}{2}+\frac{i}{2}\sqrt{3})^3$;

$\dfrac{1}{1+i}$; $5/(2-i)$;

803. Simplify your answer.

- For $z = 2+3i$ find:
  1. z^2
  2. $\bar{z}$
  3. $|z|$
  4. $\frac{1}{z}$
- For $z = 2e^{3i}$ find:
  1. $\arg(z)$
  2. $|z|$
  3. z^2
  4. $\frac{1}{z}$
- For $z = -\pi e^{\frac{\pi}{2}i}$ find:
  1. $|z|$
  2. $\arg(z)$

†396

804. Plot the following four points in the complex plane. Be sure and label them.

$$P = \sqrt{2}\, e^{\frac{5\pi}{4}i} \qquad Q = 1+2i$$
$$R = \overline{1+2i} \qquad Z = \frac{1}{1+2i}$$

†396

805. **[Deriving the addition formula for** $\tan(\theta+\phi)$**]** Let $\theta, \phi \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ be two angles.

(a) What are the arguments of

$$z = 1 + i\tan\theta \text{ and } w = 1 + i\tan\phi?$$

(Draw both z and w.)

(b) Compute zw.

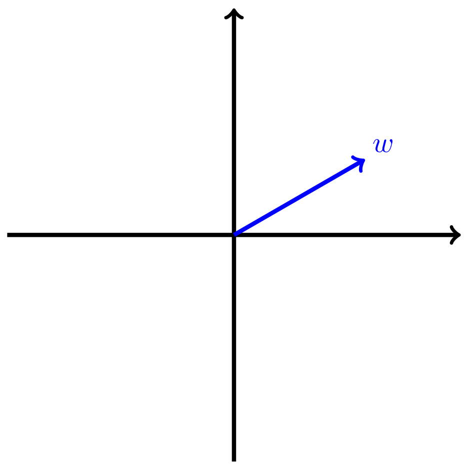
(c) What is the argument of zw?

(d) Compute $\tan(\arg zw)$.

†396

806. Find formulas for $\cos 4\theta$, $\sin 4\theta$, $\cos 5\theta$ and $\sin 6\theta$ in terms of $\cos\theta$ and $\sin\theta$, by using *de Moivre*'s formula. †396

807. In the following picture draw $2w$, $\frac{3}{4}w$, iw, $-2iw$, $(2+i)w$ and $(2-i)w$. (Try to make a nice drawing, use a ruler.)



Make a new copy of the picture, and draw $\bar{w}$, $-\bar{w}$ and $-w$.

Make yet another copy of the drawing. Draw $1/w$, $1/\bar{w}$, and $-1/w$. For this drawing you need to know where the unit circle is in your drawing: Draw a circle centered at the origin with radius of your choice, and let this be the unit circle. [Depending on which circle you draw you will get a different answer!]

808. Verify directly from the definition of addition and multiplication of complex numbers that

(a) $z + w = w + z$

(b) $zw = wz$

(c) $z(v+w) = zv + zw$

holds for *all* complex numbers v, w, and z.

809. True or False? (In mathematics this means that you should either give a proof that the statement is always true, or else give a counterexample, thereby showing that the statement is not always true.)

For any complex numbers z and w one has

(a) $\mathfrak{Re}(z)+\mathfrak{Re}(w)=\mathfrak{Re}(z+w)$

(b) $\overline{z+w}=\bar{z}+\bar{w}$

(c) $\mathfrak{Im}(z)+\mathfrak{Im}(w)=\mathfrak{Im}(z+w)$

(d) $\overline{zw}=(\bar{z})(\bar{w})$

(e) $\mathfrak{Re}(z)\mathfrak{Re}(w)=\mathfrak{Re}(zw)$

(f) $\overline{z/w}=(\bar{z})/(\bar{w})$

(g) $\mathfrak{Re}(iz)=\mathfrak{Im}(z)$

(h) $\mathfrak{Re}(iz)=i\mathfrak{Re}(z)$

(i) $\mathfrak{Re}(iz)=i\mathfrak{Im}(z)$

(j) $\mathfrak{Im}(iz)=\mathfrak{Re}(z)$

(k) $\mathfrak{Re}(\bar{z})=\mathfrak{Re}(z)$

†396

810. The imaginary part of a complex number is known to be twice its real part. The absolute value of this number is 4. Which number is this? †397

811. The real part of a complex number is known to be half the absolute value of that number. The imaginary part of the number is 1. Which number is it? †397

THE COMPLEX EXPONENTIAL

812. Compute and **draw** the following numbers in the complex plane

$e^{\pi i/3}$; $e^{\pi i/2}$; $\sqrt{2}e^{3\pi i/4}$; $e^{17\pi i/4}$.

$e^{\pi i}+1$; $e^{i\ln 2}$.

$\dfrac{1}{e^{\pi i/4}}$; $\dfrac{e^{-\pi i}}{e^{\pi i/4}}$; $\dfrac{e^{2-\pi i/2}}{e^{\pi i/4}}$

$e^{2009\pi i}$; $e^{2009\pi i/2}$.

$-8e^{4\pi i/3}$; $12e^{\pi i}+3e^{-\pi i}$.

813. Compute the absolute value and argument of $e^{(\ln 2)(1+i)}$. †397

814. Suppose z can be any complex number.

(a) Is it true that e^z is always a positive number?

(b) Is it true that $e^z\neq 0$? †397

815. Verify directly from the definition that

$$e^{-it}=\frac{1}{e^{it}}$$

holds for *all real* values of t. †397

816. Show that

$$\cos t=\frac{e^{it}+e^{-it}}{2},\qquad \sin t=\frac{e^{it}-e^{-it}}{2i}$$

817. Show that

$$\cosh x=\cos ix,\quad \sinh x=\frac{1}{i}\sin ix.$$

818. The general solution of a second order linear differential equation contains expressions of the form $Ae^{i\beta t}+Be^{-i\beta t}$. These can be rewritten as $C_1\cos\beta t+C_2\sin\beta t$.

If $Ae^{i\beta t}+Be^{-i\beta t}=2\cos\beta t+3\sin\beta t$, then what are A and B? †397

819. (a) Show that you can write a "cosine-wave" with amplitude A and phase ϕ as follows

$$A\cos(t-\phi)=\mathfrak{Re}\left(ze^{it}\right),$$

where the "complex amplitude" is given by $z=Ae^{-i\phi}$. (See §12.8.3).

(b) Show that a "sine-wave" with amplitude A and phase ϕ as follows

$$A\sin(t-\phi)=\mathfrak{Re}\left(ze^{it}\right),$$

where the "complex amplitude" is given by $z=-iAe^{-i\phi}$.

820. Find A and ϕ where $A\cos(t-\phi)=2\cos(t)+2\cos(t-\frac{2}{3}\pi)$.

821. Find A and ϕ where $A\cos(t-\phi)=12\cos(t-\frac{1}{6}\pi)+12\sin(t-\frac{1}{3}\pi)$.

822. Find A and ϕ where $A\cos(t-\phi)=12\cos(t-\pi/6)+12\cos(t-\pi/3)$.

823. Find A and ϕ such that $A\cos(t-\phi)=\cos\left(t-\frac{1}{6}\pi\right)+\sqrt{3}\cos\left(t-\frac{2}{3}\pi\right)$.

REAL AND COMPLEX SOLUTIONS OF ALGEBRAIC EQUATIONS

824. ***Find and draw*** all real and complex solutions of

(a) $z^2 + 6z + 10 = 0$

(b) $z^3 + 8 = 0$

(c) $z^3 - 125 = 0$

(d) $2z^2 + 4z + 4 = 0$

(e) $z^4 + 2z^2 - 3 = 0$

(f) $3z^6 = z^3 + 2$

(g) $z^5 - 32 = 0$

(h) $z^5 - 16z = 0$

(i) $z^4 + z^2 - 12 = 0$

†398

CALCULUS OF COMPLEX VALUED FUNCTIONS

825. Compute the derivatives of the following functions

$$f(x) = \frac{1}{x+i}$$
$$g(x) = \log x + i \arctan x$$
$$h(x) = e^{ix^2}$$

Try to simplify your answers.

†398

826. (a) Compute

$$\int \left(\cos 2x\right)^4 \, dx$$

by using $\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta})$ and expanding the fourth power.

(b) Assuming $a \in \mathbb{R}$, compute

$$\int e^{-2x} \left(\sin ax\right)^2 \, dx.$$

(same trick: write $\sin ax$ in terms of complex exponentials; make sure your final answer has no complex numbers.)

†398

827. Use $\cos\alpha = (e^{i\alpha} + e^{-i\alpha})/2$, etc. to evaluate these indefinite integrals:

(a) $\displaystyle\int \cos^2 x \, \mathrm{d}x$

(b) $\displaystyle\int \cos^4 x \, \mathrm{d}x$,

(c) $\displaystyle\int \cos^2 x \sin x \, \mathrm{d}x$,

(d) $\displaystyle\int \sin^3 x \, \mathrm{d}x$,

(e) $\displaystyle\int \cos^2 x \sin^2 x \, \mathrm{d}x$,

(f) $\displaystyle\int \sin^6 x \, \mathrm{d}x$

(g) $\displaystyle\int \sin(3x)\cos(5x) \, \mathrm{d}x$

(h) $\displaystyle\int \sin^2(2x)\cos(3x) \, \mathrm{d}x$

(i) $\displaystyle\int_0^{\pi/4} \sin(3x)\cos(x) \, \mathrm{d}x$

(j) $\displaystyle\int_0^{\pi/3} \sin^3(x)\cos^2(x) \, \mathrm{d}x$

(k) $\displaystyle\int_0^{\pi/2} \sin^2(x)\cos^2(x) \, \mathrm{d}x$

(l) $\displaystyle\int_0^{\pi/3} \sin(x)\cos^2(x) \, \mathrm{d}x$

†400

828. Compute the following integrals when $m \neq n$ are distinct integers.

(a) $\displaystyle\int_0^{2\pi} \sin(mx)\cos(nx) \, \mathrm{d}x$

(b) $\displaystyle\int_0^{2\pi} \sin(nx)\cos(nx) \, \mathrm{d}x$

(c) $\displaystyle\int_0^{2\pi} \cos(mx)\cos(nx) \, \mathrm{d}x$

(d) $\displaystyle\int_0^{\pi} \cos(mx)\cos(nx) \, \mathrm{d}x$

(e) $\displaystyle\int_0^{2\pi} \sin(mx)\sin(nx) \, \mathrm{d}x$

(f) $\int_0^\pi \sin(mx)\sin(nx)\,\mathrm{d}x$

These integrals are basic to the theory of *Fourier series,* which occurs in many applications, especially in the study of wave motion (light, sound, economic cycles, clocks, oceans, etc.). They say that different frequency waves are "independent".

829. Show that $\cos x + \sin x = C\cos(x+\beta)$ for suitable constants C and β and use this to evaluate the following integrals.

(a) $\int \frac{\mathrm{d}x}{\cos x + \sin x}$

(b) $\int \frac{\mathrm{d}x}{(\cos x + \sin x)^2}$

(c) $\int \frac{\mathrm{d}x}{A\cos x + B\sin x}$

where A and B are any constants.

830. Compute the integrals

$$\int_0^{\pi/2} \sin^2 kx \sin^2 lx \,\mathrm{d}x,$$

where k and l are positive integers.

831. Show that for any integers k, l, m

$$\int_0^\pi \sin kx \sin lx \sin mx \,\mathrm{d}x = 0$$

if and only if $k+l+m$ is even.

832. (***i***) Prove the following version of the CHAIN RULE: If $f : I \to \mathbb{C}$ is a differentiable complex valued function, and $g : J \to I$ is a differentiable real valued function, then $h = f \circ g : J \to \mathbb{C}$ is a differentiable function, and one has

$$h'(x) = f'(g(x))g'(x).$$

(***ii***) Let $n \geq 0$ be a nonnegative integer. Prove that if $f : I \to \mathbb{C}$ is a differentiable function, then $g(x) = f(x)^n$ is also differentiable, and one has

$$g'(x) = nf(x)^{n-1}f'(x).$$

Note that the chain rule from part (a) does ***not*** apply! *Why?*

COMPLEX ROOTS OF REAL POLYNOMIALS

833. For a and b complex numbers show that

(a) $\overline{a+b} = \overline{a} + \overline{b}$

(b) $\overline{a \cdot b} = \overline{a} \cdot \overline{b}$

(c) a is real iff $\overline{a} = a$

834. For $p(x) = a_0 + a_1 x + a_2 x^2 + \cdots a_n x^n$ a polynomial and z a complex number, show that

$$\overline{p(z)} = \overline{a}_0 + \overline{a}_1\ \overline{z} + \overline{a}_2\ (\overline{z})^2 + \cdots \overline{a}_n\ (\overline{z})^n$$

835. For p a real polynomial, i.e., the coefficients a_k of p are real numbers, if z is a complex root of p, i.e., $p(z) = 0$, show $\overline{z}$ is also a root of p. Hence the complex roots of p occur in conjugate pairs.

836. Using the quadratic formula show directly that the roots of a real quadratic are either both real or a complex conjugate pair.

837. Show that $2+3i$ and its conjugate $2-3i$ are the roots of a real polynomial.

838. Show that for every complex number a there is a real quadratic whose roots are a and $\overline{a}$.

839. The Fundamental theorem of Algebra states that every complex polynomial of degree n can be completely factored as a constant multiple of

$$(x-\alpha_1)(x-\alpha_2)\cdots(x-\alpha_n)$$

(The α_i may not be distinct.) It was proved by Gauss. Proofs of it are given in courses on Complex Analysis.

Use the Fundamental Theorem of Algebra to show that every real polynomial can be factored into a product real polynomials, each of degree 1 or 2.

Chapter 13

Differential Equations

13.1 What is a Differential Equation?

A ***differential equation*** is an equation involving an unknown function and its derivatives. The ***order*** of the differential equation is the order of the highest derivative which appears. A ***linear differential equation*** is one of form

$$y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_{n-1}(x)y' + a_n(x)y = k(x)$$

where the coefficients $a_1(x), \ldots, a_n(x)$ and the right hand side $k(x)$ are given functions of x and y is the unknown function. Here

$$y^{(k)} = \frac{\mathrm{d}^k y}{\mathrm{d}x^k}$$

denotes the kth derivative of y so this equation has order n. We shall mainly study the case $n = 1$ where the equation has form

$$y' + a(x)y = k(x)$$

and the case $n = 2$ with constant coefficients where the equation has form

$$y'' + ay' + by = k(x).$$

When the right hand side $k(x)$ is zero the equation is called ***homogeneous linear*** and otherwise it is called ***inhomogeneous linear*** (or ***nonhomogeneous linear*** by some people). For a homogeneous linear equation the sum of two solutions is a solution and a constant multiple of a solution is a solution. This property of linear equations is called the ***principle of superposition***.

13.2 First Order Separable Equations

A ***separable differential equation*** is a diffeq of the form

$$y'(x) = F(x)G(y(x)), \quad \text{or} \quad \frac{\mathrm{d}y}{\mathrm{d}x} = F(x)G(y). \tag{13.1}$$

To solve this equation divide by $G(y(x))$ to get

$$\frac{1}{G(y(x))}\frac{\mathrm{d}y}{\mathrm{d}x} = F(x). \tag{13.2}$$

Next find a function $H(y)$ whose derivative with respect to y is

$$H'(y) = \frac{1}{G(y)} \quad \left(\text{solution: } H(y) = \int \frac{dy}{G(y)}.\right) \tag{13.3}$$

Then the chain rule implies that (13.2) can be written as

$$\frac{\mathrm{d}H(y(x))}{\mathrm{d}x} = F(x).$$

In words: $H(y(x))$ is an antiderivative of $F(x)$, which means we can find $H(y(x))$ by integrating $F(x)$:

$$H(y(x)) = \int F(x)dx + C. \tag{13.4}$$

Once you've found the integral of $F(x)$ this gives you $y(x)$ in implicit form: the equation (13.4) gives you $y(x)$ as an *implicit function* of x. To get $y(x)$ itself you must solve the equation (13.4) for $y(x)$.

A quick way of organizing the calculation goes like this:

> To solve $\frac{\mathrm{d}y}{\mathrm{d}x} = F(x)G(y)$ you first *separate the variables*,
>
> $$\frac{\mathrm{d}y}{G(y)} = F(x)\,\mathrm{d}x,$$
>
> and then integrate,
>
> $$\int \frac{\mathrm{d}y}{G(y)} = \int F(x)\,\mathrm{d}x.$$
>
> The result is an implicit equation for the solution y with one undetermined integration constant.

Determining the constant. The solution you get from the above procedure contains an arbitrary constant C. If the value of the solution is specified at some given x_0, i.e. if $y(x_0)$ is known then you can express C in terms of $y(x_0)$ by using (13.4).

A snag: You have to divide by $G(y)$ which is problematic when $G(y) = 0$. This has as consequence that in addition to the solutions you found with the above procedure, there are at least a few more solutions: the zeroes of $G(y)$ (see Example 13.2.2 below). In addition to the zeroes of $G(y)$ there sometimes can be more solutions, as we will see in Example 13.4.2 on "Leaky Bucket Dating."

13.2.1 Example

We solve

$$\frac{\mathrm{d}z}{\mathrm{d}t} = (1 + z^2)\cos t.$$

Separate variables and integrate

$$\int \frac{\mathrm{d}z}{1 + z^2} = \int \cos t\,\mathrm{d}t,$$

to get

$$\arctan z = \sin t + C.$$

Finally solve for z and you find the general solution

$$z(t) = \tan\bigl(\sin(t) + C\bigr).$$

13.2.2 Example: The snag in action

If you apply the method to $y'(x) = Ky$ with K a constant, you get $y(x) = e^{K(x+C)}$. No matter how you choose C you never get the function $y(x) = 0$, even though $y(x) = 0$ satisfies the equation. This is because here $G(y) = Ky$, and $G(y)$ vanishes for $y = 0$.

13.3 First Order Linear Equations

There are two systematic methods which solve a first order linear inhomogeneous equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} + a(x)y = k(x). \tag{‡}$$

You can multiply the equation with an "integrating factor", or you do a substitution $y(x) = c(x)y_0(x)$, where y_0 is a solution of the *homogeneous equation* (that's the equation you get by setting $k(x) \equiv 0$).

13.3.1 The Integrating Factor

Let

$$A(x) = \int a(x)\,\mathrm{d}x, \qquad m(x) = e^{A(x)}.$$

Multiply the equation (‡) by the "integrating factor" $m(x)$ to get

$$m(x)\frac{\mathrm{d}y}{\mathrm{d}x} + a(x)m(x)y = m(x)k(x).$$

By the chain rule the integrating factor satisfies

$$\frac{\mathrm{d}m(x)}{\mathrm{d}x} = A'(x)m(x) = a(x)m(x).$$

Therefore one has

$$\frac{\mathrm{d}m(x)y}{\mathrm{d}x} = m(x)\frac{\mathrm{d}y}{\mathrm{d}x} + a(x)m(x)y = m(x)\left\{\frac{\mathrm{d}y}{\mathrm{d}x} + a(x)y\right\} = m(x)k(x).$$

Integrating and then dividing by the integrating factor gives the solution

$$y = \frac{1}{m(x)}\left(\int m(x)k(x)\,\mathrm{d}x + C\right).$$

In this derivation we have to divide by $m(x)$, but since $m(x) = e^{A(x)}$ and since exponentials never vanish we know that $m(x) \neq 0$, no matter which problem we're doing, so it's OK, we can always divide by $m(x)$.

13.3.2 Variation of constants for 1st order equations

Here is the second method of solving the inhomogeneous equation (‡). Recall again that the ***homogeneous equation*** associated with (‡) is

$$\frac{\mathrm{d}y}{\mathrm{d}x} + a(x)y = 0. \tag{†}$$

The general solution of this equation is

$$y(x) = Ce^{-A(x)}.$$

where the coefficient C is an arbitrary constant. To solve the inhomogeneous equation (‡) we replace the constant C by an unknown function $C(x)$, i.e. we look for a solution in the form

$$y = C(x)y_0(x) \text{ where } y_0(x) \stackrel{\text{def}}{=} e^{-A(x)}.$$

(This is how the method gets its name: we are allowing the constant C to vary.)
Then $y_0'(x) + a(x)y_0(x) = 0$ (because $y_0(x)$ solves (†)) and

$$y'(x) + a(x)y(x) = C'(x)y_0(x) + C(x)y_0'(x) + a(x)C(x)y_0(x) = C'(x)y_0(x)$$

so $y(x) = C(x)y_0(x)$ is a solution if $C'(x)y_0(x) = k(x)$, i.e.

$$C(x) = \int \frac{k(x)}{y_0(x)}\,\mathrm{d}x.$$

Once you notice that $y_0(x) = \dfrac{1}{m(x)}$, you realize that the resulting solution

$$y(x) = C(x)y_0(x) = y_0(x)\int \frac{k(x)}{y_0(x)}\,\mathrm{d}x$$

is the same solution we found before, using the integrating factor.
Either method implies the following:

Theorem 13.3.1. The initial value problem

$$\frac{\mathrm{d}y}{\mathrm{d}x} + a(x)y = 0, \qquad y(0) = y_0,$$

has ***exactly one*** solution. It is given by

$$y = y_0e^{-A(x)}, \text{ where } A(x) = \int_0^x a(t)\,\mathrm{d}t.$$

The theorem says three things: (1) there is a solution, (2) there is a formula for the solution, (3) there aren't any other solutions (if you insist on the initial value $y(0) = y_0$.) The last assertion is just as important as the other two, so I'll spend a whole section trying to explain why.

13.4 Dynamical Systems and Determinism

A differential equation which describes how something (e.g. the position of a particle) evolves in time is called a ***dynamical system***. In this situation the independent variable is *time*, so it is customary to call it t rather than x; the dependent variable, which depends on time is often denoted by x. In other words, one has a differential equation for a function $x = x(t)$. The simplest examples have form

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(x,t). \tag{13.5}$$

In applications such a differential equation expresses a *law* according to which the quantity $x(t)$ evolves with time (synonyms: "evolutionary law", "dynamical law", "evolution equation for x").

A good law is ***deterministic***, which means that any solution of (13.5) is completely determined by its value at one particular time t_0: if you know x at time $t = t_0$, then the "evolution law" (13.5) should predict the values of $x(t)$ at all other times, both in the past ($t < t_0$) and in the future ($t > t_0$).

Our experience with solving differential equations so far (§13.2 and §13.3) tells us that the general solution to a differential equation like (13.5) contains an unknown integration constant C. Let's call the general solution $x(t; C)$ to emphasize the presence of this constant. If the value of x at some time t_0 is known to be, say, x_0, then you get an equation

$$x(t_0; C) = x_0 \tag{13.6}$$

which you can try to solve for C. If this equation always has exactly one solution C then the evolutionary law (13.5) is deterministic (the value of $x(t_0)$ always determines $x(t)$ at all other times t); if for some prescribed value x_0 at some time t_0 the equation (13.6) has several solutions, then the evolutionary law (13.5) is not deterministic (because knowing $x(t)$ at time t_0 still does not determine the whole solution $x(t)$ at times other than t_0).

13.4.1 Example: Carbon Dating.

Suppose we have a fossil, and we want to know how old it is.

All living things contain carbon, which naturally occurs in two isotopes, C_{14} (unstable) and C_{12} (stable). A long as the living thing is alive it eats & breaths, and its ratio of C_{12} to C_{14} is kept constant. Once the thing dies the isotope C_{14} decays into C_{12} at a steady rate.

Let $x(t)$ be the ratio of C_{14} to C_{12} at time t. The laws of radioactive decay says that there is a constant $k > 0$ such that

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = -kx(t).$$

Solve this differential equation (it is both separable and first order linear: you choose your method) to find the general solution

$$x(t; C) = Ce^{-kt}.$$

After some lab work it is found that the current C_{14}/C_{12} ratio of our fossil is x_{now}. Thus we have

$$x_{\text{now}} = Ce^{-kt_{\text{now}}} \implies C = x_{\text{now}} e^{t_{\text{now}}}.$$

Therefore our fossil's C_{14}/C_{12} ratio at any other time t is/was

$$x(t) = x_{\text{now}} e^{k(t_{\text{now}} - t)}.$$

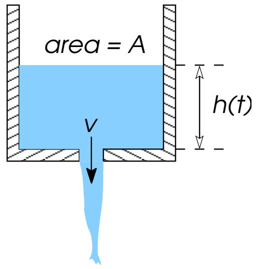
This allows you to compute the time at which the fossil died. At this time the C_{14}/C_{12} ratio must have been the common value in all living things, which can be measured, let's call it x_{life}. So at the time t_{demise} when our fossil became a fossil you would have had $x(t_{\text{demise}}) = x_{\text{life}}$. Hence the age of the fossil would be given by

$$x_{\text{life}} = x(t_{\text{demise}}) = x_{\text{now}} e^{k(t_{\text{now}} - t_{\text{demise}})} \implies \boxed{t_{\text{now}} - t_{\text{demise}} = \frac{1}{k} \ln \frac{x_{\text{life}}}{x_{\text{now}}}}$$

13.4.2 Example: On Dating a Leaky Bucket.

A bucket is filled with water. There's a hole in the bottom of the bucket so the water streams out at a certain rate.

$h(t)$	the height of water in the bucket
A	area of cross section of bucket
a	area of hole in the bucket
v	velocity with which water goes through the hole.



The amount of water in the bucket is $A \times h(t)$;
The rate at which water is leaving the bucket is $a \times v(t)$;
Hence

$$\frac{\mathrm{d}Ah(t)}{\mathrm{d}t} = -av(t).$$

In fluid mechanics it is shown that the velocity of the water as it passes through the hole only depends on the height $h(t)$ of the water, and that, for some constant K,

$$v(t) = \sqrt{Kh(t)}.$$

The last two equations together give a differential equation for $h(t)$, namely,

$$\frac{\mathrm{d}h(t)}{\mathrm{d}t} = -\frac{a}{A}\sqrt{Kh(t)}.$$

To make things a bit easier we assume that the constants are such that $\frac{a}{A}\sqrt{K} = 2$. Then $h(t)$ satisfies

$$h'(t) = -2\sqrt{h(t)}. \tag{13.7}$$

This equation is separable, and when you solve it you get

$$\frac{\mathrm{d}h}{2\sqrt{h}} = -1 \implies \sqrt{h(t)} = -t + C.$$

This formula can't be valid for *all* values of t, for if you take $t > C$, the RHS becomes negative and can't be equal to the square root in the LHS. But when $t \le C$ we do get a solution,

$$h(t; C) = (C - t)^2.$$

This solution describes a bucket which is losing water until at time C it is empty. Motivated by the physical interpretation of our solution it is natural to assume that the bucket stays empty when $t > C$, so that the solution with integration constant C is given by

$$h(t) = \begin{cases} (C-t)^2 & \text{when } t \le C \\ 0 & \text{for } t > C. \end{cases}$$

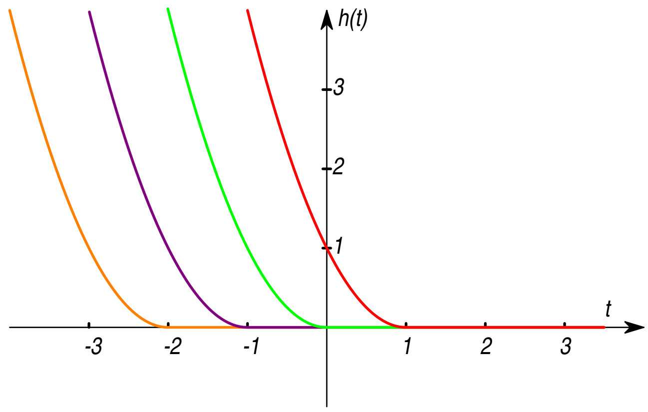


Figure 13.1: Several solutions $h(t; C)$ of the Leaking Bucket Equation (13.7). Note how they all have the same values when $t \geq 1$.

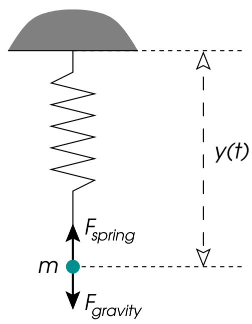
We now come to the question: is the Leaky Bucket Equation deterministic? The answer is: NO. If you let C be any negative number, then $h(t; C)$ describes the water level of a bucket which long ago had water, but emptied out at time $C < 0$. In particular, for all these solutions of the diffeq (13.7) you have $h(0) = 0$, and knowing the value of $h(t)$ at $t = 0$ in this case therefore doesn't tell you what $h(t)$ is at other times.

Once you put it in terms of the physical interpretation it is actually quite obvious why this system can't be deterministic: it's because you can't answer the question *"If you know that the bucket once had water and that it is empty now, then how much water did it hold one hour ago?"*

13.5 Higher order equations

After looking at first order differential equations we now turn to higher order equations.

13.5.1 Example: Spring with a weight.



A body of mass m is suspended by a spring. There are two forces on the body: gravity and the tension in the spring. Let F be the sum of these two forces. Newton's law says that the motion of the weight satisfies $F = ma$ where a is the acceleration. The force of gravity is mg where g=32ft/sec^2; the quantity mg is called the ***weight*** of the body.

We assume ***Hooke's law*** which says that the tension in the spring is proportional to the amount by which the spring is stretched; the constant or proportionality is called the ***spring constant***. We write k for this spring constant.

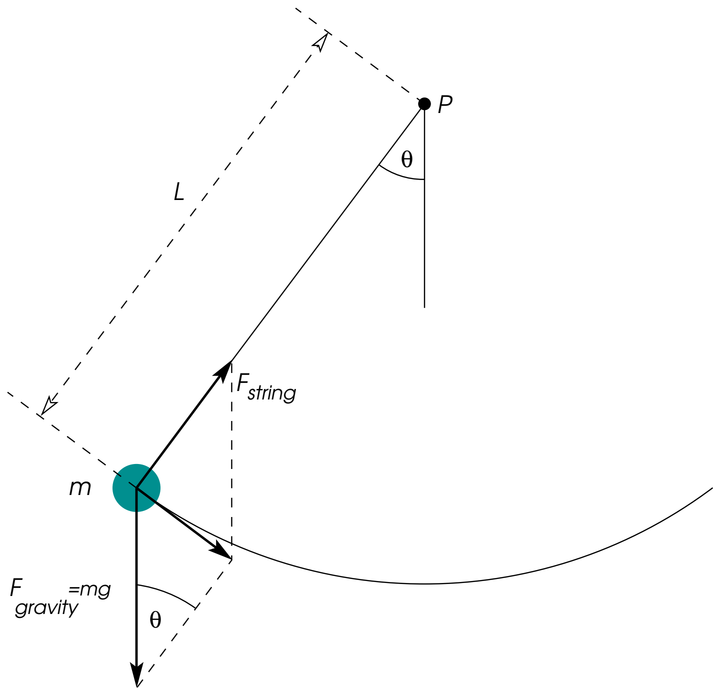
The total force acting on the body is therefore

$$F = mg - ky(t).$$

According to ***Newton's first/second/third law*** the acceleration a of the body satisfies $F = ma$. Since the acceleration a is the second derivative of position y we get the following differential equation for $y(t)$

$$m\frac{\mathrm{d}^2y}{\mathrm{d}t^2} = mg - ky(t). \tag{13.8}$$

13.5.2 Example: the pendulum.



The velocity of the weight on the pendulum is $L\frac{\mathrm{d}\theta}{\mathrm{d}t}$, hence its acceleration is $a = L\mathrm{d}^2\theta/\mathrm{d}t^2$. There are two forces acting on the weight: gravity (strength mg; direction vertically down) and the tension in the string (strength: whatever it takes to keep the weight on the circle of radius L and center P; direction parallel to the string). Together they leave a force of size $F_{\text{gravity}} \cdot \sin\theta$ which accelerates the weight. By Newton's "$F = ma$" law you get

$$mL\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} = -mg\sin\theta(t),$$

or, canceling ms,

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + \frac{g}{L}\sin\theta(t) = 0. \tag{13.9}$$

13.6 Constant Coefficient Linear Homogeneous Equations

13.6.1 Differential operators

In this section we study the homogeneous linear differential equation

$$y^{(n)} + a_1y^{(n-1)} + \cdots + a_{n-1}y' + a_ny = 0 \tag{13.10}$$

where the coefficients $a_1, \ldots, a_n$ are constants.

13.6.2 Examples.

The three equations

$$\frac{\mathrm{d}y}{\mathrm{d}x} - y = 0,$$
$$y'' - y = 0, \qquad y'' + y = 0$$
$$y^{(\mathrm{iv})} - y = 0$$

are homogeneous linear differential equations with constant coefficients. Their degrees are 1, 2, 2, and 4.

It will be handy to have an abbreviation for the Left Hand Side in (13.10), so we agree to write $\mathcal{L}[y]$ for the result of substituting a function y in the LHS of (13.10). In other words, for any given function $y = y(x)$ we set

$$\mathcal{L}[y](x) \stackrel{\text{def}}{=} y^{(n)}(x) + a_1 y^{(n-1)}(x) + \cdots + a_{n-1}y'(x) + a_n y(x).$$

We call $\mathcal{L}$ an ***operator***. An operator is like a function in that you give it an input, it does a computation and gives you an output. The difference is that ordinary functions take a number as their input, while the operator $\mathcal{L}$ takes a function $y(x)$ as its input, and gives another function (the LHS of (13.10)) as its output. Since the computation of $\mathcal{L}[y]$ involves taking derivatives of y, the operator $\mathcal{L}$ is called a ***differential operator.***

13.6.3 Example

The differential equations in the previous example correspond to the differential operators

$$\mathcal{L}_1[y] = y' - y,$$
$$\mathcal{L}_2[y] = y'' - y, \quad \mathcal{L}_3[y] = y'' + y$$
$$\mathcal{L}_4[y] = y^{(\mathrm{iv})} - y.$$

So one has

$$\mathcal{L}_3[\sin 2x] = \frac{\mathrm{d}^2 \sin 2x}{\mathrm{d}x^2} + \sin 2x = -4\sin 2x + \sin 2x = -3\sin 2x.$$

13.6.4 The superposition principle

The following theorem is the most important property of linear differential equations.

Theorem 13.6.1 (Superposition Principle)**.** For any two functions y_1 and y_2 we have

$$\mathcal{L}[y_1 + y_2] = \mathcal{L}[y_1] + \mathcal{L}[y_2].$$

For any function y and any constant c we have

$$\mathcal{L}[cy] = c\mathcal{L}[y].$$

The proof, which is rather straightforward once you know what to do, will be given in lecture. It follows from this theorem that if $y_1, \ldots, y_k$ are given functions, and $c_1, \ldots, c_k$ are constants, then

$$\mathcal{L}[c_1y_1 + \cdots + c_ky_k] = c_1\mathcal{L}[y_1] + \cdots + c_k\mathcal{L}[y_k].$$

The importance of the superposition principle is that it allows you to take old solutions to the homogeneous equation and make new ones. Namely, if $y_1, \ldots, y_k$ are solutions to the homogeneous equation $\mathcal{L}[y] = 0$, then so is $c_1 y_1 + \cdots + c_k y_k$ for any choice of constants $c_1, \ldots, c_k$.

13.6.5 Example

Consider the equation

$$y'' - 4y = 0.$$

Emily's sister Kate says that the two functions $y_1(x) = e^{2x}$ and $y_2(x) = e^{-2x}$ both are solutions to this equations. You can check that Kate is right just by substituting her solutions in the equation.

The Superposition Principle now implies that

$$y(x) = c_1 e^{2x} + c_2 e^{-2x}$$

also is a solution, for any choice of constants c_1, c_2.

13.6.6 The characteristic polynomial

This example contains in it the general method for solving linear constant coefficient ODEs. Suppose we want to solve the equation (13.10), i.e.

$$\mathcal{L}[y] \stackrel{\text{def}}{=} y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1} y' + a_n y = 0.$$

Then the first thing to do is to see if there are any exponential functions $y = e^{rx}$ which satisfy the equation. Since

$$\frac{\mathrm{d}e^{rx}}{\mathrm{d}x} = re^{rx}, \quad \frac{\mathrm{d}^2 e^{rx}}{\mathrm{d}x^2} = r^2 e^{rx}, \quad \frac{\mathrm{d}^3 e^{rx}}{\mathrm{d}x^3} = r^3 e^{rx}, \quad \text{etc.}\ldots$$

we see that

$$\mathcal{L}[e^{rx}] = \big(r^n + a_1 r^{n-1} + \cdots a_{n-1} r + a_n\big) e^{rx}. \tag{13.11}$$

The polynomial

$$P(r) = r^n + a_1 r^{n-1} + \cdots + a_{n-1} r + a_n.$$

is called the ***characteristic polynomial***.

We see that $y = e^{rx}$ is a solution of $\mathcal{L}[y] = 0$ if and only if $P(r) = 0$.

13.6.7 Example

We look for all exponential solutions of the equation

$$y'' - 4y = 0.$$

Substitution of $y = e^{rx}$ gives

$$y'' - 4y = r^2 e^{rx} - 4e^{rx} = \big(r^2 - 4\big) e^{rx}.$$

The exponential e^{rx} can't vanish, so $y'' - 4y = 0$ will hold exactly when $r^2 - 4 = 0$, i.e. when $r = \pm 2$. Therefore the only exponential functions which satisfy $y'' - 4y = 0$ are $y_1(x) = e^{2x}$ and $y_2(x) = e^{-2x}$.

Theorem 13.6.2. Suppose the polynomial $P(r)$ has n distinct roots $r_1, r_2, \ldots, r_n$. Then the general solution of $\mathcal{L}[y] = 0$ is

$$y = c_1 e^{r_1 x} + c_2 e^{r_2 x} + \cdots + c_n e^{r_n x}$$

where $c_1, c_2, \ldots, c_n$ are arbitrary constants.

Proof. We have just seen that the functions $y_1(x) = e^{r_1 x}$, $y_2(x) = e^{r_2 x}$, $y_3(x) = e^{r_3 x}$, etc. are solutions of the equation $\mathcal{L}[y] = 0$. In Math 320 (or 319, or...) you prove that these are all the solutions (it also follows from the method of variation of parameters that there aren't any other solutions).

□

13.6.8 Complex roots and repeated roots

If the characteristic polynomial has n distinct real roots then Theorem 13.6.2 tells you what the general solution to the equation $\mathcal{L}[y] = 0$ is. In general a polynomial equation like $P(r) = 0$ can have repeated roots, and it can have complex roots.

13.6.9 Example

Solve $y'' + 2y' + y = 0$.

The characteristic polynomial is $P(r) = r^2 + 2r + 1 = (r + 1)^2$, so the only root of the characteristic equation $r^2 + 2r + 1 = 0$ is $r = -1$ (it's a repeated root). This means that for this equation we only get *one* exponential solution, namely $y(x) = e^{-x}$.

It turns out that for this equation there is another solution which is not exponential. It is $y_2(x) = xe^{-x}$. You can check that it really satisfies the equation $y'' + 2y' + y = 0$.

When there are repeated roots there are other solutions: if $P(r) = 0$, then $t^j e^{rt}$ is a solution if j is a nonnegative integer less than the multiplicity of r. Also, if any of the roots are complex, the phrase *general solution* should be understood to mean *general complex solution* and the coefficients c_j should be complex. If the equation is real, the real and imaginary part of a complex solution are again solutions. We only describe the case $n = 2$ in detail.

Theorem 13.6.3. Consider the differential equation

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + a_1 \frac{\mathrm{d}y}{\mathrm{d}x} + a_2 y = 0 \tag{†}$$

and suppose that r_1 and r_2 are the solutions of the characteristic equation of $r^2 + a_1 r + a_2 = 0$. Then

(i) If r_1 and r_2 are distinct and real, the general solution of (†) is

$$y = c_1 e^{r_1 x} + c_2 e^{r_2 x}.$$

(ii) If $r_1 = r_2$, the general solution of (†) is

$$y = c_1 e^{r_1 x} + c_2 x e^{r_1 x}.$$

(iii) If $r_1 = \alpha + \beta i$ and $r_2 = \alpha - \beta i$, the general solution of (†) is

$$y = c_1 e^{\alpha x}\cos(\beta x) + c_2 e^{\alpha x}\sin(\beta x).$$

In each case c_1 and c_2 are arbitrary constants.

Case (i) and case (iii) can be subsumed into a single case using complex notation:

$$e^{(\alpha\pm\beta i)x} = e^{\alpha x}\cos\beta x \pm i e^{\alpha x}\sin\beta x,$$

$$e^{\alpha x}\cos\beta x = \frac{e^{(\alpha+\beta i)x} + e^{(\alpha-\beta i)x}}{2}, \qquad e^{\alpha x}\sin\beta x = \frac{e^{(\alpha+\beta i)x} - e^{(\alpha-\beta i)x}}{2i}.$$

13.7 Inhomogeneous Linear Equations

In this section we study the inhomogeneous linear differential equation

$$y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1}y' + a_n y = k(x)$$

where the coefficients $a_1, \ldots, a_n$ are constants and the function $k(x)$ is a given function. In the operator notation this equation may be written

$$\mathcal{L}[y] = k(x).$$

The following theorem says that once we know one particular solution y_p of the inhomogeneous equation $\mathcal{L}[y] = k(x)$ we can find all the solutions y to the inhomogeneous equation $\mathcal{L}[y] = k(x)$ by finding all the solutions y_h to the homogeneous equation $\mathcal{L}[y] = 0$.

Theorem 13.7.1 (Another Superposition Principle)**.** Assume $\mathcal{L}[y_p] = k(x)$. Then $\mathcal{L}[y] = k(x)$ if and only if $y = y_p + y_h$ where $\mathcal{L}[y_h] = 0$.

Proof. Suppose $\mathcal{L}[y_p] = k(x)$ and $y = y_p + y_h$. Then

$$\mathcal{L}[y] = \mathcal{L}[y_p + y_h] = \mathcal{L}[y_p] + \mathcal{L}[y_h] = k(x) + \mathcal{L}[y_h].$$

Hence $\mathcal{L}[y] = k(x)$ if and only if $\mathcal{L}[y_h] = 0$. □

13.8 Variation of Constants

There is a method to find the general solution of a linear inhomogeneous equation of arbitrary order, *provided you already know the solutions to the homogeneous equation.* We won't explain this method here, but merely show you the answer you get in the case of second order equations. If $y_1(x)$ and $y_2(x)$ are solutions to the homogeneous equation

$$y''(x) + a(x)y'(x) + b(x)y(x) = 0$$

for which

$$W(x) \stackrel{\text{def}}{=} y_1(x)y_2'(x) - y_1'(x)y_2(x) \neq 0,$$

then the general solution of the inhomogeneous equation

$$y''(x) + a(x)y'(x) + b(x)y(x) = f(x)$$

is given by

$$y(x) = -y_1(x)\int \frac{y_2(\xi)f(\xi)}{W(\xi)}\mathrm{d}\xi + y_2(x)\int \frac{y_1(\xi)f(\xi)}{W(\xi)}\mathrm{d}\xi.$$

For more details you should take a more advanced course like Math 319 or 320.

13.8.1 Undetermined Coefficients

The easiest way to find a particular solution y_p to the inhomogeneous equation is the method of undetermined coefficients or "educated guessing." Unlike the method of "variation of constants" which was (hardly) explained in the previous section, this method does not work for all equations. But it does give you the answer for a few equations which show up often enough to make it worth knowing the method.

The basis of the "method" is this: it turns out that many of the second order equations with you run into have the form

$$y'' + ay' + by = f(t),$$

where a and b are constants, and where the righthand side $f(t)$ comes from a fairly short list of functions. For all $f(t)$ in this list you memorize (yuck!) a particular solution y_p. With the particular solution in hand you can then find the general solution by adding it to the general solution of the homogeneous equation.

Here is the list:

$f(t) =$ **polynomial in** t In this case you try $y_p(t) =$ some other polynomial in t with the same degree as $f(t)$.

Exceptions: if $r = 0$ is a root of the characteristic equation, then you must try a polynomial $y_p(t)$ of degree one higher than $f(t)$;

if $r = 0$ is a double root then the degree of $y_p(t)$ must be two more than the degree of $f(t)$.

$f(t) = e^{at}$ try $y_p(t) = Ae^{at}$.

Exceptions: if $r = a$ is a root of the characteristic equation, then you must try $y_p(t) = Ate^{at}$;

if $r = a$ is a double root then try $y_p(t) = At^2e^{at}$.

$f(t) = \sin bt$ **or** $f(t) = \cos bt$ In both cases, try $y_p(t) = A\cos bt + B\sin bt$.

Exceptions: if $r = bi$ is a root of the characteristic equation, then you should try $y_p(t) = t(A\cos bt + B\sin bt)$.

$f(t) = e^{at}\sin bt$ **or** $f(t) = e^{at}\cos bt$ Try $y_p(t) = e^{at}(A\cos bt + B\sin bt)$.

Exceptions: if $r = a + bi$ is a root of the characteristic equation, then you should try $y_p(t) = te^{at}(A\cos bt + B\sin bt)$.

13.8.2 Example

Find the general solution to the following equations

$$y'' + xy' - y = 2e^x \tag{13.12}$$

$$y'' - 2y' + y = \sqrt{1 + x^2} \tag{13.13}$$

The first equation does not have constant coefficients so the method doesn't apply. Sorry, but we can't solve this equation in this course.[1]

[1] Who says you can't solve this equation? For equation (13.12) you *can* find a solution by computing its Taylor series! For more details you should again take a more advanced course (like MATH 319), or, in this case, give it a try yourself.

The second equation does have constant coefficients, so we can solve the homogeneous equation ($y'' - 2y' + y = 0$), but the righthand side does not appear in our list. Again, the method doesn't work.

13.8.3 A more upbeat example.

To find a particular solution of

$$y'' - y' + y = 3t^2$$

we note that (1) the equation is linear with constant coefficients, and (2) the right hand side is a polynomial, so it's in our list of "right hand sides for which we know what to guess." We try a polynomial of the same degree as the right hand side, namely 2. We don't know which polynomial, so *we leave its coefficients undetermined* (whence the name of the method.) I.e. we try

$$y_p(t) = A + Bt + Ct^2.$$

To see if this is a solution, we compute

$$y_p'(t) = B + 2Ct, \qquad y_p''(t) = 2C,$$

so that

$$y_p'' - y_p' + y_p = (A - B + 2C) + (B - 2C)t + Ct^2.$$

Thus $y_p'' - y_p' + y_p = 3t^2$ if and only if

$$A - B + 2C = 0, \quad B - 2C = 0, \quad C = 3.$$

Solving these equations leads to $C = 3$, $B = 2C = 6$ and $A = B - 2C = 0$. We conclude that

$$y_p(t) = 6t + 3t^2$$

is a particular solution.

13.8.4 Another example, which is rather long, but that's because it is meant to cover several cases

Find the general solution to the equation

$$y'' + 3y' + 2y = t + t^3 - e^t + 2e^{-2t} - e^{-t}\sin 2t.$$

Solution: First we find the characteristic equation,

$$r^2 + 3r + 2 = (r + 2)(r + 1) = 0.$$

The characteristic roots are $r_1 = -1$, and $r_2 = -2$. The general solution to the homogeneous equation is

$$y_h(t) = C_1e^{-t} + C_2e^{-2t}.$$

We now look for a particular solutions. Initially it doesn't look very good as the righthand side does not appear in our list. However, the righthand side is a sum of five terms, each of which is in our list.

Abbreviate $\mathcal{L}[y] = y'' + 3y' + 2y$. Then we will find functions $y_1, \dots, y_4$ for which one has

$$\mathcal{L}[y_1] = t + t^3, \quad \mathcal{L}[y_2] = -e^t, \quad \mathcal{L}[y_3] = 2e^{-2t}, \quad \mathcal{L}[y_4] = -e^{-t}\sin 2t.$$

Then, by the Superposition Principle (Theorem 13.6.1) you get that $y_p \stackrel{\text{def}}{=} y_1 + y_2 + y_3 + y_4$ satisfies

$$\mathcal{L}[y_p] = \mathcal{L}[y_1] + \mathcal{L}[y_2] + \mathcal{L}[y_3] + \mathcal{L}[y_4] = t + t^3 - e^t + 2e^{-2t} - e^{-t}\sin 2t.$$

So y_p (once we find it) is a particular solution.

Now let's find $y_1, \dots, y_4$.

$y_1(t)$ the righthand side $t + t^3$ is a polynomial, and $r = 0$ is not a root of the characteristic equation, so we try a polynomial of the same degree. Try

$$y_1(t) = A + Bt + Ct^2 + Dt^3.$$

Here A, B, C, D are the undetermined coefficients that give the method its name. You compute

$$\begin{aligned}\mathcal{L}[y_1] &= y_1'' + 3y_1' + 2y_1 \\ &= (2C + 6Dt) + 3(B + 2Ct + 3Dt^2) + 2(A + Bt + Ct^2 + Dt^3) \\ &= (2C + 3B + 2A) + (2B + 6C + 6D)t + (2C + 9D)t^2 + 2Dt^3.\end{aligned}$$

So to get $\mathcal{L}[y_1] = t + t^3$ we must impose the equations

$$2D = 1, \quad 2C + 9D = 0, \quad 2B + 6C + 6D = 1, \quad 2C + 6B + 2A = 0.$$

You can solve these equations one-by-one, with result

$$D = \tfrac{1}{2}, \quad C = -\tfrac{9}{4}, \quad B = -\tfrac{23}{4}, \quad A = \tfrac{87}{8},$$

and thus

$$y_1(t) = \tfrac{87}{8} - \tfrac{23}{4}t - \tfrac{9}{4}t^2 + \tfrac{1}{2}t^3.$$

$y_2(t)$ We want $y_2(t)$ to satisfy $\mathcal{L}[y_2] = -e^t$. Since $e^t = e^{at}$ with $a = 1$, and $a = 1$ is not a characteristic root, we simply try $y_2(t) = Ae^t$. A quick calculation gives

$$\mathcal{L}[y_2] = Ae^t + 3Ae^t + 2Ae^t = 6Ae^t.$$

To achieve $\mathcal{L}[y_2] = -e^t$ we therefore need $6A = -1$, i.e. $A = -\frac{1}{6}$. Thus

$$y_2(t) = -\tfrac{1}{6}e^t.$$

$y_3(t)$ We want $y_3(t)$ to satisfy $\mathcal{L}[y_3] = -e^{-2t}$. Since $e^{-2t} = e^{at}$ with $a = -2$, and $a = -2$ is a characteristic root, we can't simply try $y_3(t) = Ae^{-2t}$. Instead you have to try $y_3(t) = Ate^{-2t}$. Another calculation gives

$$\begin{aligned}\mathcal{L}[y_3] &= (4t - 4)Ae^{-2t} + 3(-2t + 2)Ae^{-2t} + 2Ate^{-2t} \qquad \text{(factor out } Ae^{-2t}\text{)} \\ &= \big[(4 + 3(-2) + 2)t + (-4 + 3)\big]Ae^{-2t} \\ &= -Ae^{-2t}.\end{aligned}$$

Note that all the terms with te^{-2t} cancel: this is no accident, but a consequence of the fact that $a = -2$ is a characteristic root.

To get $\mathcal{L}[y_3] = 2e^{-2t}$ we see we have to choose $A = -2$. We find

$$y_3(t) = -2te^{-2t}.$$

$y_4(t)$ Finally, we need a function $y_4(t)$ for which one has $\mathcal{L}[y_4] = -e^{-t}\sin 2t$. The list tells us to try

$$y_4(t) = e^{-t}\big(A\cos 2t + B\sin 2t\big).$$

(Since $-1+2i$ is not a root of the characteristic equation we are not in one of the exceptional cases.)

Diligent computation yields

$$\begin{aligned} y_4(t) &= & Ae^{-t}\cos 2t &+ Be^{-t}\sin 2t \\ y_4'(t) &= & (-A+2B)e^{-t}\cos 2t &+ (-B-2A)e^{-t}\sin 2t \\ y_4''(t) &= & (-3A-4B)e^{-t}\cos 2t &+ (-3B+4A)e^{-t}\sin 2t \end{aligned}$$

so that

$$\mathcal{L}[y_4] = (-4A+2B)e^{-t}\cos 2t + (-2A-4B)e^{-t}\sin 2t.$$

We want this to equal $-e^{-t}\sin 2t$, so we have to find A, B with

$$-4A+2B=0, \quad -2A-4B=-1.$$

The first equation implies $B=2A$, the second then gives $-10A=-1$, so $A=\frac{1}{10}$ and $B=\frac{2}{10}$. We have found

$$y_4(t) = \tfrac{1}{10}e^{-t}\cos 2t + \tfrac{2}{10}e^{-t}\sin 2t.$$

After all these calculations we get the following impressive particular solution of our differential equation,

$$y_p(t) = \tfrac{87}{8} - \tfrac{23}{4}t - \tfrac{9}{4}t^2 + \tfrac{1}{2}t^3 - \tfrac{1}{6}e^t - 2te^{-2t} + \tfrac{1}{10}e^{-t}\cos 2t + \tfrac{2}{10}e^{-t}\sin 2t$$

and the even more impressive *general* solution to the equation,

$$\begin{aligned} y(t) &= y_h(y) + y_p(t) \\ &= C_1e^{-t} + C_2e^{-2t} \\ &\quad + \tfrac{87}{8} - \tfrac{23}{4}t - \tfrac{9}{4}t^2 + \tfrac{1}{2}t^3 \\ &\quad\quad - \tfrac{1}{6}e^t - 2te^{-2t} + \tfrac{1}{10}e^{-t}\cos 2t + \tfrac{2}{10}e^{-t}\sin 2t. \end{aligned}$$

You shouldn't be put off by the fact that the result is a pretty long formula, and that the computations took up two pages. The approach is to (i) break up the right hand side into terms which are in the list at the beginning of this section, (ii) to compute the particular solutions for each of those terms and (iii) to use the Superposition Principle (Theorem 13.6.1) to add the pieces together, resulting in a particular solution for the whole right hand side you started with.

13.9 Applications of Second Order Linear Equations

13.9.1 Spring with a weight

In example 13.5.1 we showed that the height $y(t)$ a mass m suspended from a spring with constant k satisfies

$$my''(t) + ky(t) = mg, \quad \text{or} \quad y''(t) + \frac{k}{m}y(t) = g. \tag{13.14}$$

This is a Linear Inhomogeneous Equation whose homogeneous equation, $y'' + \frac{k}{m}y = 0$ has

$$y_h(t) = C_1 \cos \omega t + C_2 \sin \omega t$$

as general solution, where $\omega = \sqrt{k/m}$. The right hand side is a constant, which is a polynomial of degree zero, so the method of "educated guessing" applies, and we can find a particular solution by trying a constant $y_p = A$ as particular solution. You find that $y_p'' + \frac{k}{m}y_p = \frac{k}{m}A$, which will equal g if $A = \frac{mg}{k}$. Hence the general solution to the "spring with weight equation" is

$$y(t) = \frac{mg}{k} + C_1 \cos \omega t + C_2 \sin \omega t.$$

To solve the initial value problem $y(0) = y_0$ and $y'(0) = v_0$ we solve for the constants C_1 and C_2 and get

$$y(t) = \frac{mg}{k} + \frac{v_0}{\omega}\sin(\omega t) + \big(y_0 - \frac{mg}{k}\big)\cos(\omega t).$$

which you could rewrite as

$$y(t) = \frac{mg}{k} + Y\cos(\omega t - \phi)$$

for certain numbers Y, ϕ.

The weight in this model just oscillates up and down forever: this motion is called a ***simple harmonic oscillation,*** and the equation (13.14) is called the equation of the ***Harmonic Oscillator.***

13.9.2 The pendulum equation

In example 13.5.2 we saw that the angle $\theta(t)$ subtended by a swinging pendulum satisfies the ***pendulum equation***,

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + \frac{g}{L}\sin\theta(t) = 0. \qquad (13.9)$$

This equation is *not linear* and cannot be solved by the methods you have learned in this course. However, if the oscillations of the pendulum are small, i.e. if θ is small, then we can approximate $\sin\theta$ by θ. Remember that the error in this approximation is the remainder term in the Taylor expansion of $\sin\theta$ at $\theta = 0$. According to Lagrange this is

$$\sin\theta = \theta + R_2(\theta), \qquad R_2(\theta) = \cos\tilde\theta\ \frac{\theta^3}{3!} \text{ with } |\tilde\theta| \le \theta.$$

When θ is small, e.g. if $|\theta| \le 10^\circ \approx 0.175$ radians then compared to θ the error is at most

$$\left|\frac{R_3(\theta)}{\theta}\right| \le \frac{(0.175)^2}{3!} \approx 0.005,$$

in other words, the error is no more than half a percent.

So for small angles we will assume that $\sin\theta \approx \theta$ and hence $\theta(t)$ *almost* satisfies the equation

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + \frac{g}{L}\theta(t) = 0. \qquad (13.15)$$

In contrast to the pendulum equation (13.9), this equation is linear, and we could solve it right now.

The procedure of replacing inconvenient quantities like $\sin\theta$ by more manageable ones (like θ) in order to end up with linear equations is called ***linearization.*** Note that the solutions to the linearized equation (13.15), which we will derive in a moment, are not solutions of the Pendulum Equation (13.9). However, if the solutions we find have small angles (have $|\theta|$ small), then the Pendulum Equation and its linearized form (13.15) are almost the same, and "you would think that their solutions should also be almost the same." I put that in quotation marks, because (1) it's not a very precise statement and (2) if it were more precise, you would have to prove it, which is not easy, and not a topic for this course (or even MATH 319 – take MATH 419 or 519 for more details.)

Let's solve the linearized equation (13.15). Setting $\theta = e^{rt}$ you find the characteristic equation

$$r^2 + \frac{g}{L} = 0$$

which has two complex roots, $r_\pm = \pm i\sqrt{\frac{g}{L}}$. Therefore, the general solution to (13.15) is

$$\theta(t) = A\cos\left(\sqrt{\frac{g}{L}}t\right) + B\sin\left(\sqrt{\frac{g}{L}}t\right),$$

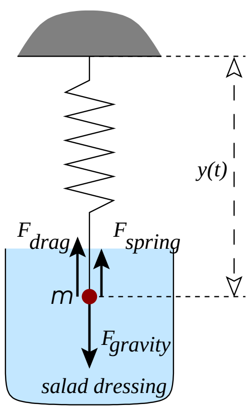
and you would expect the general solution of the Pendulum Equation (13.9) to be almost the same. So you see that a pendulum will oscillate, and that the period of its oscillation is given by

$$T = 2\pi\sqrt{\frac{L}{g}}.$$

Once again: because we have used a linearization, you should expect this statement to be valid only for small oscillations. When you study the Pendulum Equation instead of its linearization (13.15), you discover that the period T of oscillation actually depends on the amplitude of the oscillation: the bigger the swings, the longer they take.

13.9.3 The effect of friction

A real weight suspended from a real spring will of course not oscillate forever. Various kinds of friction will slow it down and bring it to a stop. As an example let's assume that air drag is noticeable, so, as the weight moves the surrounding air will exert a force on the weight (To make this more likely, assume the weight is actually moving in some viscous liquid like salad oil.) This drag is stronger as the weight moves faster.



A simple model is to assume that the friction force is proportional to the velocity of the weight,

$$F_{\text{friction}} = -hy'(t).$$

This adds an extra term to the oscillator equation (13.14), and gives

$$my''(t) = F_{\text{grav}} + F_{\text{friction}} = -ky(t) + mg - hy'(t)$$

i.e.

$$my''(t) + hy'(t) + ky(t) = mg. \tag{13.16}$$

This is a second order linear homogeneous differential equation with constant coefficients. A particular solution is easy to find, $y_p = mg/k$ works again.

To solve the homogeneous equation you try $y = e^{rt}$, which leads to the characteristic equation

$$mr^2 + hr + k = 0,$$

whose roots are

$$r_\pm = \frac{-h \pm \sqrt{h^2 - 4mk}}{2m}$$

If friction is large, i.e. if $h > \sqrt{4km}$, then the two roots $r_\pm$ are real, and all solutions are of exponential type,

$$y(t) = \frac{mg}{k} + C_+ e^{r_+ t} + C_- e^{r_- t}.$$

Both roots $r_\pm$ are negative, so all solutions satisfy

$$\lim_{t\to\infty} y(t) = 0.$$

If friction is weak, more precisely, if $h < \sqrt{4mk}$ then the two roots $r_\pm$ are complex numbers,

$$r_\pm = -\frac{h}{2m} \pm i\omega, \quad \text{with } \omega = \frac{\sqrt{4km - h^2}}{2m}.$$

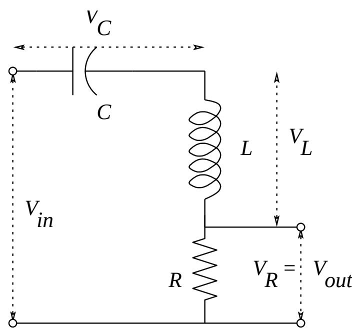
The general solution in this case is

$$y(t) = \frac{mg}{k} + e^{-\frac{h}{2m}t}\Big(A\cos\omega t + B\sin\omega t\Big).$$

These solutions also tend to zero as $t \to \infty$, but they oscillate infinitely often.

13.9.4 Electric circuits

Many equations in physics and engineering have the form (13.16). For example in the electric circuit in the diagram a time varying voltage $V_{in}(t)$ is applied to a resistor R, an inductance L and a capacitor C. This causes a current $I(t)$ to flow through the circuit. How much is this current, and how much is, say, the voltage across the resistor?



Electrical engineers will tell you that the total voltage $V_{in}(t)$ must equal the sum of the voltages $V_R(t)$, $V_L(t)$ and $V_C(t)$ across the three components. These voltages are related to the current $I(t)$ which flows through the three components as follows:

$$\begin{aligned} V_R(t) &= RI(t) \\ \frac{\mathrm{d}V_C(t)}{\mathrm{d}t} &= \frac{1}{C}I(t) \\ V_L(t) &= L\frac{\mathrm{d}I(t)}{\mathrm{d}t}. \end{aligned}$$

Surprisingly, these little electrical components know calculus! (Here R, C and L are constants depending on the particular components in the circuit. They are measured in "Ohm," "Farad," and "Henry.")

Starting from the equation

$$V_{in}(t) = V_R(t) + V_L(t) + V_C(t)$$

you get

$$\begin{aligned} V_{in}'(t) &= V_R'(t) + V_L'(t) + V_C'(t) \\ &= RI'(t) + LI''(t) + \frac{1}{C}I(t) \end{aligned}$$

In other words, for a given input voltage the current $I(t)$ satisfies a second order inhomogeneous linear differential equation

$$L\frac{\mathrm{d}^2 I}{\mathrm{d}t^2} + R\frac{\mathrm{d}I}{\mathrm{d}t} + \frac{1}{C}I = V_{in}'(t). \tag{13.17}$$

Once you know the current $I(t)$ you get the output voltage $V_{\mathrm{out}}(t)$ from

$$V_{\mathrm{out}}(t) = RI(t).$$

In general you can write down a differential equation for any electrical circuit. As you add more components the equation gets more complicated, but if you stick to resistors, inductances and capacitors the equations will always be linear, albeit of very high order.

The reader should consider watching YouTube by Zach Star for some current applications of differential equations.

13.10 PROBLEMS

GENERAL QUESTIONS

840. Classify each of the following as homogeneous linear, inhomogeneous linear, or nonlinear and specify the order. For each linear equation say whether or not the coefficients are constant.

(*i*) $y'' + y = 0$	(*ii*) $xy'' + yy' = 0$
(*iii*) $xy'' - y' = 0$	(*iv*) $xy'' + yy' = x$
(*v*) $xy'' - y' = x$	(*vi*) $y' + y = xe^x$.

841. (*i*) Show that $y = x^2 + 5$ is a solution of $xy'' - y' = 0$.

(*ii*) Show that $y = C_1x^2 + C_2$ is a solution of $xy'' - y' = 0$.

842. (***i***) Show that $y = (\tan(c_1x + c_2))/c_1$ is a solution of $yy'' = 2(y')^2 - 2y'$.

(***ii***) Show that $y_1 = \tan(x)$ and $y_2 = 1$ are solutions of this equation, but that $y_1 + y_2$ is not.

(***iii***) Is the equation linear homogeneous?

SEPARATION OF VARIABLES

843. Consider the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{4 - y^2}{4}\,.$$

(a) Find the solutions y_0, y_1, y_2, and y_3 which satisfy $y_0(0) = 0$, $y_1(0) = 1$, $y_2(0) = 2$ and $y_3(0) = 3$.

(b) Find $\lim_{t\to\infty} y_k(t)$ for $k = 1, 2, 3$.

(c) Find $\lim_{t\to-\infty} y_k(t)$ for $k = 1, 2, 3$.

(d) Graph the four solutions y_0, …, y_3.

(e) Show that the quantity $x = (y + 2)/4$ satisfies the so-called ***Logistic Equation***

$$\frac{dx}{dt} = x(1 - x).$$

(Hint: if $x = (y+2)/4$, then $y = 4x-2$; substitute this in both sides of the diffeq for y). †400

* * *

In each of the following problems you should find the function y of x which satisfies the conditions (A is an unspecified constant: you should at least indicate for which values of A your solution is valid.)

844. $\dfrac{\mathrm{d}y}{\mathrm{d}x} + x^2y = 0, y(1) = 5.$ †400

845. $\dfrac{\mathrm{d}y}{\mathrm{d}x} + (1 + 3x^2)y = 0, y(1) = 1.$ †400

846. $\dfrac{\mathrm{d}y}{\mathrm{d}x} + x\cos^2 y = 0, y(0) = \frac{\pi}{3}.$ †400

847. $\dfrac{\mathrm{d}y}{\mathrm{d}x} + \dfrac{1 + x}{1 + y} = 0, y(0) = A.$ †401

848. $\dfrac{\mathrm{d}y}{\mathrm{d}x} + 1 - y^2 = 0, y(0) = A.$ †401

849. $\dfrac{\mathrm{d}y}{\mathrm{d}x} + 1 + y^2 = 0, y(0) = A.$ †401

850. Find the function y of x which satisfies the **initial** value problem:

$$\frac{dy}{dx} + \frac{x^2 - 1}{y} = 0 \qquad y(0) = 1$$

†401

851. Find the **general** solution of

$$\frac{dy}{dx} + 2y + e^x \equiv 0$$

†401

852. $\dfrac{\mathrm{d}y}{\mathrm{d}x} - (\cos x)y = e^{\sin x}, y(0) = A.$ †401

853. $y^2\frac{\mathrm{d}y}{\mathrm{d}x} + x^3 = 0, y(0) = A.$ †401

854. Read Example 13.4.2 on "Leaky bucket dating" again. In that example we assumed that $\frac{a}{A}\sqrt{K} = 2$.

(a) Solve diffeq for $h(t)$ without assuming $\frac{a}{A}\sqrt{K} = 2$. Abbreviate $C = \frac{a}{A}\sqrt{K}$.

(b) If in an experiment one found that the bucket empties in 20 seconds after being filled to height 20 cm, then how much is the constant C?

LINEAR HOMOGENEOUS

855. (a) Show that $y = 4e^x + 7e^{2x}$ is a solution of $y'' - 3y' + 2y = 0$.

(b) Show that $y = C_1e^x + C_2e^{2x}$ is a solution of $y'' - 3y' + 2y = 0$.

(c) Find a solution of $y'' - 3y' + 2y = 0$ such that $y(0) = 7$ and $y'(0) = 9$.

856. (a) Find all solutions of $\frac{\mathrm{d}y}{\mathrm{d}x} + 2y = 0$.

(b) Find all solutions of $\frac{\mathrm{d}y}{\mathrm{d}x} + 2y = e^{-x}$.

(c) Find y if $\frac{\mathrm{d}y}{\mathrm{d}x} + 2y = e^{-x}$ and $y = 7$ when $x = 0$.

857. (a) Find all real solutions of

$$\frac{\mathrm{d}^2y}{\mathrm{d}t^2} - 6\frac{\mathrm{d}y}{\mathrm{d}t} + 10y = 0.$$

(b) Find y if

$$y'' - 6y' + 10y = 0,$$

and in addition y satisfies the initial conditions $y(0) = 7$, and $y'(0) = 11$. †401

858. Solve the **initial** value problem:

$$y'' - 5y' + 4y \equiv 0$$

$$y(0) = 2$$

$$y'(0) = -1$$

†401

859. For y as a function of x, find the **general** solution of the equation:

$$y'' - 2y' + 10y \equiv 0$$

†401

* * *

Find the general solution $y = y(x)$ of the following differential equations

860. $\frac{\mathrm{d}^4y}{\mathrm{d}x^4} = y$ †401

861. $\frac{\mathrm{d}^4y}{\mathrm{d}x^4} + y = 0$ †401

862. $\frac{\mathrm{d}^4y}{\mathrm{d}x^4} - \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 0$ †401

863. $\frac{\mathrm{d}^4y}{\mathrm{d}x^4} + \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 0$ †401

864. $\dfrac{d^3y}{dx^3} + y = 0$ †401

865. $\dfrac{d^3y}{dx^3} - y = 0$ †402

866. $y^{(4)}(t) - 2y''(t) - 3y(t) = 0$ †402

867. $y^{(4)}(t) + 4y''(t) + 3y(t) = 0.$ †402

868. $y^{(4)}(t) + 2y''(t) + 2y(t) = 0.$ †402

869. $y^{(4)}(t) + y''(t) - 6y(t) = 0.$

870. $y^{(4)}(t) - 8y''(t) + 15y(t) = 0.$

871. $f'''(x) - 125f(x) = 0.$ †403

872. $u^{(5)}(x) - 32u(x) = 0.$ †403

873. $u^{(5)}(x) + 32u(x) = 0.$

874. $y'''(t) - 5y''(t) + 6y'(t) - 2y(t) = 0.$ †404

875. $h^{(4)}(t) - h^{(3)}(t) + 4h''(t) - 4h(t) = 0.$

876. $z'''(x) - 5z''(x) + 4z(x) = 0.$ †404

* * *

Solve each of the following initial value problems. Your final answer should not use complex numbers, but you may use complex numbers to find it.

877. $y''+9y = 0,\ y(0) = 0, y'(0) = -3.$ †404

878. $y''+9y = 0,\ y(0) = -3, y'(0) = 0.$ †404

879. $y'' - 5y' + 6y = 0,\ y(0) = 0, y'(0) = 1.$ †404

880. $y'' + 5y' + 6y = 0,\ y(0) = 1, y'(0) = 0.$ †404

881. $y'' + 5y' + 6y = 0,\ y(0) = 0, y'(0) = 1.$ †404

882. $y'' - 6y' + 5y = 0,\ y(0) = 1, y'(0) = 0.$ †404

883. $y'' - 6y' + 5y = 0,\ y(0) = 0, y'(0) = 1.$ †404

884. $y'' + 6y' + 5y = 0,\ y(0) = 1, y'(0) = 0.$ †404

885. $y'' + 6y' + 5y = 0,\ y(0) = 0, y'(0) = 1.$ †404

886. $y'' - 4y' + 5y = 0,\ y(0) = 1, y'(0) = 0.$ †404

887. $y'' - 4y' + 5y = 0,\ y(0) = 0, y'(0) = 1.$ †404

888. $y'' + 4y' + 5y = 0,\ y(0) = 1, y'(0) = 0.$ †404

889. $y'' + 4y' + 5y = 0,\ y(0) = 0, y'(0) = 1.$ †404

890. $y'' - 5y' + 6y = 0,\ y(0) = 1, y'(0) = 0.$ †404

891. $f'''(t) + f''(t) - f'(t) + 15f(t) = 0,$ with initial conditions $f(0) = 0, f'(0) = 1, f''(0) = 0.$

†405

LINEAR INHOMOGENEOUS

892. Find particular solutions of

$$y'' - 3y' + 2y = e^{3x}$$
$$y'' - 3y' + 2y = e^{x}$$
$$y'' - 3y' + 2y = 4e^{3x} + 5e^{x}$$

* * *

893. Find a **particular** solution of the equation:

$$y'' + y' + 2y = e^{x} + x + 1$$

†405

Find the general solution $y(t)$ of the following differential equations

894. $\dfrac{d^2y}{dt^2} - y = 2$ †405

895. $\dfrac{d^2y}{dt^2} - y = 2e^t$ †405

896. $\dfrac{d^2y}{dt^2} + 9y = \cos 3t$ †405

897. $\dfrac{d^2y}{dt^2} + 9y = \cos t$ †405

898. $\dfrac{d^2y}{dt^2} + y = \cos t$ †405

899. $\dfrac{d^2y}{dt^2} + y = \cos 3t.$ †405

900. Find y if

(a) $\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} + y = 0$ $\quad y(0) = 2,\quad y'(0) = 3$

(b) $\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} + y = e^{-x}$ $\quad y(0) = 0,\quad y'(0) = 0$

(c) $\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} + y = xe^{-x}$ $\quad y(0) = 0,\quad y'(0) = 0$

(d) $\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} + y = e^{-x} + xe^{-x}$ $\quad y(0) = 2,\quad y'(0) = 3.$

Hint: Use the Superposition Principle to save work.

901. (i) Find the general solution of

$$z'' + 4z' + 5z = e^{it}$$

using complex exponentials. †405

(ii) Solve

$$z'' + 4z' + 5z = \sin t$$

using your solution to question (i).

(iii) Find a solution for the equation

$$z'' + 2z' + 2z = 2e^{-(1-i)t}$$

in the form $z(t) = u(t)e^{-(1-i)t}$.

(iv) Find a solution for the equation

$$x'' + 2x' + 2x = 2e^{-t}\cos t.$$

Hint: Take the real part of the previous answer.

(v) Find a solution for the equation

$$y'' + 2y' + 2y = 2e^{-t}\sin t.$$

APPLICATIONS

902. A population of bacteria grows at a rate proportional to its size. Write and solve a differential equation which expresses this. If there are 1000 bacteria after one hour and 2000 bacteria after two hours, how many bacteria are there after three hours?

903. Rabbits in Madison have a birth rate of 5% per year and a death rate (from old age) of 2% per year. Each year 1000 rabbits get run over and 700 rabbits move in from Sun Prairie. †405

(***i***) Write a differential equation which describes Madison's rabbit population at time t.

(***ii***) If there were 12,000 rabbits in Madison in 1991, how many are there in 1994?

904. According to ***Newton's law of cooling*** the rate $\mathrm{d}T/\mathrm{d}t$ at which an object cools is proportional to the difference $T - A$ between its temperature T and the ambient temperature A. The differential equation which expresses this is

$$\frac{\mathrm{d}T}{\mathrm{d}t} = k(T - A)$$

where $k < 0$ and A are constants.

(***i***) Solve this equation and show that every solution satisfies

$$\lim_{t\to\infty} T = A.$$

(***ii***) A cup of coffee at a temperature of 180^oF sits in a room whose temperature is 75^0F. In five minutes its temperature has dropped to 150°F. When will its temperature be 90°F? What is the limit of the temperature as $t \to \infty$?

†406

905. Retaw is a mysterious living liquid; it grows at a rate of 5% of its volume per hour. A scientist has a tank initially holding y_0 gallons of retaw and removes retaw from the tank continuously at the rate of 3 gallons per hour. †406

(***i***) Find a differential equation for the number $y(t)$ of gallons of retaw in the tank at time t.

(***ii***) Solve this equation for y as a function of t. (The initial volume y_0 will appear in your answer.)

(***iii***) What is $\lim_{t\to\infty} y(t)$ if $y_0 = 100$?

(***iv***) What should the value of y_0 be so that $y(t)$ remains constant?

906. A 1000 gallon vat is full of 25% solution of acid. Starting at time $t = 0$ a 40% solution of acid is pumped into the vat at 20 gallons per minute. The solution is kept well mixed and drawn off at 20 gallons per minute so as to maintain the total value of 1000 gallons. Derive an expression for the acid concentration at times $t > 0$. As $t \to \infty$ what percentage solution is approached? †406

907. The volume of a lake is $V = 10^9$ cubic feet. Pollution P runs into the lake at 3 cubic feet per minute, and clean water runs in at 21 cubic feet per minute. The lake drains at a rate of 24 cubic feet per minute so its volume is constant. Let C be the concentration of pollution in the lake; i.e. $C = P/V$.

($\boldsymbol{i}$) Give a differential equation for C.

($\boldsymbol{ii}$) Solve the differential equation. Use the initial condition $C = C_0$ when $t = 0$ to evaluate the constant of integration.

($\boldsymbol{iii}$) There is a critical value C^* with the property that for any solution $C = C(t)$ we have

$$\lim_{t\to\infty} C = C^*.$$

Find C^*. If $C_0 = C^*$, what is $C(t)$?

908. A philanthropist endows a chair. This means that she donates an amount of money B_0 to the university. The university invests the money (it earns interest) and pays the salary of a professor. Denote the interest rate on the investment by r (e.g. if $r = .06$, then the investment earns interest at a rate of 6% per year) the salary of the professor by a (e.g. $a = \$50,000$ per year), and the balance in the investment account at time t by B.

($\boldsymbol{i}$) Give a differential equation for B.

($\boldsymbol{ii}$) Solve the differential equation. Use the initial condition $B = B_0$ when $t = 0$ to evaluate the constant of integration.

($\boldsymbol{iii}$) There is a critical value B^* with the property that (1) if $B_0 < B^*$, then there is a $t > 0$ with $B(t) = 0$ (i.e. the account runs out of money) while (2) if $B_0 > B^*$, then $\lim_{t\to\infty} B = \infty$. Find B^*.

($\boldsymbol{iv}$) This problem is like the pollution problem except for the signs of r and a. Explain.

909. A citizen pays social security taxes of a dollars per year for T_1 years, then retires, then receives payments of b dollars per year for T_2 years, then dies. The account which receives and dispenses the money earns interest at a rate of r% per year and has no money at time $t = 0$ and no money at the time $t = T_1 + T_2$ of death. Find two differential equations for the balance $B(t)$ at time t; one valid for $0 \le t \le T_1$, the other valid for $T_1 \le t \le T_1 + T_2$. Express the ratio b/a in terms of T_1, T_2, and r. Reasonable values for T_1, T_2, and r are $T_1 = 40$, $T_2 = 20$, and $r = 5\% = .05$. This model ignores inflation. Notice that $0 < dB/\mathrm{d}t$ for $0 < t < T_1$, that $dB/\mathrm{d}t < 0$ for $T_1 < t < T_1 + T_2$, and that the account earns interest *even for* $T_1 < t < T_1 + T_2$.

910. A 300 gallon tank is full of milk containing 2% butterfat. Milk containing 1% butterfat is pumped in a 10 gallons per minute starting at 10:00 AM and the well mixed milk is drained off at 15 gallons per minute. What is the percent butterfat in the milk in the tank 5 minutes later at 10:05 AM? Hint: How much milk is in the tank at time t? How much butterfat is in the milk at time $t = 0$?

911. A sixteen pound weight is suspended from the lower end of a spring whose upper end is attached to a rigid support. The weight extends the spring by half a foot. It is struck by a sharp blow which gives it an initial downward velocity of eight feet per second. Find its position as a function of time.

912. A sixteen pound weight is suspended from the lower end of a spring whose upper end is attached to a rigid support. The weight extends the spring by half a foot. The weight is pulled down one feet and released. Find its position as a function of time.

913. The equation for the displacement $y(t)$ from equilibrium of a spring subject to a forced vibration of frequency ω is

$$\frac{d^2y}{\mathrm{d}t^2} + 4y = \sin(\omega t). \tag{13.18}$$

(***i***) Find the solution $y = y(\omega, t)$ of (13.18) for $\omega \neq 2$ if $y(0) = 0$ and $y'(0) = 0$.

(***ii***) What is $\lim_{\omega\to 2} y(\omega, t)$?

(***iii***) Find the solution $y(t)$ of

$$\frac{d^2y}{dt^2} + 4y = \sin(2t) \tag{13.19}$$

if $y(0) = 0$ and $y'(0) = 0$. (Hint: Compare with (13.18).)

914. Suppose that an undamped spring is subjected to an external periodic force so that its position y at time t satisfies the differential equation

$$\frac{d^2y}{dt^2} + \omega_0^2 y = c\sin(\omega t).$$

(***i***) Show that the general solution is

$$y = C_1 \cos\omega_0 t + C_2 \sin\omega_0 t + \frac{c}{\omega_0^2 - \omega^2}\sin\omega t.$$

when $\omega_0 \neq \omega$.

(***ii***) Solve the equation when $\omega = \omega_0$.

(***iii***) Show that in part (i) the solution remains bounded as $t \to \infty$ but in part (ii) this is not so. (This phenomenon is called ***resonance***. To see an example of resonance try Googling "Tacoma Bridge Disaster.")

915. Have look at the electrical circuit equation (13.17) from §13.9.4.

(***i***) Find the general solution of (13.17), assuming that $V_{\text{in}}(t)$ does not depend on time t. What is $\lim_{t\to\infty} I(t)$?

(***ii***) Assume for simplicity that $L = C = 1$, and that the resistor has been short circuited, i.e. that $R = 0$. If the input voltage is a sinusoidal wave,

$$V_{\text{in}}(t) = A\sin\omega t, \qquad (\omega \neq 1)$$

then find a particular solution, and then the general solution.

(***iii***) Repeat problem (ii) with $\omega = 1$.

(***iv***) Suppose again that $L = C = 1$, but now assume that $R > 0$. Find the general solution when $V_{\text{in}}(t)$ is constant.

(***v***) Still assuming $L = C = 1$, $R > 0$ find a particular solution of the equation when the input voltage is a sinusoidal wave

$$V_{\text{in}}(t) = A\sin\omega t.$$

916. You are watching a buoy bobbing up and down in the water. Assume that the buoy height with respect to the surface level of the water satisfies the damped oscillator equation: $z'' + bz' + kz \equiv 0$ where b and k are positive constants.

Something has initially disturbed the buoy causing it to go up and down, but friction will gradually cause its motion to die out.

You make the following observations: At time zero the center of the buoy is at $z(0) = 0$, i.e., the position it would be in if it were at rest. It then rises up to a peak and falls down so that at time $t = 2$ it again at zero, $z(2) = 0$ descends downward and then comes back to 0 at time 4, i.e, $z(4) = 0$. Suppose $z(1) = 25$ and $z(3) = -16$.

(a) How high will z be at time $t = 5$?

(b) What are b and k?

Hint: Use that $z = Ae^{\alpha t}\sin(\omega t + B)$.

†407

917. Contrary to what one may think the buoy does not reach its peak at time $t = 1$ which is midway between its first two zeros, at $t = 0$ and $t = 2$. For example, suppose $z = e^{-t}\sin t$. Then z is zero at both $t = 0$ and $t = \pi$. Does z have a local maximum at $t = \frac{\pi}{2}$?

†407

918. In the buoy problem 916 suppose you make the following observations:

It rises up to its first peak at $t = 1$ where $z(1) = 25$ and then descends downward to a local minimum at at $t = 3$ where $z(3) = -16$.

(a) When will the buoy reach its second peak and how high will that be?

(b) What are b and k?

Note: It will not be the case that $z(0) = 0$.

†407

Linear operators

Given a polynomial $p = p(x) = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n$ and $z = z(t)$ an infinitely differentiable function of t define

$$\mathcal{L}_p(z) = a_0z + a_1z^{(1)} + a_2z^{(2)} + \ldots + a_nz^{(n)}$$

where $z^{(k)} = \frac{d^kz}{dt^k}$ is the k^{th} derivative of z with respect to t.

919. Show for $\mathcal{L} = \mathcal{L}_p$ that

(a) $\mathcal{L}(z_1 + z_2) = \mathcal{L}(z_1) + \mathcal{L}(z_2)$

(b) $\mathcal{L}(Cz) = C\mathcal{L}(z)$ where C is any constant

Such an $\mathcal{L}$ is called a linear operator. Operator because it takes as input a function and then ouputs another function. Linear refers to properties (a) and (b).

920. Let r be any constant and p any polynomial. Show that $\mathcal{L}_p(e^{rt}) = p(r)e^{rt}$.

921. For p and q polynomials show that

$$\mathcal{L}_{p+q}(z) = \mathcal{L}_p(z) + \mathcal{L}_q(z)$$

922. For p and q polynomials show that

$$\mathcal{L}_{p\cdot q}(z) = \mathcal{L}_p(\mathcal{L}_q(z))$$

Here $p \cdot q$ refers to the ordinary product of the two polynomials.

923. Let α be a constant. For any u an infinitely differentiable function of t show that

(a) $\mathcal{L}_{x-\alpha}(u \cdot e^{\alpha t}) = u^{(1)}e^{\alpha t}$

(b) $\mathcal{L}_{(x-\alpha)^n}(u \cdot e^{\alpha t}) = u^{(n)}e^{\alpha t}$

†407

924. Let α be any constant, p a polynomial, and suppose that $(x-\alpha)^n$ divides p. Show that for any $k < n$

$$\mathcal{L}_p(t^k e^{\alpha t}) \equiv 0$$

925. Suppose that $p(x) = (x-\alpha_1)^{n_1} \cdots (x-\alpha_m)^{n_m}$ where the α_i are distinct constants. Suppose that

$$z = C_1^1 e^{\alpha_1 t} + C_1^2 t e^{\alpha_1 t} + \cdots C_1^{n_1} t^{n_1 - 1} e^{\alpha_1 t} + \cdots + C_m^1 e^{\alpha_m t} + C_m^2 t e^{\alpha_m t} + \cdots C_m^{n_m} t^{n_m - 1} e^{\alpha_m t}$$

Show that $\mathcal{L}_p(z) \equiv 0$.

In a more advanced course in the theory of differential equations it would be proved that every solution of $\mathcal{L}_p(z) \equiv 0$ has this form, i.e., z satisfies the above formula for some choice of the constants C_j^i.

926. Suppose $\mathcal{L}$ is a linear operator and $b = b(t)$ is a fixed function of t. Suppose that z_P is one particular solution of $\mathcal{L}(z) = b$, i.e., $\mathcal{L}(z_P) = b$. Suppose that z is any other solution of $\mathcal{L}(z) = b$. Show that $\mathcal{L}(z - z_P) \equiv 0$. Show that for any solution of the equation $\mathcal{L}(z) = b$ there is a solution z_H of the associated homogenous equation such that $z = z_P + z_H$.

Variations of Parameters

927. Given the equation

$$\mathcal{L}(z) = z'' + a_0 z' + a_1 z \equiv b$$

where a_0, a_1, b are given functions of t. Then

$$\mathcal{L}(f z_1 + g z_2) = b$$

where

$$z_h = C_1 z_1 + C_2 z_2$$

is the general solution of the associated homogenous equation $\mathcal{L}(z) \equiv 0$ and the derivatives of f and g satisfy:

$$f' = \frac{\det \begin{pmatrix} 0 & z_2 \\ b & z_2' \end{pmatrix}}{\det \begin{pmatrix} z_1 & z_2 \\ z_1' & z_2' \end{pmatrix}} \qquad g' = \frac{\det \begin{pmatrix} z_1 & 0 \\ z_1' & b \end{pmatrix}}{\det \begin{pmatrix} z_1 & z_2 \\ z_1' & z_2' \end{pmatrix}}$$

Use these formulas to find the general solution of

$$z'' + z \equiv (\cos t)^2$$

928. Use these formulas to find the general solution of

$$z'' + z \equiv \frac{1}{\sin t}$$

†407

929. Solve the initial value problem:

$$z'' + z \equiv (\tan t)^2$$

$$z(0) = 1$$
$$z'(0) = -1$$

†407

930. Find the general solution of

$$z'' - z \equiv \frac{1}{e^t + e^{-t}}$$

931.

Given a system of linear equations

$ax + by = r$

$cx + dy = s$

show that the solution is given by:

$$x = \frac{\det\begin{pmatrix} r & b \\ s & d \end{pmatrix}}{\det\begin{pmatrix} a & b \\ c & d \end{pmatrix}} \qquad y = \frac{\det\begin{pmatrix} a & r \\ c & s \end{pmatrix}}{\det\begin{pmatrix} a & b \\ c & d \end{pmatrix}}$$

You may assume the determinant:

$$\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$$

is not zero.

†407

932. Given the linear operator $\mathcal{L}(z) = z'' + a_0 z' + a_1 z$ suppose $\mathcal{L}(z_1) \equiv 0$ and $\mathcal{L}(z_2) \equiv 0$ and that f and g are functions of t which satisfy $f' z_1 + g' z_2 \equiv 0$. Show that

$$\mathcal{L}(f z_1 + g z_2) = f' z_1' + g' z_2'$$

933. Prove that the formulas given problem 927 work.

†407

Chapter 14

Vectors

14.1 Introduction to vectors

Definition 14.1.1. A vector is a column of two, three, or more numbers, written as

$$\vec{\boldsymbol{a}} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \quad \text{or} \quad \vec{\boldsymbol{a}} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \quad \text{or} \quad \vec{\boldsymbol{a}} = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$$

in general.
The ***length of a vector*** $\vec{\boldsymbol{a}} = \left(\begin{smallmatrix} a_1 \\ a_2 \\ a_3 \end{smallmatrix}\right)$ is defined by

$$\|\vec{\boldsymbol{a}}\| = \left\| \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \right\| = \sqrt{a_1^2 + a_2^2 + a_3^2}.$$

We will always deal with either the two or three dimensional cases, in other words, the cases $n = 2$ or $n = 3$, respectively. For these cases there is a geometric description of vectors which is very useful. In fact, the two and three dimensional theories have their origins in mechanics and geometry. In higher dimensions the geometric description fails, simply because we cannot visualize a four dimensional space, let alone a higher dimensional space. Instead of a geometric description of vectors there is an abstract theory called ***Linear Algebra*** which deals with "vector spaces" of any dimension (even infinite!). This theory of vectors in higher dimensional spaces is very useful in science, engineering and economics. You can learn about it in courses like MATH 320 or 340/341.

14.1.1 Basic arithmetic of vectors

You can add and subtract vectors, and you can multiply them with arbitrary real numbers. this section tells you how.
The ***sum of two vectors*** is defined by

$$\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} a_1 + b_1 \\ a_2 + b_2 \end{pmatrix}, \tag{14.1}$$

and

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} a_1 + b_1 \\ a_2 + b_2 \\ a_3 + b_3 \end{pmatrix}.$$

The ***zero vector*** is defined by

$$\vec{\mathbf{0}} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \textit{or} \quad \vec{\mathbf{0}} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

It has the property that

$$\vec{\boldsymbol{a}} + \vec{\mathbf{0}} = \vec{\mathbf{0}} + \vec{\boldsymbol{a}} = \vec{\boldsymbol{a}}$$

no matter what the vector $\vec{\boldsymbol{a}}$ is.

You can multiply a vector $\vec{\boldsymbol{a}} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ with a real number t according to the rule

$$t\vec{\boldsymbol{a}} = \begin{pmatrix} ta_1 \\ ta_2 \\ ta_3 \end{pmatrix}.$$

In particular, "minus a vector" is defined by

$$-\vec{\boldsymbol{a}} = (-1)\vec{\boldsymbol{a}} = \begin{pmatrix} -a_1 \\ -a_2 \\ -a_3 \end{pmatrix}.$$

The difference of two vectors is defined by

$$\vec{\boldsymbol{a}} - \vec{\boldsymbol{b}} = \vec{\boldsymbol{a}} + (-\vec{\boldsymbol{b}}).$$

So, to subtract two vectors you subtract their components,

$$\vec{\boldsymbol{a}} - \vec{\boldsymbol{b}} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} - \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} a_1 - b_1 \\ a_2 - b_2 \\ a_3 - b_3 \end{pmatrix}$$

14.1.2 Some GOOD examples.

$$\begin{pmatrix} 2 \\ 3 \end{pmatrix} + \begin{pmatrix} -3 \\ \pi \end{pmatrix} = \begin{pmatrix} -1 \\ 3+\pi \end{pmatrix} \qquad 2\begin{pmatrix} 1 \\ 0 \end{pmatrix} + 3\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

$$\left(\begin{smallmatrix} 1 \\ 0 \\ 3 \end{smallmatrix}\right) - \left(\begin{smallmatrix} -1 \\ 12 \\ \sqrt{2} \end{smallmatrix}\right) = \left(\begin{smallmatrix} 2 \\ -12 \\ 3-\sqrt{2} \end{smallmatrix}\right) \qquad a\left(\begin{smallmatrix} 1 \\ 0 \\ 0 \end{smallmatrix}\right) + b\left(\begin{smallmatrix} 0 \\ 1 \\ 0 \end{smallmatrix}\right) + c\left(\begin{smallmatrix} 0 \\ 0 \\ 1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} a \\ b \\ c \end{smallmatrix}\right)$$

$$0 \cdot \begin{pmatrix} 12\sqrt{39} \\ \pi^2 - \ln 3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \vec{\mathbf{0}} \qquad \begin{pmatrix} t + t^2 \\ 1 - t^2 \end{pmatrix} = (1+t)\begin{pmatrix} t \\ 1 - t \end{pmatrix}$$

14.1.3 Two very, very BAD examples.

Vectors must have the same size to be added, therefore

$$\begin{pmatrix}2\\3\end{pmatrix} + \begin{pmatrix}1\\3\\2\end{pmatrix} = \textbf{undefined!!!}$$

Vectors and numbers are different things, so an equation like

$$\vec{a} = 3 \quad \text{is nonsense!}$$

This equation says that some vector ($\vec{a}$) is equal to some number (in this case: 3). ***Vectors and numbers are never equal!***

14.1.4 Algebraic properties of vector addition and multiplication

Addition of vectors and multiplication of numbers and vectors were defined in such a way that the following always hold for any vectors $\vec{a}$, $\vec{b}$, $\vec{c}$ (of the same size) and any real numbers s, t

$$\vec{a} + \vec{b} = \vec{b} + \vec{a} \qquad \text{[vector addition is commutative]} \qquad (14.2)$$

$$\vec{a} + (\vec{b} + \vec{c}) = (\vec{a} + \vec{b}) + \vec{c} \qquad \text{[vector addition is associative]} \qquad (14.3)$$

$$t(\vec{a} + \vec{b}) = t\vec{a} + t\vec{b} \qquad \text{[first distributive property]} \qquad (14.4)$$

$$(s + t)\vec{a} = s\vec{a} + t\vec{a} \qquad \text{[second distributive property]} \qquad (14.5)$$

14.1.5 Prove (14.2).

Let $\vec{a} = \left(\begin{smallmatrix}a_1\\a_2\\a_3\end{smallmatrix}\right)$ and $\vec{b} = \left(\begin{smallmatrix}b_1\\b_2\\b_3\end{smallmatrix}\right)$ be two vectors, and consider both possible ways of adding them:

$$\begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix} + \begin{pmatrix}b_1\\b_2\\b_3\end{pmatrix} = \begin{pmatrix}a_1+b_1\\a_2+b_2\\a_3+b_3\end{pmatrix} \quad \text{and} \quad \begin{pmatrix}b_1\\b_2\\b_3\end{pmatrix} + \begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix} = \begin{pmatrix}b_1+a_1\\b_2+a_2\\b_3+a_3\end{pmatrix}$$

We know (or we have assumed long ago) that addition of real numbers is commutative, so that $a_1 + b_1 = b_1 + a_1$, etc. Therefore

$$\vec{a} + \vec{b} = \left(\begin{smallmatrix}a_1+b_1\\a_2+b_2\\a_3+b_3\end{smallmatrix}\right) = \left(\begin{smallmatrix}b_1+a_1\\b_2+a_2\\b_3+a_3\end{smallmatrix}\right) = \vec{b} + \vec{a}.$$

This proves (14.2).

14.1.6 Example

If $\vec{v}$ and $\vec{w}$ are two vectors, we define

$$\vec{a} = 2\vec{v} + 3\vec{w}, \quad \vec{b} = -\vec{v} + \vec{w}.$$

Problem: Compute $\vec{a} + \vec{b}$ and $2\vec{a} - 3\vec{b}$ in terms of $\vec{v}$ and $\vec{w}$.

Solution:

$$\vec{\boldsymbol{a}} + \vec{\boldsymbol{b}} = (2\vec{\boldsymbol{v}} + 3\vec{\boldsymbol{w}}) + (-\vec{\boldsymbol{v}} + \vec{\boldsymbol{w}}) = (2-1)\vec{\boldsymbol{v}} + (3+1)\vec{\boldsymbol{w}} = \vec{\boldsymbol{v}} + 4\vec{\boldsymbol{w}}$$
$$2\vec{\boldsymbol{a}} - 3\vec{\boldsymbol{b}} = 2(2\vec{\boldsymbol{v}} + 3\vec{\boldsymbol{w}}) - 3(-\vec{\boldsymbol{v}} + \vec{\boldsymbol{w}}) = 4\vec{\boldsymbol{w}} + 6\vec{\boldsymbol{w}} + 3\vec{\boldsymbol{v}} - 3\vec{\boldsymbol{w}} = 7\vec{\boldsymbol{v}} + 3\vec{\boldsymbol{w}}.$$

Problem: Find s, t so that $s\vec{\boldsymbol{a}} + t\vec{\boldsymbol{b}} = \vec{\boldsymbol{v}}$.
Solution: Simplifying $s\vec{\boldsymbol{a}} + t\vec{\boldsymbol{b}}$ you find

$$s\vec{\boldsymbol{a}} + t\vec{\boldsymbol{b}} = s(2\vec{\boldsymbol{v}} + 3\vec{\boldsymbol{w}}) + t(-\vec{\boldsymbol{v}} + \vec{\boldsymbol{w}}) = (2s-t)\vec{\boldsymbol{v}} + (3s+t)\vec{\boldsymbol{w}}.$$

One way to ensure that $s\vec{\boldsymbol{a}} + t\vec{\boldsymbol{b}} = \vec{\boldsymbol{v}}$ holds is therefore to choose s and t to be the solutions of

$$2s - t = 1$$
$$3s + t = 0$$

The second equation says $t = -3s$. The first equation then leads to $2s + 3s = 1$, i.e. $s = \frac{1}{5}$. Since $t = -3s$ we get $t = -\frac{3}{5}$. The solution we have found is therefore

$$\tfrac{1}{5}\vec{\boldsymbol{a}} - \tfrac{3}{5}\vec{\boldsymbol{b}} = \vec{\boldsymbol{v}}.$$

14.1.7 Geometric description of vectors

Vectors originally appeared in mechanics, where they represented forces: a force acting on some object has a ***magnitude*** and a ***direction***. Thus a force can be thought of as an arrow, where the length of the arrow indicates how strong the force is (how hard it pushes or pulls).

So we will think of vectors as ***arrows***: if you specify two points P and Q, then the arrow pointing from P to Q is a vector and we denote this vector by $\overrightarrow{PQ}$.

The precise mathematical definition is as follows:

Definition 14.1.2. For any pair of points P and Q whose coordinates are (p_1, p_2, p_3) and (q_1, q_2, q_3) one defines a vector $\overrightarrow{PQ}$ by

$$\overrightarrow{PQ} = \begin{pmatrix} q_1 - p_1 \\ q_2 - p_2 \\ q_3 - p_3 \end{pmatrix}.$$

If the initial point of an arrow is the origin O, and the final point is any point Q, then the vector $\overrightarrow{OQ}$ is called the ***position vector*** of the point Q.

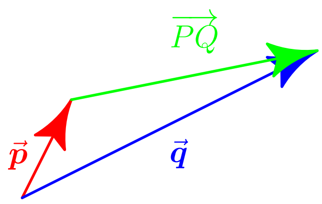


Figure 14.1: A vector from two position vectors

If $\vec{\boldsymbol{p}}$ and $\vec{\boldsymbol{q}}$ are the position vectors of P and Q, then one can write $\overrightarrow{PQ}$ as

$$\overrightarrow{PQ} = \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} - \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} = \vec{\boldsymbol{q}} - \vec{\boldsymbol{p}}.$$

For plane vectors we define $\overrightarrow{PQ}$ similarly, namely, $\overrightarrow{PQ} = \left(\begin{smallmatrix} q_1 - p_1 \\ q_2 - p_2 \end{smallmatrix}\right)$. The formula for the distance between two points P and Q in the plane

$$\text{distance from } P \text{ to } Q = \sqrt{(q_1 - p_1)^2 + (q_2 - p_2)^2}$$

says that the length of the vector $\overrightarrow{PQ}$ is just the distance between the points P and Q, i.e.

$$\text{distance from } P \text{ to } Q = \left\| \overrightarrow{PQ} \right\|.$$

This formula generalises if P and Q are points in 3D space.

$$\text{distance from } P \text{ to } Q = \sqrt{(q_1 - p_1)^2 + (q_2 - p_2)^2 + (q_3 - p_3)^2}$$

14.1.8 Example

The point P has coordinates $(2, 3)$; the point Q has coordinates $(8, 6)$. The vector $\overrightarrow{PQ}$ is therefore

$$\overrightarrow{PQ} = \begin{pmatrix} 8 - 2 \\ 6 - 3 \end{pmatrix} = \begin{pmatrix} 6 \\ 3 \end{pmatrix}.$$

This vector is the position vector of the point R whose coordinates are $(6, 3)$. Thus

$$\overrightarrow{PQ} = \overrightarrow{OR} = \begin{pmatrix} 6 \\ 3 \end{pmatrix}.$$

The distance from P to Q is the length of the vector $\overrightarrow{PQ}$, i.e.

$$\text{distance } P \text{ to } Q = \left\| \begin{pmatrix} 6 \\ 3 \end{pmatrix} \right\| = \sqrt{6^2 + 3^2} = 3\sqrt{5}.$$

14.1.9 Example

Find the distance between the points A and B whose position vectors are $\vec{\boldsymbol{a}} = \left(\begin{smallmatrix} 1 \\ 1 \\ 0 \end{smallmatrix}\right)$ and $\vec{\boldsymbol{b}} = \left(\begin{smallmatrix} 0 \\ 1 \\ 1 \end{smallmatrix}\right)$ respectively.

Solution: One has

$$\text{distance } A \text{ to } B = \|\overrightarrow{AB}\| = \|\vec{\boldsymbol{b}} - \vec{\boldsymbol{a}}\| = \left\| \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\| = \sqrt{(-1)^2 + 0^2 + 1^2} = \sqrt{2}$$

14.1.10 Geometric interpretation of vector addition and multiplication

Suppose you have two vectors $\vec{\boldsymbol{a}}$ and $\vec{\boldsymbol{b}}$. Consider them as position vectors, i.e. represent them by vectors that have the origin as initial point:

$$\vec{\boldsymbol{a}} = \overrightarrow{OA}, \quad \vec{\boldsymbol{b}} = \overrightarrow{OB}.$$

Then the origin and the three endpoints of the vectors $\vec{\boldsymbol{a}}$, $\vec{\boldsymbol{b}}$ and $\vec{\boldsymbol{a}} + \vec{\boldsymbol{b}}$ form a parallelogram. See figure 14.2.

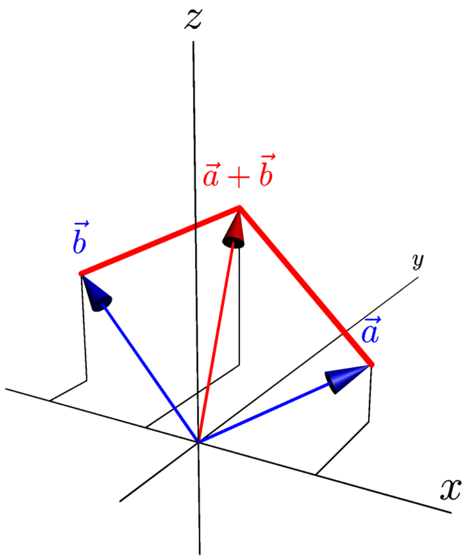


Figure 14.2: three dimensional vector addition by completing the parallelogram.

To multiply a vector $\vec{\boldsymbol{a}}$ with a real number t you multiply its length with $|t|$; if $t < 0$ you reverse the direction of $\vec{\boldsymbol{a}}$.

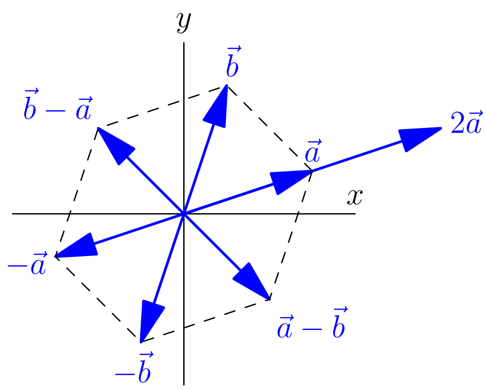


Figure 14.3: Vector negation and scalar multiplication.

14.1.11 Example 14.1.6, geometrically

In example 14.1.6 we assumed two vectors $\vec{\boldsymbol{v}}$ and $\vec{\boldsymbol{w}}$ were given, and then defined $\vec{\boldsymbol{a}} = 2\vec{\boldsymbol{v}} + 3\vec{\boldsymbol{w}}$ and $\vec{\boldsymbol{b}} = -\vec{\boldsymbol{v}} + \vec{\boldsymbol{w}}$. In figure 14.4 the vectors $\vec{\boldsymbol{a}}$ and $\vec{\boldsymbol{b}}$ are constructed geometrically from some arbitrarily chosen $\vec{\boldsymbol{v}}$ and $\vec{\boldsymbol{w}}$. We also found algebraically in example 14.1.6 that $\vec{\boldsymbol{a}} + \vec{\boldsymbol{b}} = \vec{\boldsymbol{v}} + 4\vec{\boldsymbol{w}}$. The drawing in figure 14.4 illustrates this.

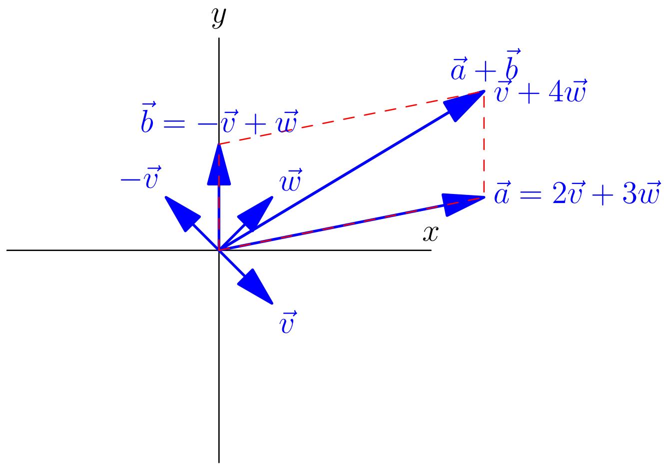


Figure 14.4: Picture proof that $\vec{\boldsymbol{a}} + \vec{\boldsymbol{b}} = \vec{\boldsymbol{v}} + 4\vec{\boldsymbol{w}}$ in example 14.1.11

For a good introduction to vector addition and multiplication by a scaler the reader should consider watching YouTube by 3Blue1Brown .

14.2 Parametric equations for lines and planes

Given two ***distinct*** points A and B we consider the line segment AB. If X is any given point on AB then we will now find a formula for the position vector of X.

Define t to be the ratio between the lengths of the line segments AX and AB,

$$t = \frac{\text{length } AX}{\text{length } AB}.$$

Then the vectors $\overrightarrow{AX}$ and $\overrightarrow{AB}$ are related by $\overrightarrow{AX} = t\overrightarrow{AB}$. Since AX is shorter than AB we have $0 < t < 1$.

The position vector of the point X on the line segment AB is

$$\overrightarrow{OX} = \overrightarrow{OA} + \overrightarrow{AX} = \overrightarrow{OA} + t\overrightarrow{AB}.$$

If we write $\vec{\boldsymbol{a}}, \vec{\boldsymbol{b}}, \vec{\boldsymbol{x}}$ for the position vectors of A, B, X, then we get

$$\vec{\boldsymbol{x}} = (1-t)\vec{\boldsymbol{a}} + t\vec{\boldsymbol{b}} = \vec{\boldsymbol{a}} + t(\vec{\boldsymbol{b}} - \vec{\boldsymbol{a}}). \tag{14.6}$$

This equation is called the ***parametric equation for the line through A and B.*** In our derivation the parameter t satisfied $0 \le t \le 1$, but there is nothing that keeps us from substituting negative values of t, or numbers $t > 1$ in (14.6). The resulting vectors $\vec{\boldsymbol{x}}$ are position vectors of points X which lie on the line ℓ through A and B.

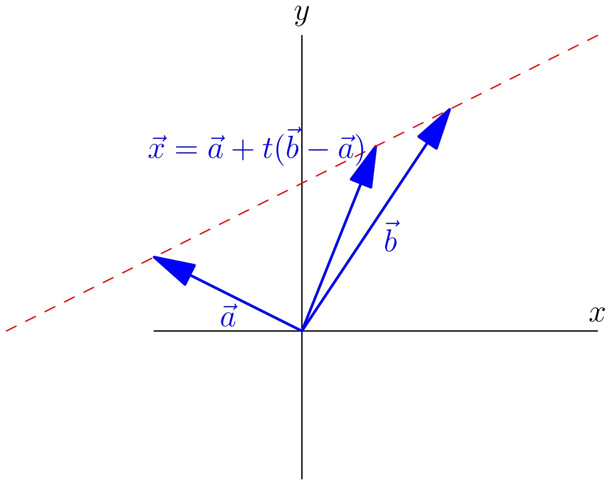


Figure 14.5: Constructing points on the line through A and B.

14.2.1 Example

[Find the parametric equation for the line ℓ through the points $A(1,2)$ and $B(3,-1)$, and determine where ℓ intersects the x_1 axis.]

Solution: The position vectors of A, B are $\vec{\boldsymbol{a}} = \left(\begin{smallmatrix}1\\2\end{smallmatrix}\right)$ and $\vec{\boldsymbol{b}} = \left(\begin{smallmatrix}3\\-1\end{smallmatrix}\right)$, so the position vector of an arbitrary point on ℓ is given by

$$\vec{\boldsymbol{x}} = \vec{\boldsymbol{a}} + t(\vec{\boldsymbol{b}} - \vec{\boldsymbol{a}}) = \begin{pmatrix}1\\2\end{pmatrix} + t\begin{pmatrix}3-1\\-1-2\end{pmatrix} = \begin{pmatrix}1\\2\end{pmatrix} + t\begin{pmatrix}2\\-3\end{pmatrix} = \begin{pmatrix}1+2t\\2-3t\end{pmatrix}$$

where t is an arbitrary real number.

This vector points to the point $X = (1+2t, 2-3t)$. By definition, a point lies on the x_1-axis if its x_2 component vanishes. Thus if the point

$$X = (1+2t, 2-3t)$$

lies on the x_1-axis, then $2-3t = 0$, i.e. $t = \frac{2}{3}$. The intersection point X of ℓ and the x_1-axis is therefore $X|_{t=2/3} = (1 + 2\cdot\frac{2}{3}, 0) = (\frac{5}{3}, 0)$.

14.2.2 Midpoint of a line segment.

If M is the midpoint of the line segment AB, then the vectors $\overrightarrow{AM}$ and $\overrightarrow{MB}$ are both parallel and have the same direction and length (namely, half the length of the line segment AB). Hence they are equal: $\overrightarrow{AM} = \overrightarrow{MB}$. If $\vec{\boldsymbol{a}}$, $\vec{\boldsymbol{m}}$, and $\vec{\boldsymbol{b}}$ are the position vectors of A , M and B, then this means

$$\vec{\boldsymbol{m}} - \vec{\boldsymbol{a}} = \overrightarrow{AM} = \overrightarrow{MB} = \vec{\boldsymbol{b}} - \vec{\boldsymbol{m}}.$$

Add $\vec{\boldsymbol{m}}$ and $\vec{\boldsymbol{a}}$ to both sides, and divide by 2 to get

$$\vec{\boldsymbol{m}} = \tfrac{1}{2}\vec{\boldsymbol{a}} + \tfrac{1}{2}\vec{\boldsymbol{b}} = \frac{\vec{\boldsymbol{a}} + \vec{\boldsymbol{b}}}{2}.$$

14.2.3 Parametric equations for planes in space*

You can specify a plane in three dimensional space by naming a point A on the plane $\mathcal{P}$, and two vectors $\vec{\boldsymbol{v}}$ and $\vec{\boldsymbol{w}}$ parallel to the plane $\mathcal{P}$, but not parallel to each other. Then any point on the plane $\mathcal{P}$ has position vector $\vec{\boldsymbol{x}}$ given by

$$\vec{\boldsymbol{x}} = \vec{\boldsymbol{a}} + s\vec{\boldsymbol{v}} + t\vec{\boldsymbol{w}}. \tag{14.7}$$

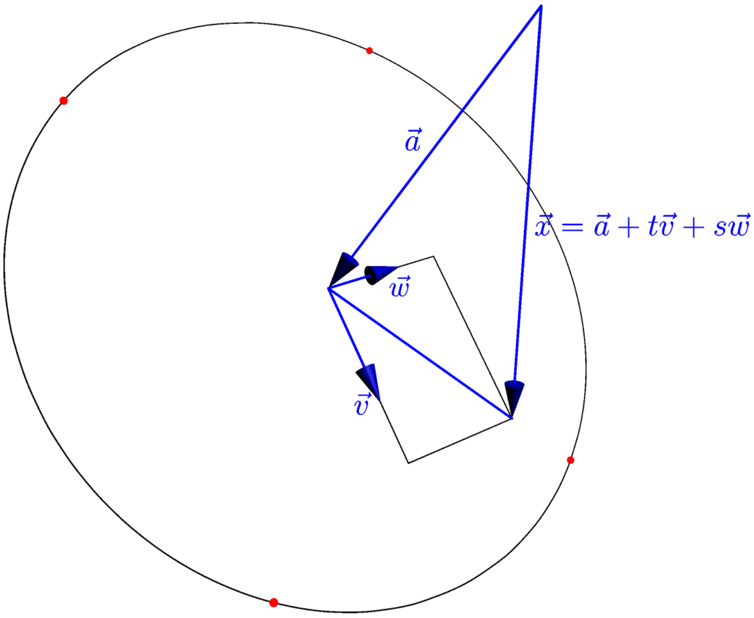


Figure 14.6: Generating points on a plane $\mathcal{P}$

The following construction explains why equation (14.7) will give you any point on the plane through A, parallel to $\vec{\boldsymbol{v}}, \vec{\boldsymbol{w}}$.

Let A, $\vec{\boldsymbol{v}}$, $\vec{\boldsymbol{w}}$ be given, and suppose we want to express the position vector of some other point X on the plane $\mathcal{P}$ in terms of $\vec{\boldsymbol{a}} = \overrightarrow{OA}$, $\vec{\boldsymbol{v}}$, and $\vec{\boldsymbol{w}}$.

First we note that

$$\overrightarrow{OX} = \overrightarrow{OA} + \overrightarrow{AX}.$$

Next, you draw a parallelogram in the plane $\mathcal{P}$ whose sides are parallel to the vectors $\vec{\boldsymbol{v}}$ and $\vec{\boldsymbol{w}}$, and whose diagonal is the line segment AX. The sides of this parallelogram represent vectors which are multiples of $\vec{\boldsymbol{v}}$ and $\vec{\boldsymbol{w}}$ and which add up to $\overrightarrow{AX}$. So, if one side of the parallelogram is $s\vec{\boldsymbol{v}}$ and the other $t\vec{\boldsymbol{w}}$ then we have $\overrightarrow{AX} = s\vec{\boldsymbol{v}} + t\vec{\boldsymbol{w}}$. With $\overrightarrow{OX} = \overrightarrow{OA} + \overrightarrow{AX}$ this implies (14.7).

14.3 Vector Bases

14.3.1 The Standard Basis Vectors

The notation for vectors which we have been using so far is not the most traditional. In the late 19th century GIBBS and HEAVYSIDE adapted HAMILTON's theory of Quaternions to deal with vectors. Their notation is still popular in texts on electromagnetism and fluid mechanics.

Define the following three vectors:

$$\vec{\boldsymbol{i}} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \vec{\boldsymbol{j}} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \vec{\boldsymbol{k}} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Then *every vector can be written as a linear combination of* $\vec{\boldsymbol{i}}$*,* $\vec{\boldsymbol{j}}$ *and* $\vec{\boldsymbol{k}}$, namely as follows:

$$\vec{\boldsymbol{a}} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = a_1\vec{\boldsymbol{i}} + a_2\vec{\boldsymbol{j}} + a_3\vec{\boldsymbol{k}}.$$

Moreover, *there is only one way to write a given vector as a linear combination of* $\{\vec{\boldsymbol{i}}, \vec{\boldsymbol{j}}, \vec{\boldsymbol{k}}\}$. This means that

$$a_1\vec{\boldsymbol{i}} + a_2\vec{\boldsymbol{j}} + a_3\vec{\boldsymbol{k}} = b_1\vec{\boldsymbol{i}} + b_2\vec{\boldsymbol{j}} + b_3\vec{\boldsymbol{k}} \iff \begin{cases} a_1 = b_1 \\ a_2 = b_2 \\ a_3 = b_3 \end{cases}$$

For plane vectors one defines

$$\vec{\boldsymbol{i}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \vec{\boldsymbol{j}} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

and just as for three dimensional vectors one can write every (plane) vector $\vec{\boldsymbol{a}}$ as a linear combination of $\vec{\boldsymbol{i}}$ and $\vec{\boldsymbol{j}}$,

$$\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = a_1\vec{\boldsymbol{i}} + a_2\vec{\boldsymbol{j}}.$$

Just as for space vectors, there is only one way to write a given vector as a linear combination of $\vec{\boldsymbol{i}}$ and $\vec{\boldsymbol{j}}$.

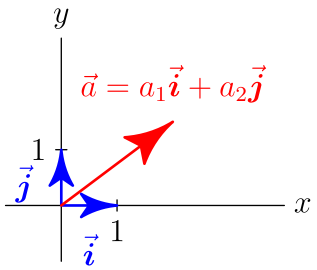


Figure 14.7: Expressing a vector as the sum of scalar multiples of unit vectors

If we know the length of the vector and the angle it makes with the axes unit vectors we can deduce the components of the vector:

In 2 dimensions we have:

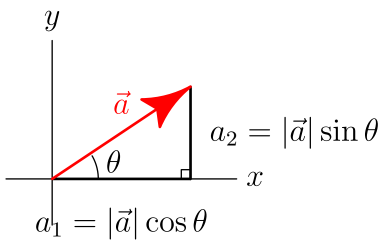


Figure 14.8: a vector in the plane is decomposed into its components

whilst in three dimensions we have:

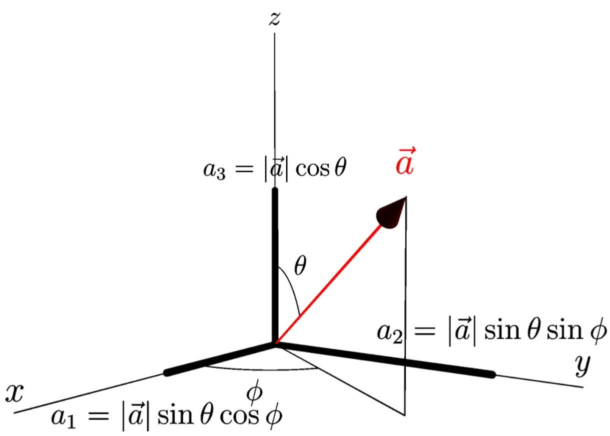


Figure 14.9: a three dimensional vector is decomposed into its components

14.3.2 A Basis of Vectors (in general)*

The vectors $\vec{\boldsymbol{i}}, \vec{\boldsymbol{j}}, \vec{\boldsymbol{k}}$ are called the ***standard basis vectors***. They are an example of what is called a "basis". Here is the definition in the case of space vectors:

Definition 14.3.1. A triple of space vectors $\{\vec{\boldsymbol{u}}, \vec{\boldsymbol{v}}, \vec{\boldsymbol{w}}\}$ is a ***basis*** if every space vector $\vec{\boldsymbol{a}}$ can be written as a linear combination of $\{\vec{\boldsymbol{u}}, \vec{\boldsymbol{v}}, \vec{\boldsymbol{w}}\}$, i.e.

$$\vec{\boldsymbol{a}} = a_u\vec{\boldsymbol{u}} + a_v\vec{\boldsymbol{v}} + a_w\vec{\boldsymbol{w}},$$

and if there is only one way to do so for any given vector $\vec{\boldsymbol{a}}$ (i.e. the vector $\vec{\boldsymbol{a}}$ determines the coefficients a_u, a_v, a_w).

For plane vectors the definition of a basis is almost the same, except that a basis consists of two vectors rather than three:

Definition 14.3.2. A pair of plane vectors $\{\vec{\boldsymbol{u}}, \vec{\boldsymbol{v}}\}$ is a ***basis*** if every plane vector $\vec{\boldsymbol{a}}$ can be written as a linear combination of $\{\vec{\boldsymbol{u}}, \vec{\boldsymbol{v}}\}$, i.e. $\vec{\boldsymbol{a}} = a_u\vec{\boldsymbol{u}} + a_v\vec{\boldsymbol{v}}$, and if there is only one way to do so for any given vector $\vec{\boldsymbol{a}}$ (i.e. the vector $\vec{\boldsymbol{a}}$ determines the coefficients a_u, a_v).

To understand Basis vectors and the span of a collection of vectors the reader should watch YouTube by 3Blue1Brown

14.4 Dot Product

Definition 14.4.1. The "inner product" or "dot product" of two vectors is given by

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \cdot \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = a_1 b_1 + a_2 b_2 + a_3 b_3.$$

Note that the dot-product of two vectors is a number!

The dot product of two plane vectors is (predictably) defined by

$$\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \cdot \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = a_1 b_1 + a_2 b_2.$$

An important property of the dot product is its relation with the length of a vector:

$$\|\vec{\boldsymbol{a}}\|^2 = \vec{\boldsymbol{a}} \cdot \vec{\boldsymbol{a}}. \tag{14.8}$$

14.4.1 Algebraic properties of the dot product

The dot product satisfies the following rules,

$$\vec{\boldsymbol{a}} \cdot \vec{\boldsymbol{b}} = \vec{\boldsymbol{b}} \cdot \vec{\boldsymbol{a}} \tag{14.9}$$

$$\vec{\boldsymbol{a}} \cdot (\vec{\boldsymbol{b}} + \vec{\boldsymbol{c}}) = \vec{\boldsymbol{a}} \cdot \vec{\boldsymbol{b}} + \vec{\boldsymbol{a}} \cdot \vec{\boldsymbol{c}} \tag{14.10}$$

$$(\vec{\boldsymbol{b}} + \vec{\boldsymbol{c}}) \cdot \vec{\boldsymbol{a}} = \vec{\boldsymbol{b}} \cdot \vec{\boldsymbol{a}} + \vec{\boldsymbol{c}} \cdot \vec{\boldsymbol{a}} \tag{14.11}$$

$$t(\vec{\boldsymbol{a}} \cdot \vec{\boldsymbol{b}}) = (t\vec{\boldsymbol{a}}) \cdot \vec{\boldsymbol{b}} \tag{14.12}$$

which hold for all vectors $\vec{\boldsymbol{a}}, \vec{\boldsymbol{b}}, \vec{\boldsymbol{c}}$ and any real number t.

14.4.2 Example

Simplify $\|\vec{\boldsymbol{a}} + \vec{\boldsymbol{b}}\|^2$.

One has

$$\begin{aligned} \|\vec{\boldsymbol{a}} + \vec{\boldsymbol{b}}\|^2 &= (\vec{\boldsymbol{a}} + \vec{\boldsymbol{b}}) \cdot (\vec{\boldsymbol{a}} + \vec{\boldsymbol{b}}) \\ &= \vec{\boldsymbol{a}} \cdot (\vec{\boldsymbol{a}} + \vec{\boldsymbol{b}}) + \vec{\boldsymbol{b}} \cdot (\vec{\boldsymbol{a}} + \vec{\boldsymbol{b}}) \\ &= \vec{\boldsymbol{a}} \cdot \vec{\boldsymbol{a}} + \underbrace{\vec{\boldsymbol{a}} \cdot \vec{\boldsymbol{b}} + \vec{\boldsymbol{b}} \cdot \vec{\boldsymbol{a}}}_{=2\vec{\boldsymbol{a}} \cdot \vec{\boldsymbol{b}} \text{ by } (14.9)} + \vec{\boldsymbol{b}} \cdot \vec{\boldsymbol{b}} \\ &= \|\vec{\boldsymbol{a}}\|^2 + 2\vec{\boldsymbol{a}} \cdot \vec{\boldsymbol{b}} + \|\vec{\boldsymbol{b}}\|^2 \end{aligned}$$

14.4.3 The diagonals of a parallelogram

Here is an example of how you can use the algebra of the dot product to prove something in geometry.

Suppose you have a parallelogram one of whose vertices is the origin. Label the vertices, starting at the origin and going around counterclockwise, O, A, C and B. Let $\vec{\boldsymbol{a}} = \overrightarrow{OA}$, $\vec{\boldsymbol{b}} = \overrightarrow{OB}$, $\vec{\boldsymbol{c}} = \overrightarrow{OC}$. One has

$$\overrightarrow{OC} = \vec{\boldsymbol{c}} = \vec{\boldsymbol{a}} + \vec{\boldsymbol{b}}, \quad \text{and} \quad \overrightarrow{AB} = \vec{\boldsymbol{b}} - \vec{\boldsymbol{a}}.$$

These vectors correspond to the diagonals OC and AB

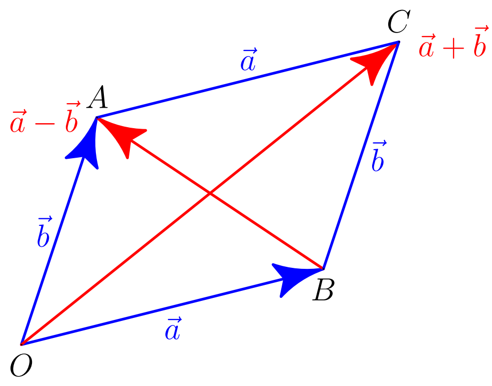


Figure 14.10: two vectors form a parallelogram

Theorem 14.4.1. In a parallelogram $OACB$ the sum of the squares of the lengths of the two diagonals equals the sum of the squares of the lengths of all four sides.

Proof. The squared lengths of the diagonals are

$$\|\overrightarrow{OC}\|^2 = \|\vec{\boldsymbol{a}} + \vec{\boldsymbol{b}}\|^2 = \|\vec{\boldsymbol{a}}\|^2 + 2\vec{\boldsymbol{a}}\cdot\vec{\boldsymbol{b}} + \|\vec{\boldsymbol{b}}\|^2$$
$$\|\overrightarrow{AB}\|^2 = \|\vec{\boldsymbol{a}} - \vec{\boldsymbol{b}}\|^2 = \|\vec{\boldsymbol{a}}\|^2 - 2\vec{\boldsymbol{a}}\cdot\vec{\boldsymbol{b}} + \|\vec{\boldsymbol{b}}\|^2$$

Adding both these equations you get

$$\|\overrightarrow{OC}\|^2 + \|\overrightarrow{AB}\|^2 = 2\left(\|\vec{\boldsymbol{a}}\|^2 + \|\vec{\boldsymbol{b}}\|^2\right).$$

The squared lengths of the sides are

$$\|\overrightarrow{OA}\|^2 = \|\vec{\boldsymbol{a}}\|^2, \quad \|\overrightarrow{AB}\|^2 = \|\vec{\boldsymbol{b}}\|^2, \quad \|\overrightarrow{BC}\|^2 = \|\vec{\boldsymbol{a}}\|^2, \quad \|\overrightarrow{OC}\|^2 = \|\vec{\boldsymbol{b}}\|^2.$$

Together these also add up to $2\left(\|\vec{\boldsymbol{a}}\|^2 + \|\vec{\boldsymbol{b}}\|^2\right)$. □

14.4.4 The Law of cosines

We will need *the law of cosines* from high-school trigonometry.

Theorem 14.4.2. Recall that for a triangle OAB with angle θ at the point O, and with sides OA and OB of lengths a and b, the length c of the opposing side AB is given by

$$c^2 = a^2 + b^2 - 2ab\cos\theta. \tag{14.13}$$

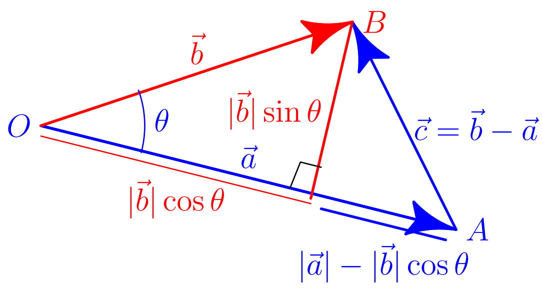


Figure 14.11: Law of cosines, $|\vec{c}|^2 = |\vec{a}|^2 + |\vec{b}|^2 - 2|\vec{a}||\vec{b}|\cos\theta$.

Proof. Consider figure 14.11. In trigonometry this law is proved by dropping a perpendicular line from B onto the side OA. The triangle OAB gets divided into two right triangles, one of which has AB as hypotenuse. Pythagoras then implies

$$c^2 = (b\sin\theta)^2 + (a - b\cos\theta)^2 .$$

After simplification you get (14.13). □

14.4.5 The dot product and the angle between two vectors

Here is the most important interpretation of the dot product:

Theorem 14.4.3. If the angle between two vectors $\vec{\boldsymbol{a}}$ and $\vec{\boldsymbol{b}}$ is θ, then one has

$$\vec{\boldsymbol{a}}\cdot\vec{\boldsymbol{b}} = \|\vec{\boldsymbol{a}}\| \cdot \|\vec{\boldsymbol{b}}\|\cos\theta.$$

Proof. To prove the theorem consider figure 14.11 again. Let O be the origin, and then observe that the length of the side AB is the length of the vector $\overrightarrow{AB} = \vec{\boldsymbol{b}} - \vec{\boldsymbol{a}}$. Here $\vec{\boldsymbol{a}} = \overrightarrow{OA}$, $\vec{\boldsymbol{b}} = \overrightarrow{OB}$, and hence

$$c^2 = \|\vec{\boldsymbol{b}} - \vec{\boldsymbol{a}}\|^2 = (\vec{\boldsymbol{b}} - \vec{\boldsymbol{a}})\cdot(\vec{\boldsymbol{b}} - \vec{\boldsymbol{a}}) = \|\vec{\boldsymbol{b}}\|^2 + \|\vec{\boldsymbol{a}}\|^2 - 2\vec{\boldsymbol{a}}\cdot\vec{\boldsymbol{b}}.$$

Compare this with (14.13), keeping in mind that $a = \|\vec{\boldsymbol{a}}\|$ and $b = \|\vec{\boldsymbol{b}}\|$: you are led to conclude that $-2\vec{\boldsymbol{a}}\cdot\vec{\boldsymbol{b}} = -2ab\cos\theta$, and thus $\vec{\boldsymbol{a}}\cdot\vec{\boldsymbol{b}} = \|\vec{\boldsymbol{a}}\| \cdot \|\vec{\boldsymbol{b}}\|\cos\theta$.

□

14.4.6 Orthogonal projection of one vector onto another

The following construction comes up very often. Let $\vec{\boldsymbol{a}} \neq \vec{\boldsymbol{0}}$ be a given vector. Then for any other vector $\vec{\boldsymbol{x}}$ there is a number λ such that

$$\vec{\boldsymbol{x}} = \lambda\vec{\boldsymbol{a}} + \vec{\boldsymbol{y}}$$

where $\vec{\boldsymbol{y}} \perp \vec{\boldsymbol{a}}$. In other words, you can write any vector $\vec{\boldsymbol{x}}$ as the sum of one vector parallel to $\vec{\boldsymbol{a}}$ and another vector orthogonal to $\vec{\boldsymbol{a}}$.

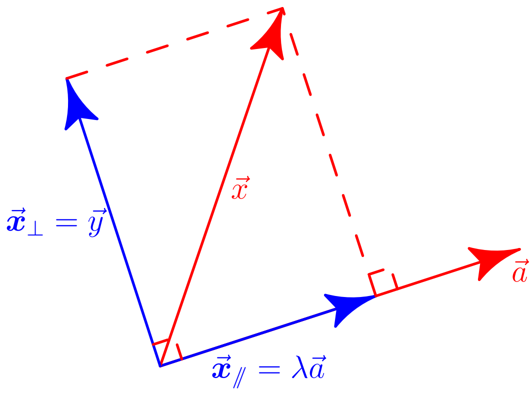


Figure 14.12: given $\vec{\boldsymbol{x}}$ and $\vec{\boldsymbol{a}}$, find $\vec{\boldsymbol{x}}_{/\!/}$ and $\vec{\boldsymbol{x}}_\perp$.

The two vectors $\lambda\vec{\boldsymbol{a}}$ and $\vec{\boldsymbol{y}}$ are called the ***parallel*** and ***orthogonal components*** of the vector $\vec{\boldsymbol{x}}$ (with respect to $\vec{\boldsymbol{a}}$), and sometimes the following notation is used

$$\vec{\boldsymbol{x}}^{/\!/} = \lambda\vec{\boldsymbol{a}}, \qquad \vec{\boldsymbol{x}}^{\perp} = \vec{\boldsymbol{y}},$$

so that

$$\vec{\boldsymbol{x}} = \vec{\boldsymbol{x}}^{/\!/} + \vec{\boldsymbol{x}}^{\perp}.$$

There are moderately simple formulas for $\vec{\boldsymbol{x}}^{/\!/}$ and $\vec{\boldsymbol{x}}^{\perp}$, but it is better to remember the following derivation of these formulas.

Assume that the vectors $\vec{\boldsymbol{a}}$ and $\vec{\boldsymbol{x}}$ are given. Then we look for a number λ such that $\vec{\boldsymbol{y}} = \vec{\boldsymbol{x}} - \lambda\vec{\boldsymbol{a}}$ is perpendicular to $\vec{\boldsymbol{a}}$. Recall that $\vec{\boldsymbol{a}} \perp (\vec{\boldsymbol{x}} - \lambda\vec{\boldsymbol{a}})$ if and only if

$$\vec{\boldsymbol{a}}\cdot(\vec{\boldsymbol{x}} - \lambda\vec{\boldsymbol{a}}) = 0.$$

Expand the dot product and you get this equation for λ

$$\vec{\boldsymbol{a}}\cdot\vec{\boldsymbol{x}} - \lambda\vec{\boldsymbol{a}}\cdot\vec{\boldsymbol{a}} = 0,$$

whence

$$\lambda = \frac{\vec{\boldsymbol{a}}\cdot\vec{\boldsymbol{x}}}{\vec{\boldsymbol{a}}\cdot\vec{\boldsymbol{a}}} = \frac{\vec{\boldsymbol{a}}\cdot\vec{\boldsymbol{x}}}{\|\vec{\boldsymbol{a}}\|^2} \tag{14.14}$$

To compute the parallel and orthogonal components of $\vec{\boldsymbol{x}}$ w.r.t. $\vec{\boldsymbol{a}}$ you first compute λ according to (14.14), which tells you that the parallel component is given by

$$\vec{\boldsymbol{x}}^{/\!/} = \lambda\vec{\boldsymbol{a}} = \frac{\vec{\boldsymbol{a}}\cdot\vec{\boldsymbol{x}}}{\vec{\boldsymbol{a}}\cdot\vec{\boldsymbol{a}}}\,\vec{\boldsymbol{a}}.$$

The orthogonal component is then "the rest," i.e. by definition $\vec{\boldsymbol{x}}^{\perp} = \vec{\boldsymbol{x}} - \vec{\boldsymbol{x}}^{/\!/}$, so

$$\vec{\boldsymbol{x}}^{\perp} = \vec{\boldsymbol{x}} - \vec{\boldsymbol{x}}^{/\!/} = \vec{\boldsymbol{x}} - \frac{\vec{\boldsymbol{a}}\cdot\vec{\boldsymbol{x}}}{\vec{\boldsymbol{a}}\cdot\vec{\boldsymbol{a}}}\,\vec{\boldsymbol{a}}.$$

To fully understand the connection between dot products and projections consider watching YouTube by 3Blue1Brown .

14.4.7 Defining equations of lines

In § 14.2 we saw how to generate points on a line given two points on that line by means of a "parametrization." I.e. given points A and B on the line ℓ the point whose position vector is $\vec{\boldsymbol{x}} = \vec{\boldsymbol{a}} + t(\vec{\boldsymbol{b}} - \vec{\boldsymbol{a}})$ will be on ℓ for any value of the "parameter" t.

In this section we will use the dot-product to give a different description of lines in the plane (and planes in three dimensional space.) We will derive an equation for a line. Rather than generating points on the line ℓ this equation tells us if any given point X in the plane is on the line or not.

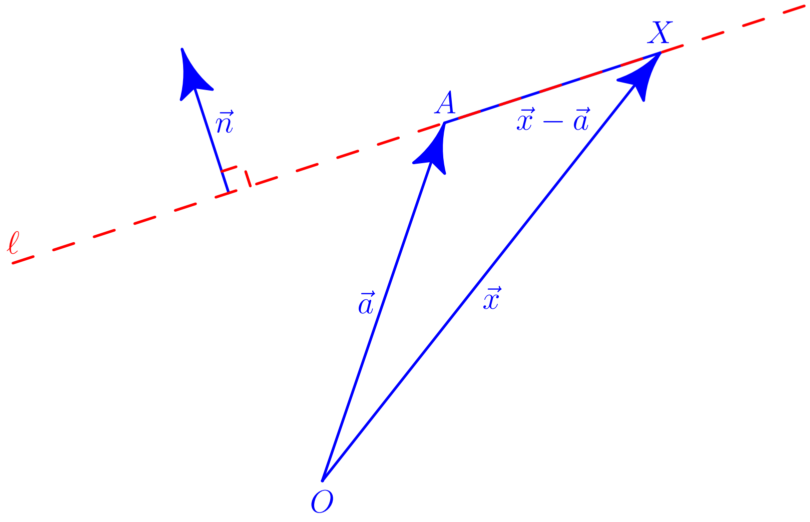


Figure 14.13: Is X on ℓ

Here is the derivation of the equation of a line in the plane. To produce the equation you need two ingredients:

1. One particular point on the line (let's call this point A, and write $\vec{\boldsymbol{a}}$ for its position vector),

2. a ***normal vector*** $\vec{\boldsymbol{n}}$ for the line, i.e. a nonzero vector which is perpendicular to the line.

Now let X be any point in the plane, and consider the line segment AX.

- Clearly, X will be on the line if and only if AX is parallel to ℓ [1]
- Since ℓ is perpendicular to $\vec{\boldsymbol{n}}$, the segment AX and the line ℓ will be parallel if and only if $AX \perp \vec{\boldsymbol{n}}$.
- $AX \perp \vec{\boldsymbol{n}}$ holds if and only if $\overrightarrow{AX} \cdot \vec{\boldsymbol{n}} = 0$.

So in the end we see that X lies on the line ℓ if and only if the following vector equation is satisfied:

$$\overrightarrow{AX} \cdot \vec{\boldsymbol{n}} = 0 \quad \text{or} \quad (\vec{\boldsymbol{x}} - \vec{\boldsymbol{a}}) \cdot \vec{\boldsymbol{n}} = 0 \tag{14.15}$$

This equation is called a ***defining equation for the line*** ℓ**.**

Any given line has many defining equations. Just by changing the length of the normal you get a different equation, which still describes the same line.

[1] From plane Euclidean geometry: parallel lines either don't intersect or they coincide.

14.4.7.1 Example, find a line through one point and perpendicular to another.

Find a defining equation for the line ℓ which goes through $A(1,1)$ and is perpendicular to the line segment AB where B is the point $(3,-1)$.

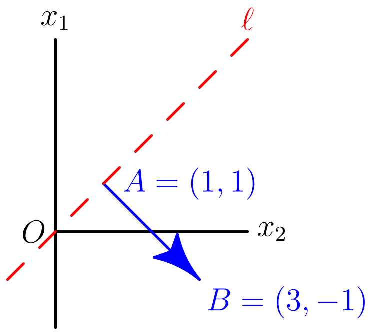


Figure 14.14: example, find an equation for ℓ

Solution. We already know a point on the line, namely A, but we still need a normal vector. The line is required to be perpendicular to AB, so $\vec{\boldsymbol{n}} = \overrightarrow{AB}$ is a normal vector:

$$\vec{\boldsymbol{n}} = \overrightarrow{AB} = \begin{pmatrix} 3-1 \\ (-1)-1 \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \end{pmatrix}$$

Of course any multiple of $\vec{\boldsymbol{n}}$ is also a normal vector, for instance

$$\vec{\boldsymbol{m}} = \tfrac{1}{2}\vec{\boldsymbol{n}} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

is a normal vector.

With $\vec{\boldsymbol{a}} = \left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right)$ we then get the following equation for ℓ

$$\vec{\boldsymbol{n}}\cdot(\vec{\boldsymbol{x}} - \vec{\boldsymbol{a}}) = \begin{pmatrix} 2 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} x_1 - 1 \\ x_2 - 1 \end{pmatrix} = 2x_1 - 2x_2 = 0.$$

If you choose the normal $\vec{\boldsymbol{m}}$ instead, you get

$$\vec{\boldsymbol{m}}\cdot(\vec{\boldsymbol{x}} - \vec{\boldsymbol{a}}) = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} x_1 - 1 \\ x_2 - 1 \end{pmatrix} = x_1 - x_2 = 0.$$

Both equations $2x_1 - 2x_2 = 0$ and $x_1 - x_2 = 0$ are equivalent.

14.4.8 Distance to a line

Let ℓ be a line in the plane and assume a point A on the line as well as a vector $\vec{\boldsymbol{n}}$ perpendicular to ℓ are known. Using the dot product one can easily compute the distance from the line to any other given point P in the plane. Here is how:

Draw the line m through A perpendicular to ℓ, and drop a perpendicular line from P onto m. let Q be the projection of P onto m. The distance from P to ℓ is then equal to the length of the line segment AQ. Since AQP is a right triangle one has

$$AQ = AP\cos\theta.$$

Here θ is the angle between the normal $\vec{\boldsymbol{n}}$ and the vector $\overrightarrow{AP}$. One also has

$$\vec{\boldsymbol{n}}\cdot(\vec{\boldsymbol{p}} - \vec{\boldsymbol{a}}) = \vec{\boldsymbol{n}}\cdot\overrightarrow{AP} = \|\overrightarrow{AP}\| \ \|\vec{\boldsymbol{n}}\| \cos\theta = AP \, \|\vec{\boldsymbol{n}}\| \cos\theta.$$

Hence we get

$$\mathrm{dist}(P, \ell) = \frac{\vec{\boldsymbol{n}}\cdot(\vec{\boldsymbol{p}} - \vec{\boldsymbol{a}})}{\|\vec{\boldsymbol{n}}\|}.$$

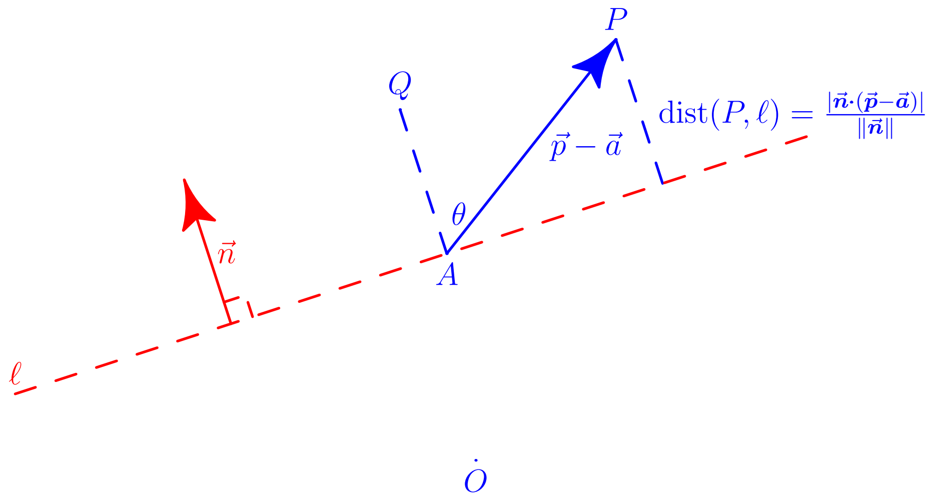


Figure 14.15: Distance from a point P to a line ℓ

This argument from a drawing contains a hidden assumption, namely that the point P lies on the side of the line ℓ pointed to by the vector $\vec{\boldsymbol{n}}$. If this is not the case, so that $\vec{\boldsymbol{n}}$ and $\overrightarrow{AP}$ point to opposite sides of ℓ, then the angle between them exceeds 90°, i.e. $\theta > \pi/2$. In this case $\cos\theta < 0$, and one has $AQ = -AP\cos\theta$. the distance formula therefore has to be modified to

$$\mathrm{dist}(P, \ell) = -\frac{\vec{\boldsymbol{n}}\cdot(\vec{\boldsymbol{p}} - \vec{\boldsymbol{a}})}{\|\vec{\boldsymbol{n}}\|}.$$

14.4.9 Defining equation of a plane

Just as we have seen how you can form the defining equation for a line in the plane from just one point on the line and one normal vector to the line, you can also form the defining equation for a plane in space, again knowing only one point on the plane, and a vector perpendicular to it.

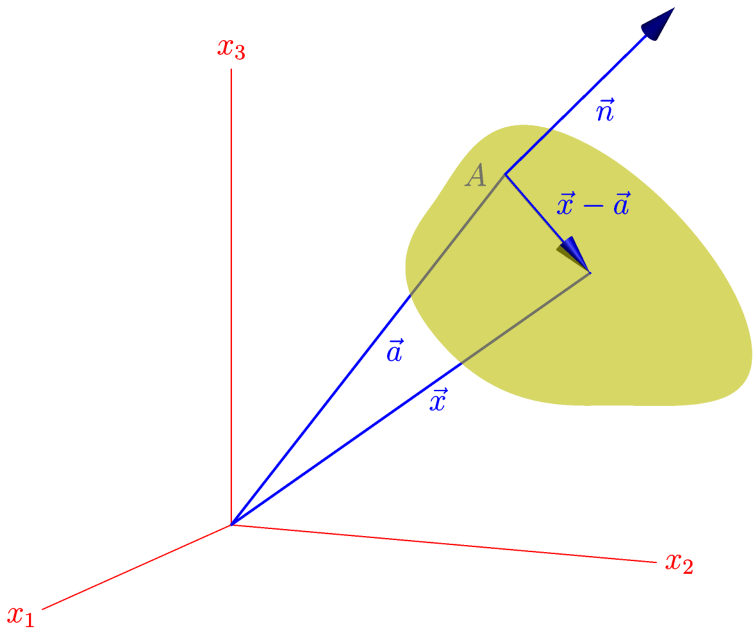


Figure 14.16: Equation for a plane from point A and normal $\vec{n}$.

If A is a point on some plane $\mathcal{P}$ and $\vec{\boldsymbol{n}}$ is a vector perpendicular to $\mathcal{P}$, then any other point X lies on $\mathcal{P}$ if and only if $\overrightarrow{AX} \perp \vec{\boldsymbol{n}}$. In other words, in terms of the position vectors $\vec{\boldsymbol{a}}$ and $\vec{\boldsymbol{x}}$ of A and X,

$$\text{the point } X \text{ is on } \mathcal{P} \iff \vec{\boldsymbol{n}}\boldsymbol{\cdot}(\vec{\boldsymbol{x}} - \vec{\boldsymbol{a}}) = 0.$$

Arguing just as in § 14.4.8 you find that the distance of a point X in space to the plane $\mathcal{P}$ is

$$\text{dist}(X, \mathcal{P}) = \pm \frac{\vec{\boldsymbol{n}}\boldsymbol{\cdot}(\vec{\boldsymbol{x}} - \vec{\boldsymbol{a}})}{\|\vec{\boldsymbol{n}}\|}. \tag{14.16}$$

Here the sign is "+" if X and the normal $\vec{\boldsymbol{n}}$ are on the same side of the plane $\mathcal{P}$; otherwise the sign is "−".

14.4.9.1 Example

Find the defining equation for the plane $\mathcal{P}$ through the point $A(1, 0, 2)$ which is perpendicular to the vector $\left(\begin{smallmatrix}1\\2\\1\end{smallmatrix}\right)$.

Solution: We know a point (A) and a normal vector $\vec{\boldsymbol{n}} = \left(\begin{smallmatrix}1\\2\\1\end{smallmatrix}\right)$ for $\mathcal{P}$. Then any point X with coordinates (x_1, x_2, x_3), or, with position vector $\vec{\boldsymbol{x}} = \left(\begin{smallmatrix} x_1 \\ x_2 \\ x_3 \end{smallmatrix}\right)$, will lie on the plane $\mathcal{P}$ if and only if

$$\begin{aligned}
\vec{\boldsymbol{n}}\cdot(\vec{\boldsymbol{x}} - \vec{\boldsymbol{a}}) = 0 &\iff \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \cdot \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} \right\} = 0 \\
&\iff \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} x_1 - 1 \\ x_2 \\ x_3 - 2 \end{pmatrix} = 0 \\
&\iff 1 \cdot (x_1 - 1) + 2 \cdot (x_2) + 1 \cdot (x_3 - 2) = 0 \\
&\iff x_1 + 2x_2 + x_3 - 3 = 0.
\end{aligned}$$

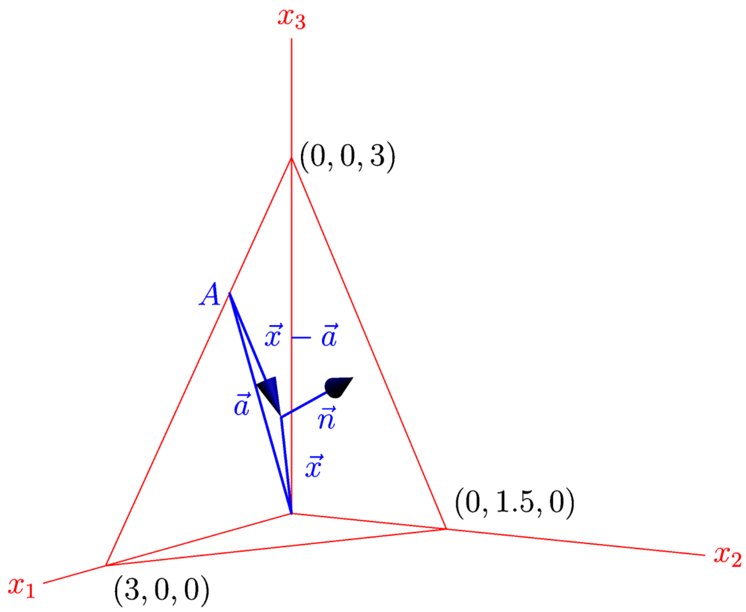


Figure 14.17: equation for a plane from point on plane and a normal to plane.

14.4.9.2 Example continued

Let $\mathcal{P}$ be the plane from the previous example. Which of the points $P(0, 0, 1)$, $Q(0, 0, 2)$, $R(-1, 2, 0)$ and $S(-1, 0, 5)$ lie on $\mathcal{P}$? Compute the distances from the points P, Q, R, S to the plane $\mathcal{P}$. Separate the points which do not lie on $\mathcal{P}$ into two groups of points which lie on the same side of $\mathcal{P}$.

Solution: We apply (14.16) to the position vectors $\vec{\boldsymbol{p}}, \vec{\boldsymbol{q}}, \vec{\boldsymbol{r}}, \vec{\boldsymbol{s}}$ of the points P, Q, R, S. For each calculation we need

$$\|\vec{\boldsymbol{n}}\| = \sqrt{1^2 + 2^2 + 1^2} = \sqrt{6}.$$

The third component of the given normal $\vec{\boldsymbol{n}} = \left(\begin{smallmatrix} 1 \\ 2 \\ 1 \end{smallmatrix}\right)$ is positive, so $\vec{\boldsymbol{n}}$ points "upwards." Therefore, if a point lies on the side of $\mathcal{P}$ pointed to by $\vec{\boldsymbol{n}}$, we shall say that the point lies *above the plane.*

P $\vec{\boldsymbol{p}} = \left(\begin{smallmatrix}0\\0\\1\end{smallmatrix}\right)$, $\vec{\boldsymbol{p}} - \vec{\boldsymbol{a}} = \left(\begin{smallmatrix}-1\\0\\-1\end{smallmatrix}\right)$, $\vec{\boldsymbol{n}}\cdot(\vec{\boldsymbol{p}} - \vec{\boldsymbol{a}}) = 1\cdot(-1) + 2\cdot(0) + 1\cdot(-1) = -2$

$$\frac{\vec{\boldsymbol{n}}\cdot(\vec{\boldsymbol{p}} - \vec{\boldsymbol{a}})}{\|\vec{\boldsymbol{n}}\|} = -\frac{2}{\sqrt{6}} = -\frac{1}{3}\sqrt{6}.$$

This quantity is negative, so P lies below $\mathcal{P}$. Its distance to $\mathcal{P}$ is $\frac{1}{3}\sqrt{6}$.

Q $\vec{\boldsymbol{q}} = \left(\begin{smallmatrix}0\\0\\2\end{smallmatrix}\right)$, $\vec{\boldsymbol{p}} - \vec{\boldsymbol{a}} = \left(\begin{smallmatrix}-1\\0\\0\end{smallmatrix}\right)$, $\vec{\boldsymbol{n}}\cdot(\vec{\boldsymbol{p}} - \vec{\boldsymbol{a}}) = 1\cdot(-1) + 2\cdot(0) + 1\cdot(0) = -1$

$$\frac{\vec{\boldsymbol{n}}\cdot(\vec{\boldsymbol{p}} - \vec{\boldsymbol{a}})}{\|\vec{\boldsymbol{n}}\|} = -\frac{1}{\sqrt{6}} = -\frac{1}{6}\sqrt{6}.$$

This quantity is negative, so Q also lies below $\mathcal{P}$. Its distance to $\mathcal{P}$ is $\frac{1}{6}\sqrt{6}$.

R $\vec{\boldsymbol{r}} = \left(\begin{smallmatrix}-1\\2\\0\end{smallmatrix}\right)$, $\vec{\boldsymbol{p}} - \vec{\boldsymbol{a}} = \left(\begin{smallmatrix}-2\\2\\-2\end{smallmatrix}\right)$, $\vec{\boldsymbol{n}}\cdot(\vec{\boldsymbol{p}} - \vec{\boldsymbol{a}}) = 1\cdot(-2) + 2\cdot(2) + 1\cdot(-2) = 0$

$$\frac{\vec{\boldsymbol{n}}\cdot(\vec{\boldsymbol{p}} - \vec{\boldsymbol{a}})}{\|\vec{\boldsymbol{n}}\|} = 0.$$

Thus R lies on the plane $\mathcal{P}$, and its distance to $\mathcal{P}$ is of course 0.

S $\vec{\boldsymbol{s}} = \left(\begin{smallmatrix}-1\\0\\5\end{smallmatrix}\right)$, $\vec{\boldsymbol{p}} - \vec{\boldsymbol{a}} = \left(\begin{smallmatrix}-2\\0\\3\end{smallmatrix}\right)$, $\vec{\boldsymbol{n}}\cdot(\vec{\boldsymbol{p}} - \vec{\boldsymbol{a}}) = 1\cdot(-1) + 2\cdot(0) + 1\cdot(3) = 2$

$$\frac{\vec{\boldsymbol{n}}\cdot(\vec{\boldsymbol{p}} - \vec{\boldsymbol{a}})}{\|\vec{\boldsymbol{n}}\|} = \frac{2}{\sqrt{6}} = \frac{1}{3}\sqrt{6}.$$

This quantity is positive, so S lies above $\mathcal{P}$. Its distance to $\mathcal{P}$ is $\frac{1}{3}\sqrt{6}$.

We have found that P and Q lie below the plane, R lies on the plane, and S is above the plane.

14.4.9.3 Example continued

Where does the line through the points $B(2,0,0)$ and $C(0,1,2)$ intersect the plane $\mathcal{P}$ from example 14.4.9.1?

Solution: Let ℓ be the line through B and C. We set up the parametric equation for ℓ. According to §14.2, (14.6) every point X on ℓ has position vector $\vec{\boldsymbol{x}}$ given by

$$\vec{\boldsymbol{x}} = \vec{\boldsymbol{b}} + t(\vec{\boldsymbol{c}} - \vec{\boldsymbol{b}}) = \begin{pmatrix}2\\0\\0\end{pmatrix} + t\begin{pmatrix}0-2\\1-0\\2-0\end{pmatrix} = \begin{pmatrix}2-2t\\t\\2t\end{pmatrix} \tag{14.17}$$

for some value of t.

The point X whose position vector $\vec{\boldsymbol{x}}$ is given above lies on the plane $\mathcal{P}$ if $\vec{\boldsymbol{x}}$ satisfies the defining equation of the plane. In example 14.4.9.1 we found this defining equation. It was

$$\vec{\boldsymbol{n}}\cdot(\vec{\boldsymbol{x}} - \vec{\boldsymbol{a}}) = 0, \text{ i.e. } x_1 + 2x_2 + x_3 - 3 = 0. \tag{14.18}$$

So to find the point of intersection of ℓ and $\mathcal{P}$ you substitute the parametrization (14.17) in the defining equation (14.18):

$$0 = x_1 + 2x_2 + x_3 - 3 = (2-2t) + 2(t) + (2t) - 3 = 2t - 1.$$

This implies $t = \frac{1}{2}$, and thus the intersection point has position vector

$$\vec{\boldsymbol{x}} = \vec{\boldsymbol{b}} + \tfrac{1}{2}(\vec{\boldsymbol{c}} - \vec{\boldsymbol{b}}) = \begin{pmatrix}2-2t\\t\\2t\end{pmatrix} = \begin{pmatrix}1\\\frac{1}{2}\\1\end{pmatrix},$$

i.e. ℓ and $\mathcal{P}$ intersect at $X(1, \frac{1}{2}, 1)$.

14.5 Cross Product

14.5.1 Algebraic definition of the cross product

Here is the definition of the cross-product of two vectors. The definition looks a bit strange and arbitrary at first sight – it really makes you wonder who thought of this. We will just put up with that for now and explore the properties of the cross product. Later on we will see a geometric interpretation of the cross product which will show that this particular definition is really useful. We will also find a few tricks that will help you reproduce the formula without memorizing it.

Definition 14.5.1. The "outer product" or "cross product" of two vectors is given by

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \times \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{pmatrix}$$

Note that the cross-product of two vectors is again a vector!

14.5.1.1 Note

If you set $\vec{\boldsymbol{b}} = \vec{\boldsymbol{a}}$ in the definition you find the following important fact: *The cross product of any vector with itself is the zero vector:*

$$\vec{\boldsymbol{a}} \times \vec{\boldsymbol{a}} = \vec{\mathbf{0}} \quad \text{for } \textbf{\textit{any}} \text{ vector } \vec{\boldsymbol{a}}.$$

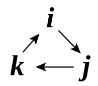
14.5.1.2 Example

Let $\vec{\boldsymbol{a}} = \left(\begin{smallmatrix} 1 \\ 2 \\ 3 \end{smallmatrix}\right)$, $\vec{\boldsymbol{b}} = \left(\begin{smallmatrix} -2 \\ 1 \\ 0 \end{smallmatrix}\right)$ and compute the cross product of these vectors.
Solution:

$$\vec{\boldsymbol{a}} \times \vec{\boldsymbol{b}} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \times \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \cdot 0 - 3 \cdot 1 \\ 3 \cdot (-2) - 1 \cdot 0 \\ 1 \cdot 1 - 2 \cdot (-2) \end{pmatrix} = \begin{pmatrix} -3 \\ -6 \\ 5 \end{pmatrix}$$

In terms of the standard basis vectors you can check the *multiplication table*. An easy way to remember the multiplication table is to put the vectors $\vec{\boldsymbol{i}}, \vec{\boldsymbol{j}}, \vec{\boldsymbol{k}}$ clockwise in a circle. Given two of the three vectors their product is either plus or minus the remaining vector. To determine the sign you step from the first vector to the second, to the third: if this makes you go clockwise you have a plus sign, if you have to go counterclockwise, you get a minus.

$\times$	$\vec{\boldsymbol{i}}$	$\vec{\boldsymbol{j}}$	$\vec{\boldsymbol{k}}$
$\vec{\boldsymbol{i}}$	$\vec{0}$	$\vec{\boldsymbol{k}}$	$-\vec{\boldsymbol{j}}$
$\vec{\boldsymbol{j}}$	$-\vec{\boldsymbol{k}}$	$\vec{0}$	$\vec{\boldsymbol{i}}$
$\vec{\boldsymbol{k}}$	$\vec{\boldsymbol{j}}$	$-\vec{\boldsymbol{i}}$	$\vec{0}$



The products of $\vec{\boldsymbol{\imath}}, \vec{\boldsymbol{\jmath}}$ and $\vec{\boldsymbol{k}}$ are all you need to know to compute the cross product. Given two vectors $\vec{\boldsymbol{a}}$ and $\vec{\boldsymbol{b}}$ write them as $\vec{\boldsymbol{a}} = a_1\vec{\boldsymbol{\imath}} + a_2\vec{\boldsymbol{\jmath}} + a_3\vec{\boldsymbol{k}}$ and $\vec{\boldsymbol{b}} = b_1\vec{\boldsymbol{\imath}} + b_2\vec{\boldsymbol{\jmath}} + b_3\vec{\boldsymbol{k}}$, and multiply as follows

$$\begin{aligned}
\vec{\boldsymbol{a}} \times \vec{\boldsymbol{b}} =& (a_1\vec{\boldsymbol{\imath}} + a_2\vec{\boldsymbol{\jmath}} + a_3\vec{\boldsymbol{k}}) \times (b_1\vec{\boldsymbol{\imath}} + b_2\vec{\boldsymbol{\jmath}} + b_3\vec{\boldsymbol{k}}) \\
=& \quad a_1\vec{\boldsymbol{\imath}} \times (b_1\vec{\boldsymbol{\imath}} + b_2\vec{\boldsymbol{\jmath}} + b_3\vec{\boldsymbol{k}}) \\
& +a_2\vec{\boldsymbol{\jmath}} \times (b_1\vec{\boldsymbol{\imath}} + b_2\vec{\boldsymbol{\jmath}} + b_3\vec{\boldsymbol{k}}) \\
& +a_3\vec{\boldsymbol{k}} \times (b_1\vec{\boldsymbol{\imath}} + b_2\vec{\boldsymbol{\jmath}} + b_3\vec{\boldsymbol{k}}) \\
=& \ a_1b_1\vec{\boldsymbol{\imath}} \times \vec{\boldsymbol{\imath}} + a_1b_2\vec{\boldsymbol{\imath}} \times \vec{\boldsymbol{\jmath}} + a_1b_3\vec{\boldsymbol{\imath}} \times \vec{\boldsymbol{k}} + \\
& a_2b_1\vec{\boldsymbol{\jmath}} \times \vec{\boldsymbol{\imath}} + a_2b_2\vec{\boldsymbol{\jmath}} \times \vec{\boldsymbol{\jmath}} + a_2b_3\vec{\boldsymbol{\jmath}} \times \vec{\boldsymbol{k}} + \\
& a_3b_1\vec{\boldsymbol{k}} \times \vec{\boldsymbol{\imath}} + a_3b_2\vec{\boldsymbol{k}} \times \vec{\boldsymbol{\jmath}} + a_3b_3\vec{\boldsymbol{k}} \times \vec{\boldsymbol{k}} \\
=& \ a_1b_1\vec{\boldsymbol{0}} + a_1b_2\vec{\boldsymbol{k}} - a_1b_3\vec{\boldsymbol{\jmath}} \\
& -a_2b_1\vec{\boldsymbol{k}} + a_2b_2\vec{\boldsymbol{0}} + a_2b_3\vec{\boldsymbol{\imath}} + \\
& a_3b_1\vec{\boldsymbol{\jmath}} - a_3b_2\vec{\boldsymbol{\imath}} + a_3b_3\vec{\boldsymbol{0}} \\
=& (a_2b_3 - a_3b_2)\vec{\boldsymbol{\imath}} + (a_3b_1 - a_1b_3)\vec{\boldsymbol{\jmath}} + (a_1b_2 - a_2b_1)\vec{\boldsymbol{k}}
\end{aligned}$$

This is a useful way of remembering how to compute the cross product, particularly when many of the components a_i and b_j are zero.

14.5.1.3 Example

Compute $\vec{\boldsymbol{k}} \times (p\vec{\boldsymbol{\imath}} + q\vec{\boldsymbol{\jmath}} + r\vec{\boldsymbol{k}})$:

$$\vec{\boldsymbol{k}} \times (p\vec{\boldsymbol{\imath}} + q\vec{\boldsymbol{\jmath}} + r\vec{\boldsymbol{k}}) = p(\vec{\boldsymbol{k}} \times \vec{\boldsymbol{\imath}}) + q(\vec{\boldsymbol{k}} \times \vec{\boldsymbol{\jmath}}) + r(\vec{\boldsymbol{k}} \times \vec{\boldsymbol{k}}) = -q\vec{\boldsymbol{\imath}} + p\vec{\boldsymbol{\jmath}}.$$

There is another way of remembering how to find $\vec{\boldsymbol{a}} \times \vec{\boldsymbol{b}}$. It involves the "triple product" and determinants. See § 14.5.3.

14.5.2 Algebraic properties of the cross product

Unlike the dot product, the cross product of two vectors behaves much less like ordinary multiplication. To begin with, the product is ***not commutative*** – instead one has

$$\vec{\boldsymbol{a}} \times \vec{\boldsymbol{b}} = -\vec{\boldsymbol{b}} \times \vec{\boldsymbol{a}} \quad \text{for } \textbf{\textit{all}} \text{ vectors } \vec{\boldsymbol{a}} \text{ and } \vec{\boldsymbol{b}}. \tag{14.19}$$

This property is sometimes called "anti-commutative."

$\vec{\boldsymbol{\imath}} \times (\vec{\boldsymbol{\imath}} \times \vec{\boldsymbol{\jmath}}) = \vec{\boldsymbol{\imath}} \times \vec{\boldsymbol{k}} = -\vec{\boldsymbol{\jmath}}$, $(\vec{\boldsymbol{\imath}} \times \vec{\boldsymbol{\imath}}) \times \vec{\boldsymbol{\jmath}} = \vec{\boldsymbol{0}} \times \vec{\boldsymbol{\jmath}} = \vec{\boldsymbol{0}}$ so "×" is not associative

Since the crossproduct of two vectors is again a vector you can compute the cross product of three vectors $\vec{\boldsymbol{a}}, \vec{\boldsymbol{b}}, \vec{\boldsymbol{c}}$. You now have a choice: do you first multiply $\vec{\boldsymbol{a}}$ and $\vec{\boldsymbol{b}}$, or $\vec{\boldsymbol{b}}$ and $\vec{\boldsymbol{c}}$, or $\vec{\boldsymbol{a}}$ and $\vec{\boldsymbol{c}}$? With numbers it makes no difference (e.g. $2 \times (3 \times 5) = 2 \times 15 = 30$ and $(2 \times 3) \times 5 = 6 \times 5 =$ also 30) but with the cross product of vectors it does matter: the cross product is ***not associative,*** i.e.

$$\vec{\boldsymbol{a}} \times (\vec{\boldsymbol{b}} \times \vec{\boldsymbol{c}}) \neq (\vec{\boldsymbol{a}} \times \vec{\boldsymbol{b}}) \times \vec{\boldsymbol{c}} \quad \text{for } \textbf{\textit{most}} \text{ vectors } \vec{\boldsymbol{a}}, \vec{\boldsymbol{b}}, \vec{\boldsymbol{c}}.$$

The ***distributive law*** does hold, i.e.

$$\vec{\boldsymbol{a}} \times (\vec{\boldsymbol{b}} + \vec{\boldsymbol{c}}) = \vec{\boldsymbol{a}} \times \vec{\boldsymbol{b}} + \vec{\boldsymbol{a}} \times \vec{\boldsymbol{c}}, \quad \text{and} \quad (\vec{\boldsymbol{b}} + \vec{\boldsymbol{c}}) \times \vec{\boldsymbol{a}} = \vec{\boldsymbol{b}} \times \vec{\boldsymbol{a}} + \vec{\boldsymbol{c}} \times \vec{\boldsymbol{a}}$$

is true for all vectors $\vec{a}, \vec{b}, \vec{c}$.

Also, an associative law, where one of the factors is a number and the other two are vectors, does hold. I.e.

$$t(\vec{a} \times \vec{b}) = (t\vec{a}) \times \vec{b} = \vec{a} \times (t\vec{b})$$

holds for all vectors $\vec{a}, \vec{b}$ and any number t. We were already using these properties when we multiplied $(a_1\vec{i} + a_2\vec{j} + a_3\vec{k}) \times (b_1\vec{i} + b_2\vec{j} + b_3\vec{k})$ in the previous section.

Finally, the cross product is only defined for space vectors, not for plane vectors.

14.5.3 The triple product and determinants

Definition 14.5.2. The triple product of three given vectors $\vec{a}, \vec{b}$, and $\vec{c}$ is defined to be

$$\vec{a} \cdot (\vec{b} \times \vec{c}).$$

In terms of the components of $\vec{a}, \vec{b}$, and $\vec{c}$ one has

$$\begin{aligned} \vec{a} \cdot (\vec{b} \times \vec{c}) &= \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \cdot \begin{pmatrix} b_2c_3 - b_3c_2 \\ b_3c_1 - b_1c_3 \\ b_1c_2 - b_2c_1 \end{pmatrix} \\ &= a_1b_2c_3 - a_1b_3c_2 + a_2b_3c_1 - a_2b_1c_3 + a_3b_1c_2 - a_3b_2c_1. \end{aligned}$$

This quantity is called a ***determinant,*** and is written as follows

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1b_2c_3 - a_1b_3c_2 + a_2b_3c_1 - a_2b_1c_3 + a_3b_1c_2 - a_3b_2c_1 \tag{14.20}$$

To compute the cross product of two given vectors $\vec{a}$ and $\vec{b}$ you arrange their components in the following determinant

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & a_1 & b_1 \\ \vec{j} & a_2 & b_2 \\ \vec{k} & a_3 & b_3 \end{vmatrix} = (a_2b_3 - a_3b_2)\vec{i} + (a_3b_1 - a_1b_3)\vec{j} + (a_1b_2 - a_2b_1)\vec{k}. \tag{14.21}$$

This is not a normal determinant since some of its entries are vectors, but if you ignore that odd circumstance and simply compute the determinant according to the definition (14.20), you get (14.21).

An important property of the triple product is that it is much more symmetric in the factors $\vec{a}, \vec{b}, \vec{c}$ than the notation $\vec{a} \cdot (\vec{b} \times \vec{c})$ suggests.

Theorem 14.5.1. For any triple of vectors $\vec{a}, \vec{b}, \vec{c}$ one has

$$\vec{a} \cdot (\vec{b} \times \vec{c}) = \vec{b} \cdot (\vec{c} \times \vec{a}) = \vec{c} \cdot (\vec{a} \times \vec{b}),$$

and

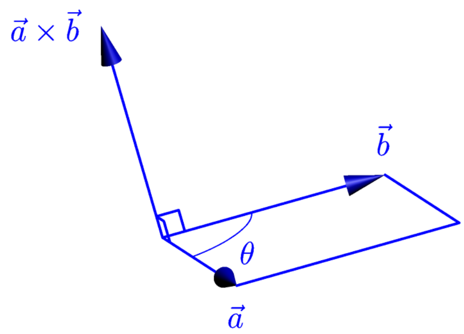
$$\vec{a} \cdot (\vec{b} \times \vec{c}) = -\vec{b} \cdot (\vec{a} \times \vec{c}) = -\vec{c} \cdot (\vec{b} \times \vec{a}).$$

In other words, if you exchange two factors in the product $\vec{a} \cdot (\vec{b} \times \vec{c})$ it changes its sign. If you "rotate the factors," i.e. if you replace $\vec{a}$ by $\vec{b}$, $\vec{b}$ by $\vec{c}$ and $\vec{c}$ by $\vec{a}$, the product doesn't change at all.

14.5.4 Geometric description of the cross product

Theorem 14.5.2.

$$\vec{a} \times \vec{b} \perp \vec{a}, \vec{b}$$



Proof.

Figure 14.18: the cross product of two vectors

We use the triple product:

$$\vec{a}\cdot(\vec{a} \times \vec{b}) = \vec{b}\cdot(\vec{a} \times \vec{a}) = \vec{0}$$

since $\vec{a} \times \vec{a} = \vec{0}$ for any vector $\vec{a}$. It follows that $\vec{a} \times \vec{b}$ is perpendicular to $\vec{a}$. Similarly, $\vec{b}\cdot(\vec{a} \times \vec{b}) = \vec{a}\cdot(\vec{b} \times \vec{b}) = \vec{0}$ shows that $\vec{a} \times \vec{b}$ is perpendicular to $\vec{b}$. □

Theorem 14.5.3.

$$\|\vec{a} \times \vec{b}\| = \|\vec{a}\| \, \|\vec{b}\| \, \sin\theta$$

Proof. Emily[2] just slipped us a piece of paper with the following formula on it:

$$\|\vec{a} \times \vec{b}\|^2 + (\vec{a}\cdot\vec{b})^2 = \|\vec{a}\|^2 \|\vec{b}\|^2. \tag{14.22}$$

After setting $\vec{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ and $\vec{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ and diligently computing both sides we find that this formula actually holds for any pair of vectors $\vec{a}, \vec{b}$! The (long) computation which implies this identity will be presented in class (maybe).

If we assume that Lagrange's identity holds then we get

$$\|\vec{a} \times \vec{b}\|^2 = \|\vec{a}\|^2 \|\vec{b}\|^2 - (\vec{a}\cdot\vec{b})^2 = \|\vec{a}\|^2 \|\vec{b}\|^2 - \|\vec{a}\|^2 \|\vec{b}\|^2 \cos^2\theta = \|\vec{a}\|^2 \|\vec{b}\|^2 \sin^2\theta$$

since $1 - \cos^2\theta = \sin^2\theta$. The theorem is proved. □

These two theorems *almost* allow you to construct the cross product of two vectors geometrically. If $\vec{a}$ and $\vec{b}$ are two vectors, then their cross product satisfies the following description:

1. If $\vec{a}$ and $\vec{b}$ are parallel, then the angle θ between them vanishes, and so their cross product is the zero vector. Assume from here on that $\vec{a}$ and $\vec{b}$ are not parallel.

2. $\vec{a} \times \vec{b}$ is perpendicular to both $\vec{a}$ and $\vec{b}$. In other words, since $\vec{a}$ and $\vec{b}$ are not parallel, they determine a plane, and their cross product is a vector perpendicular to this plane.

[2]It's actually called ***Lagrange's identity***. Yes, the same Lagrange who found the formula for the remainder term.

3. the length of the cross product $\vec{a} \times \vec{b}$ is $\|\vec{a}\| \cdot \|\vec{b}\| \sin\theta$.

There are only two vectors that satisfy conditions 2 and 3: to determine which one of these is the cross product you must apply the ***Right Hand Rule*** (screwdriver rule, corkscrew rule, etc.) for $\vec{a}, \vec{b}, \vec{a} \times \vec{b}$: if you turn a screw whose axis is perpendicular to $\vec{a}$ and $\vec{b}$ in the direction from $\vec{a}$ to $\vec{b}$, the screw moves in the direction of $\vec{a} \times \vec{b}$.

Alternatively, without seriously injuring yourself, you should be able to make a fist with your ***right*** hand, and then stick out your thumb, index and middle fingers so that your thumb is $\vec{a}$, your index finger is $\vec{b}$ and your middle finger is $\vec{a} \times \vec{b}$. Only people with the most flexible joints can do this with their left hand.

For a more in depth description of the *cross product* the student should watch YouTube by 3Blue1Brown but be warned that you should really watch the whole series on linear algebra for this one episode to make sense.

14.6 A few applications of the cross product

14.6.1 Area of a parallelogram

Let $ABCD$ be a parallelogram. Its area is given by "height times base," a formula which should be familiar from high school geometry.

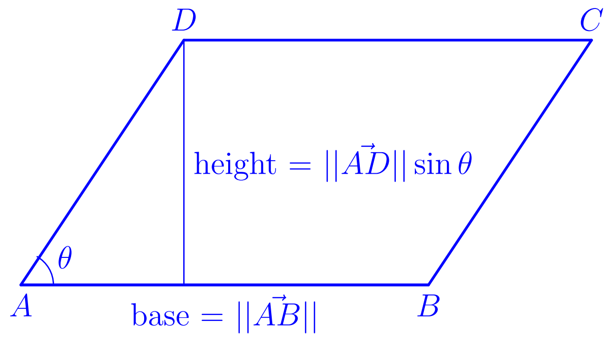


Figure 14.19: Area of a parallelogram, $||\vec{AB} \times \vec{AD}||$.

If the angle between the sides AB and AD is θ, then the height of the parallelogram is $\|\overrightarrow{AD}\| \sin\theta$, so that the area of $ABCD$ is

$$\text{area of } ABCD = \|\overrightarrow{AB}\| \cdot \|\overrightarrow{AD}\| \sin\theta = \|\overrightarrow{AB} \times \overrightarrow{AD}\|. \tag{14.23}$$

The area of the triangle ABD is of course half as much,

$$\text{area of triangle } ABD = \tfrac{1}{2} \|\overrightarrow{AB} \times \overrightarrow{AD}\|.$$

These formulae are valid even when the points A, B, C, and D are points in space. Of course they must lie in one plane for otherwise $ABCD$ couldn't be a parallelogram.

14.6.2 Example

Let the points $A(1,0,2)$, $B(2,0,0)$, $C(3,1,-1)$ and $D(2,1,1)$ be given.

Show that $ABCD$ is a parallelogram, and compute its area.

Solution: $ABCD$ will be a parallelogram if and only if $\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{AD}$. In terms of the position vectors $\vec{\boldsymbol{a}}$,$\vec{\boldsymbol{b}}$, $\vec{\boldsymbol{c}}$ and $\vec{\boldsymbol{d}}$ of A, B, C, D this boils down to

$$\vec{\boldsymbol{c}} - \vec{\boldsymbol{a}} = (\vec{\boldsymbol{b}} - \vec{\boldsymbol{a}}) + (\vec{\boldsymbol{d}} - \vec{\boldsymbol{a}}), \quad \text{i.e.} \quad \vec{\boldsymbol{a}} + \vec{\boldsymbol{c}} = \vec{\boldsymbol{b}} + \vec{\boldsymbol{d}}.$$

For our points we get

$$\vec{\boldsymbol{a}} + \vec{\boldsymbol{c}} = \begin{pmatrix}1\\0\\2\end{pmatrix} + \begin{pmatrix}3\\1\\-1\end{pmatrix} = \begin{pmatrix}4\\1\\1\end{pmatrix}, \qquad \vec{\boldsymbol{b}} + \vec{\boldsymbol{d}} = \begin{pmatrix}2\\0\\0\end{pmatrix} + \begin{pmatrix}2\\1\\1\end{pmatrix} = \begin{pmatrix}4\\1\\1\end{pmatrix}.$$

So $ABCD$ is indeed a parallelogram. Its area is the length of

$$\overrightarrow{AB} \times \overrightarrow{AD} = \begin{pmatrix}2-1\\0\\0-2\end{pmatrix} \times \begin{pmatrix}2-1\\1-0\\1-2\end{pmatrix} = \begin{pmatrix}1\\0\\-2\end{pmatrix} \times \begin{pmatrix}1\\-1\\-1\end{pmatrix} = \begin{pmatrix}-2\\-1\\-1\end{pmatrix}.$$

So the area of $ABCD$ is $\sqrt{(-2)^2 + (-1)^2 + (-1)^2} = \sqrt{6}$.

14.6.3 Finding the normal to a plane

If you know two vectors $\vec{\boldsymbol{a}}$ and $\vec{\boldsymbol{b}}$ which are parallel to a given plane $\mathcal{P}$ but not parallel to each other, then you can find a normal vector for the plane $\mathcal{P}$ by computing

$$\vec{\boldsymbol{n}} = \vec{\boldsymbol{a}} \times \vec{\boldsymbol{b}}.$$

We have just seen that the vector $\vec{\boldsymbol{n}}$ must be perpendicular to both $\vec{\boldsymbol{a}}$ and $\vec{\boldsymbol{b}}$, and hence it is perpendicular to the plane $\mathcal{P}$.

This trick is especially useful when you have three points A, B and C, and you want to find the defining equation for the plane $\mathcal{P}$ through these points. We will assume that the three points do not all lie on one line, for otherwise there are many planes through A, B and C.

To find the defining equation we need one point on the plane (we have three of them), and a normal vector to the plane. A normal vector can be obtained by computing the cross product of two vectors parallel to the plane. Since $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are both parallel to $\mathcal{P}$, the vector $\vec{\boldsymbol{n}} = \overrightarrow{AB} \times \overrightarrow{AC}$ is such a normal vector.

Thus the defining equation for the plane through three given points A, B and C is

$$\vec{\boldsymbol{n}} \cdot (\vec{\boldsymbol{x}} - \vec{\boldsymbol{a}}) = 0, \quad \text{with} \quad \vec{\boldsymbol{n}} = \overrightarrow{AB} \times \overrightarrow{AC} = (\vec{\boldsymbol{b}} - \vec{\boldsymbol{a}}) \times (\vec{\boldsymbol{c}} - \vec{\boldsymbol{a}}).$$

14.6.4 Example

Find the defining equation of the plane $\mathcal{P}$ through the points $A(2,-1,0)$, $B(2,1,-1)$ and $C(-1,1,1)$. Find the intersections of $\mathcal{P}$ with the three coordinate axes, and find the distance from the origin to $\mathcal{P}$.

Solution: We have

$$\overrightarrow{AB} = \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix} \quad \text{and} \quad \overrightarrow{AC} = \begin{pmatrix} -3 \\ 2 \\ 1 \end{pmatrix}$$

so that

$$\vec{\boldsymbol{n}} = \overrightarrow{AB} \times \overrightarrow{AC} = \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix} \times \begin{pmatrix} -3 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 3 \\ 6 \end{pmatrix}$$

is a normal to the plane. The defining equation for $\mathcal{P}$ is therefore

$$0 = \vec{\boldsymbol{n}}\cdot(\vec{\boldsymbol{x}} - \vec{\boldsymbol{a}}) = \begin{pmatrix} 4 \\ 3 \\ 6 \end{pmatrix} \cdot \begin{pmatrix} x_1 - 2 \\ x_2 + 1 \\ x_3 - 0 \end{pmatrix}$$

i.e.

$$4x_1 + 3x_2 + 6x_3 - 5 = 0.$$

The plane intersects the x_1 axis when $x_2 = x_3 = 0$ and hence $4x_1 - 5 = 0$, i.e. in the point $(\frac{5}{4}, 0, 0)$. The intersections with the other two axes are $(0, \frac{5}{3}, 0)$ and $(0, 0, \frac{5}{6})$.
The distance from any point with position vector $\vec{\boldsymbol{x}}$ to $\mathcal{P}$ is given by

$$\text{dist} = \pm\frac{\vec{\boldsymbol{n}}\cdot(\vec{\boldsymbol{x}} - \vec{\boldsymbol{a}})}{\|\vec{\boldsymbol{n}}\|},$$

so the distance from the origin (whose position vector is $\vec{\boldsymbol{x}} = \vec{\boldsymbol{0}} = \left(\begin{smallmatrix} 0 \\ 0 \\ 0 \end{smallmatrix}\right)$) to $\mathcal{P}$ is

$$\text{distance origin to } \mathcal{P} = \pm\frac{\vec{\boldsymbol{a}}\cdot\vec{\boldsymbol{n}}}{\|\vec{\boldsymbol{n}}\|} = \pm\frac{2 \cdot 4 + (-1) \cdot 3 + 0 \cdot 6}{\sqrt{4^2 + 3^2 + 6^2}} = \frac{5}{\sqrt{61}} (\approx 1.024\cdots).$$

14.6.5 Volume of a parallelepiped

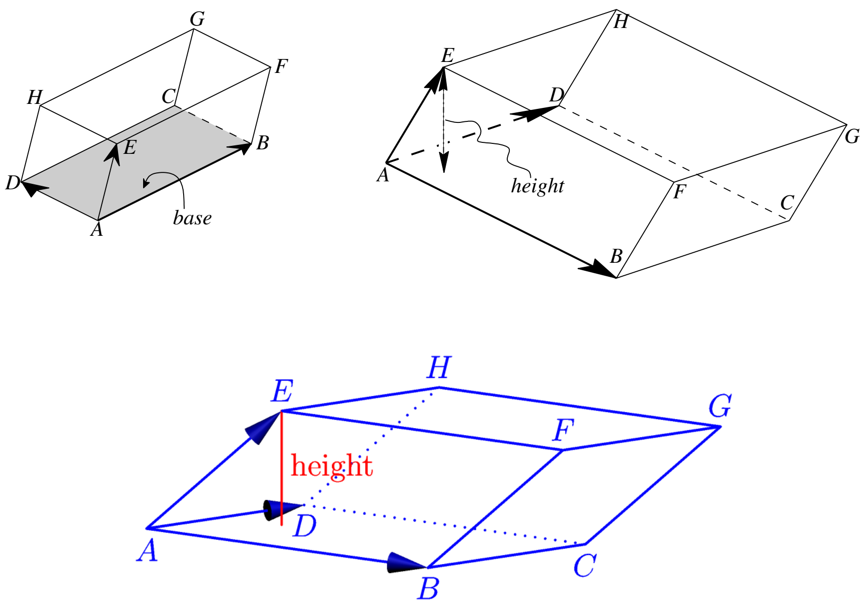


Figure 14.20: the volume of a parallelepiped

A ***parallelepiped*** is a three dimensional body whose sides are parallelograms. For instance, a cube is an example of a parallelepiped; a rectangular block (whose faces are rectangles, meeting at right angles) is also a parallelepiped. Any parallelepiped has 8 vertices (corner points), 12 edges and 6 faces.

Let $\genfrac{}{}{0pt}{}{ABCD}{EFGH}$ be a parallelepiped. If we call one of the faces, say $ABCD$, the base of the parallelepiped, then the other face $EFGH$ is parallel to the base. The ***height of the parallelepiped*** is the distance from any point in $EFGH$ to the base, e.g. to compute the height of $\genfrac{}{}{0pt}{}{ABCD}{EFGH}$ one could compute the distance from the point E (or F, or G, or H) to the plane through $ABCD$.

The volume of the parallelepiped $\genfrac{}{}{0pt}{}{ABCD}{EFGH}$ is given by the formula

$$\text{Volume } \genfrac{}{}{0pt}{}{ABCD}{EFGH} = \text{Area of base} \times \text{height}.$$

Since the base is a parallelogram we know its area is given by

$$\text{Area of base} ABCD = \|\overrightarrow{AB} \times \overrightarrow{AD}\|$$

We also know that $\vec{\boldsymbol{n}} = \overrightarrow{AB} \times \overrightarrow{AD}$ is a vector perpendicular to the plane through $ABCD$, i.e. perpendicular to the base of the parallelepiped. If we let the angle between the edge AE and the normal $\vec{\boldsymbol{n}}$ be ψ, then the height of the parallelepiped is given by

$$\text{height} = \|\overrightarrow{AE}\| \cos\psi.$$

Therefore the triple product of $\overrightarrow{AB}, \overrightarrow{AD}, \overrightarrow{AE}$ is

$$\begin{aligned} \text{Volume } \genfrac{}{}{0pt}{}{ABCD}{EFGH} &= \text{height} \times \text{Area of base} \\ &= \|\overrightarrow{AE}\| \cos\psi \quad \|\overrightarrow{AB} \times \overrightarrow{AD}\|, \end{aligned}$$

i.e.

$$\boxed{\text{Volume } \genfrac{}{}{0pt}{}{ABCD}{EFGH} = \overrightarrow{AE} \cdot (\overrightarrow{AB} \times \overrightarrow{AD}).}$$

14.7 Notation

In the next chapter we will be using vectors, so let's take a minute to summarize the concepts and notation we have been using.

Given a point in the plane, or in space you can form its position vector. So associated to a point we have three different objects: the point, its position vector and its coordinates. here is the notation we use for these:

OBJECT	NOTATION
Point....................	Upper case letters, A, B, etc.
Position vector	Lowercase letters with an arrow on top. The position vector $\overrightarrow{OA}$ of the point A should be $\vec{\boldsymbol{a}}$, so that letters match across changes from upper to lower case.
Coordinates of a point...	The coordinates of the point A are the same as the components of its position vector $\vec{\boldsymbol{a}}$: we use lower case letters with a subscript to indicate which coordinate we have in mind: (a_1, a_2).

14.8 PROBLEMS

COMPUTING AND DRAWING VECTORS

934. Simplify the following

$$\vec{\boldsymbol{a}} = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} + 3 \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix};$$

$$\vec{\boldsymbol{b}} = 12 \begin{pmatrix} 1 \\ 1/3 \end{pmatrix} - 3 \begin{pmatrix} 4 \\ 1 \end{pmatrix};$$

$$\vec{\boldsymbol{c}} = (1+t) \begin{pmatrix} 1 \\ 1-t \end{pmatrix} - t \begin{pmatrix} 1 \\ -t \end{pmatrix},$$

$$\vec{\boldsymbol{d}} = t \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + t^2 \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

935. If $\vec{\boldsymbol{a}}, \vec{\boldsymbol{b}}, \vec{\boldsymbol{c}}$ are as in the previous problem, then which of the following expressions mean anything? Compute those expressions that are well defined.

(***i***) $\vec{\boldsymbol{a}} + \vec{\boldsymbol{b}}$ (***ii***) $\vec{\boldsymbol{b}} + \vec{\boldsymbol{c}}$ (***iii***) $\pi\vec{\boldsymbol{a}}$
(***iv***) $\vec{\boldsymbol{b}}^2$ (***v***) $\vec{\boldsymbol{b}}/\vec{\boldsymbol{c}}$ (***vi***) $\|\vec{\boldsymbol{a}}\| + \|\vec{\boldsymbol{b}}\|$
(***vii***) $\|\vec{\boldsymbol{b}}\|^2$ (***viii***) $\vec{\boldsymbol{b}}/\|\vec{\boldsymbol{c}}\|$

936. Let $\vec{a} = \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}$ and $\vec{b} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$.

Compute:

1. $||\vec{a}||$
2. $2\vec{a}$
3. $||2\vec{a}||^2$
4. $\vec{a} + \vec{b}$
5. $3\vec{a} - \vec{b}$

†408

937. Let $\vec{\boldsymbol{u}}, \vec{\boldsymbol{v}}, \vec{\boldsymbol{w}}$ be three given vectors, and suppose

$$\vec{\boldsymbol{a}} = \vec{\boldsymbol{v}} + \vec{\boldsymbol{w}}, \quad \vec{\boldsymbol{b}} = 2\vec{\boldsymbol{u}} - \vec{\boldsymbol{w}}, \quad \vec{\boldsymbol{c}} = \vec{\boldsymbol{u}} + \vec{\boldsymbol{v}} + \vec{\boldsymbol{w}}.$$

(a) Simplify $\vec{\boldsymbol{p}} = \vec{\boldsymbol{a}} + 3\vec{\boldsymbol{b}} - \vec{\boldsymbol{c}}$ and $\vec{\boldsymbol{q}} = \vec{\boldsymbol{c}} - 2(\vec{\boldsymbol{u}} + \vec{\boldsymbol{a}})$.

(b) Find numbers r, s, t such that $r\vec{\boldsymbol{a}} + s\vec{\boldsymbol{b}} + t\vec{\boldsymbol{c}} = \vec{\boldsymbol{u}}$.

(c) Find numbers k, l, m such that $k\vec{\boldsymbol{a}} + l\vec{\boldsymbol{b}} + m\vec{\boldsymbol{c}} = \vec{\boldsymbol{v}}$.

938. Prove the Algebraic Properties (14.2), (14.3), (14.4), and (14.5) in section 14.1.4.

939. (a) Does there exist a number x such that

$$\begin{pmatrix} 1 \\ 2 \end{pmatrix} + \begin{pmatrix} x \\ x \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}?$$

(b) Make a drawing of all points P whose position vectors are given by

$$\vec{\boldsymbol{p}} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \begin{pmatrix} x \\ x \end{pmatrix}.$$

(c) Do there exist a numbers x and y such that

$$x\begin{pmatrix}1\\2\end{pmatrix}+y\begin{pmatrix}1\\1\end{pmatrix}=\begin{pmatrix}2\\1\end{pmatrix}?$$

†408

940. Given points $A(2,1)$ and $B(-1,4)$ compute the vector $\overrightarrow{AB}$. *Is* $\overrightarrow{AB}$ *a position vector?* †408

941. Given: points $A(2,1)$, $B(3,2)$, $C(4,4)$ and $D(5,2)$. *Is ABCD a parallelogram?* †408

942. Given: points $A(0,2,1)$, $B(0,3,2)$, $C(4,1,4)$ and D.

(a) If $ABCD$ is a parallelogram, then what are the coordinates of the point D?

(b) If $ABDC$ is a parallelogram, then what are the coordinates of the point D? †408

943. You are given three points in the plane: A has coordinates $(2,3)$, B has coordinates $(-1,2)$ and C has coordinates $(4,-1)$.

(a) Compute the vectors $\overrightarrow{AB}$, $\overrightarrow{BA}$, $\overrightarrow{AC}$, $\overrightarrow{CA}$, $\overrightarrow{BC}$ and $\overrightarrow{CB}$.

(b) Find the points P, Q, R and S whose position vectors are $\overrightarrow{AB}$, $\overrightarrow{BA}$, $\overrightarrow{AC}$, and $\overrightarrow{BC}$, respectively. *hint: make a precise drawing.*

944. Have a look at figure 14.21

(a) Draw the vectors $2\vec{\boldsymbol{v}}+\frac{1}{2}\vec{\boldsymbol{w}}$, $-\frac{1}{2}\vec{\boldsymbol{v}}+\vec{\boldsymbol{w}}$, and $\frac{3}{2}\vec{\boldsymbol{v}}-\frac{1}{2}\vec{\boldsymbol{w}}$

(b) Find real numbers s, t such that $s\vec{\boldsymbol{v}}+t\vec{\boldsymbol{w}}=\vec{\boldsymbol{a}}$.

(c) Find real numbers p, q such that $p\vec{\boldsymbol{v}}+q\vec{\boldsymbol{w}}=\vec{\boldsymbol{b}}$.

(d) Find real numbers k, l, m, n such that $\vec{\boldsymbol{v}}=k\vec{\boldsymbol{a}}+l\vec{\boldsymbol{b}}$, and $\vec{\boldsymbol{w}}=m\vec{\boldsymbol{a}}+n\vec{\boldsymbol{w}}$.

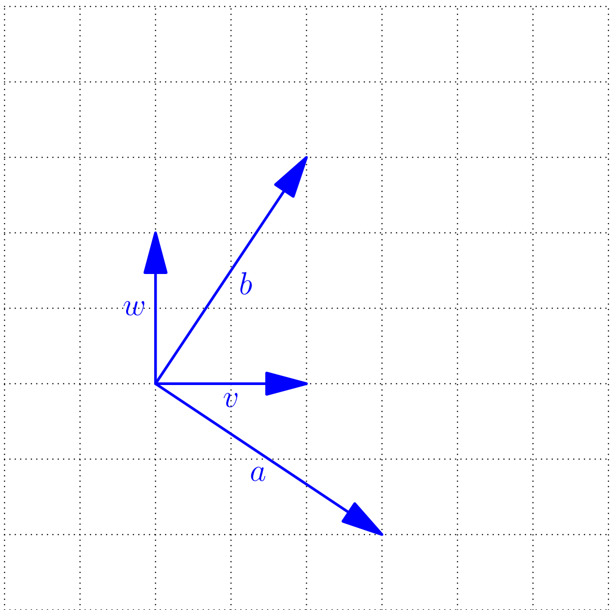


Figure 14.21: Drawing for problem 944

PARAMETRIC EQUATIONS FOR A LINE

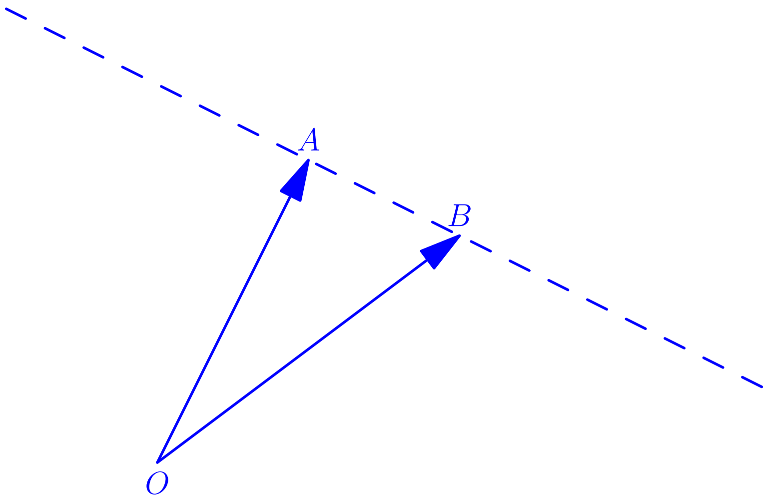


Figure 14.22: Parametric equations for a line.

945. In the figure above draw the points whose position vectors are given by $\vec{\boldsymbol{x}} = \vec{\boldsymbol{a}} + t(\vec{\boldsymbol{b}} - \vec{\boldsymbol{a}})$ for $t = 0, 1, \frac{1}{3}, \frac{3}{4}, -1, 2$. (as always, $\vec{\boldsymbol{a}} = \overrightarrow{OA}$, etc.)

946. In the figure above also draw the points whose position vector are given by $\vec{\boldsymbol{x}} = \vec{\boldsymbol{b}} + s(\vec{\boldsymbol{a}} - \vec{\boldsymbol{b}})$ for $s = 0, 1, \frac{1}{3}, \frac{3}{4}, -1, 2$.

947. (a) Find a parametric equation for the line ℓ through the points $A(3, 0, 1)$ and $B(2, 1, 2)$.

(b) Where does ℓ intersect the coordinate planes? †408

948. (a) Find a parametric equation for the line which contains the two vectors

$$\vec{a} = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} \text{ and } \vec{b} = \begin{pmatrix} 3 \\ 2 \\ 3 \end{pmatrix}.$$

(b) The vector $\vec{c} = \begin{pmatrix} c_1 \\ 1 \\ c_3 \end{pmatrix}$ is on this line. What is $\vec{c}$?

†408

949. Consider a triangle ABC and let $\vec{\boldsymbol{a}}, \vec{\boldsymbol{b}}, \vec{\boldsymbol{c}}$ be the position vectors of A, B, and C.

(a) Compute the position vector of the midpoint P of the line segment BC. Also compute the position vectors of the midpoints Q of AC and R of AB. (Make a drawing.)

(b) Let M be the point on the line segment AP which is twice as far from A as it is from P. Find the position vector of M.

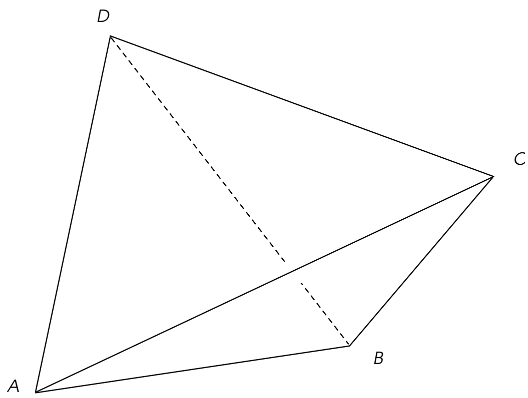
(c) Show that M also lies on the line segments BQ and CR.

†408

950. Let $ABCD$ be a tetrahedron, and let $\vec{\boldsymbol{a}}, \vec{\boldsymbol{b}}, \vec{\boldsymbol{c}}, \vec{\boldsymbol{d}}$ be the position vectors of the points A, B, C, D.

(*i*) Find position vectors of the midpoint P of AB, the midpoint Q of CD and the midpoint M of PQ.

(*ii*) Find position vectors of the midpoint R of BC, the midpoint S of AD and the midpoint N of RS.



ORTHOGONAL DECOMPOSITION OF ONE VECTOR WITH RESPECT TO ANOTHER

951. Given the vectors $\vec{\boldsymbol{a}} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$ and $\vec{\boldsymbol{b}} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ find $\vec{\boldsymbol{a}}^{/\!/}$, $\vec{\boldsymbol{a}}^{\perp}$, $\vec{\boldsymbol{b}}^{/\!/}$, $\vec{\boldsymbol{b}}^{\perp}$ for which

$$\vec{\boldsymbol{a}} = \vec{\boldsymbol{a}}^{/\!/} + \vec{\boldsymbol{a}}^{\perp}, \text{ with } a^{/\!/} /\!/ \vec{\boldsymbol{b}}, a^{\perp} \perp \vec{\boldsymbol{b}},$$

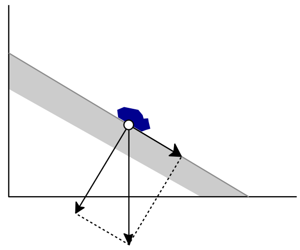
and

$$\vec{\boldsymbol{b}} = \vec{\boldsymbol{b}}^{/\!/} + \vec{\boldsymbol{b}}^{\perp}, \text{ with } b^{/\!/} /\!/ \vec{\boldsymbol{a}}, b^{\perp} \perp \vec{\boldsymbol{a}}.$$

†409

952. Emily left her backpack on a hill, which in some coordinate system happens to be the line with equation $12x_1 + 5x_2 = 130$.

The force exerted by gravity on the backpack is $\vec{\boldsymbol{f}}_{\text{grav}} = \begin{pmatrix} 0 \\ -mg \end{pmatrix}$. Decompose this force into a part perpendicular to the hill, and a part parallel to the hill.



†409

953. An eraser is lying on the plane $\mathcal{P}$ with equation $x_1 + 3x_2 + x_3 = 6$. Gravity pulls the eraser down, and exerts a force given by

$$\vec{\boldsymbol{f}}_{\text{grav}} = \begin{pmatrix} 0 \\ 0 \\ -mg \end{pmatrix}.$$

(a) Find a normal $\vec{\boldsymbol{n}}$ for the plane $\mathcal{P}$.

(b) Decompose the force $\vec{\boldsymbol{f}}$ into a part perpendicular to the plane $\mathcal{P}$ and a part perpendicular to $\vec{\boldsymbol{n}}$.

THE DOT PRODUCT

954. (i) Simplify $\|\vec{\boldsymbol{a}} - \vec{\boldsymbol{b}}\|^2$.

(ii) Simplify $\|2\vec{\boldsymbol{a}} - \vec{\boldsymbol{b}}\|^2$.

(iii) If $\vec{\boldsymbol{a}}$ has length 3, $\vec{\boldsymbol{b}}$ has length 7 and $\vec{\boldsymbol{a}}\cdot\vec{\boldsymbol{b}} = -2$, then compute $\|\vec{\boldsymbol{a}} + \vec{\boldsymbol{b}}\|$, $\|\vec{\boldsymbol{a}} - \vec{\boldsymbol{b}}\|$ and $\|2\vec{\boldsymbol{a}} - \vec{\boldsymbol{b}}\|$. †409

955. Simplify $(\vec{\boldsymbol{a}} + \vec{\boldsymbol{b}})\cdot(\vec{\boldsymbol{a}} - \vec{\boldsymbol{b}})$.

956. Find the lengths of the sides, and the angles in the triangle ABC whose vertices are $A(2,1)$, $B(3,2)$, and $C(1,4)$.

†409

957. Given: $A(1,1)$, $B(3,2)$ and a point C which lies on the line with parametric equation $\vec{\boldsymbol{c}} = \begin{pmatrix} 0 \\ 3 \end{pmatrix} + t\begin{pmatrix} 1 \\ -1 \end{pmatrix}$. If $\triangle ABC$ is a right triangle, then where is C? (There are three possible answers, depending on whether you assume A, B or C is the right angle.) †409

958. (***i***) Find the defining equation and a normal vector $\vec{\boldsymbol{n}}$ for the line ℓ which is the graph of $y = 1 + \frac{1}{2}x$. †410

(***ii***) What is the distance from the origin to ℓ? †410

(***iii***) Answer the same two questions for the line m which is the graph of $y = 2 - 3x$. †410

(***iv***) What is the angle between ℓ and m? †410

959. Let ℓ and m be the lines with parametrizations

$$\ell : \vec{\boldsymbol{x}} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} + t\begin{pmatrix} 1 \\ 2 \end{pmatrix},$$
$$m : \vec{\boldsymbol{x}} = \begin{pmatrix} 0 \\ -1 \end{pmatrix} + s\begin{pmatrix} -2 \\ 3 \end{pmatrix}$$

Where do they intersect, and find the angle between ℓ and m.

960. Let ℓ and m be the lines with parametrizations

$$\ell : \vec{\boldsymbol{x}} = \begin{pmatrix} 2 \\ 1 \\ -4 \end{pmatrix} + t \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix},$$
$$m : \vec{\boldsymbol{x}} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} + s \begin{pmatrix} -2 \\ 0 \\ 3 \end{pmatrix}$$

Do ℓ and m intersect? Find the angle between ℓ and m.

961. Let ℓ and m be the lines with parametrizations

$$\ell : \vec{\boldsymbol{x}} = \begin{pmatrix} 2 \\ \alpha \\ 1 \end{pmatrix} + t \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix},$$
$$m : \vec{\boldsymbol{x}} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} + s \begin{pmatrix} -2 \\ 0 \\ 3 \end{pmatrix}$$

Here α is some unknown number.

If it is known that the lines ℓ and m intersect, what can you say about α?

THE CROSS PRODUCT

962. Compute the following cross products

(*i*) $\begin{pmatrix}3\\1\\2\end{pmatrix} \times \begin{pmatrix}3\\2\\1\end{pmatrix}$

(*ii*) $\begin{pmatrix}12\\-71\\3\frac{1}{2}\end{pmatrix} \times \begin{pmatrix}12\\-71\\3\frac{1}{2}\end{pmatrix}$

(*iii*) $\begin{pmatrix}1\\0\\0\end{pmatrix} \times \begin{pmatrix}1\\1\\0\end{pmatrix}$

(*iv*) $\begin{pmatrix}\sqrt{2}\\1\\0\end{pmatrix} \times \begin{pmatrix}0\\\sqrt{2}\\0\end{pmatrix}$

963. Compute the following cross products

(*i*) $\vec{i} \times (\vec{i} + \vec{j})$

(*ii*) $(\sqrt{2}\vec{i} + \vec{j}) \times \sqrt{2}\vec{j}$

(*iii*) $(2\vec{i} + \vec{k}) \times (\vec{j} - \vec{k})$

(*iv*) $(\cos\theta\vec{i} + \sin\theta\vec{k}) \times (\sin\theta\vec{i} - \cos\theta\vec{k})$

964. (*i*) Simplify $(\vec{a} + \vec{b}) \times (\vec{a} + \vec{b})$. †410

(*ii*) Simplify $(\vec{a} - \vec{b}) \times (\vec{a} - \vec{b})$.

(*iii*) Simplify $(\vec{a} + \vec{b}) \times (\vec{a} - \vec{b})$. †410

965. *True or False:* If $\vec{a} \times \vec{b} = \vec{c} \times \vec{b}$ and $\vec{b} \neq \vec{0}$ then $\vec{a} = \vec{c}$? †410

966. Given $A(2,0,0)$, $B(0,0,2)$ and $C(2,2,2)$. Let $\mathcal{P}$ be the plane through A, B and C.

(*i*) Find a normal vector for $\mathcal{P}$. †411

(*ii*) Find a defining equation for $\mathcal{P}$. †411

(*iii*) What is the distance from $D(0,2,0)$ to $\mathcal{P}$? What is the distance from the origin $O(0,0,0)$ to $\mathcal{P}$? †411

(*iv*) Do D and O lie on the same side of $\mathcal{P}$? †411

(*v*) Find the area of the triangle ABC. †411

(*vi*) Where does the plane $\mathcal{P}$ intersect the three coordinate axes? †411

967. (*i*) Does $D(2,1,3)$ lie on the plane $\mathcal{P}$ through the points $A(-1,0,0)$, $B(0,2,1)$ and $C(0,3,0)$? †411

(*ii*) The point $E(1,1,\alpha)$ lies on $\mathcal{P}$. What is α? †411

968. Given points $A(1,-1,1)$, $B(2,0,1)$ and $C(1,2,0)$.

(*i*) Where is the point D which makes $ABCD$ into a parallelogram? †411

(*ii*) What is the area of the parallelogram $ABCD$? †411

(*iii*) Find a defining equation for the plane $\mathcal{P}$ containing the parallelogram $ABCD$. †411

(*iv*) Where does $\mathcal{P}$ intersect the coordinate axes? †411

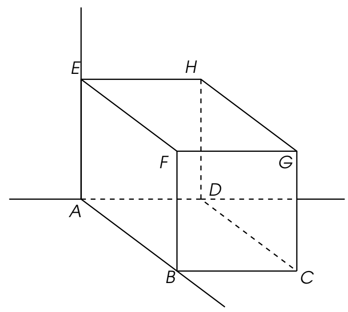
969. Given points $A(1,0,0)$, $B(0,2,0)$ and $D(-1,0,1)$ and $E(0,0,2)$.

(*i*) If $\mathfrak{P} = {}^{ABCD}_{EFGH}$ is a parallelepiped, then where are the points C, F, G and H? †412

(*ii*) Find the area of the base $ABCD$ of $\mathfrak{P}$. †412

(*iii*) Find the height of $\mathfrak{P}$. †412

(*iv*) Find the volume of $\mathfrak{P}$. †412



970. Let ${}^{ABCD}_{EFGH}$ be the cube with A at the origin, $B(1,0,0)$, $D(0,1,0)$ and $E(0,0,1)$.

(*i*) Find the coordinates of all the points A, B, C, D, E, F, G, H.

(*ii*) Find the position vectors of the midpoints of the line segments AG, BH, CE

and DF. Make a drawing of the cube with these line segments.

(***iii***) Find the defining equation for the plane BDE. Do the same for the plane CFH. Show that these planes are parallel.

(***iv***) Find the parametric equation for the line through AG.

(***v***) Where do the planes BDE and CFH intersect the line AG?

(***vi***) Find the angle between the planes BDE and BGH.

(***vii***) Find the angle between the planes BDE and BCH. Draw these planes.

Chapter 15

Vector Functions and Parametrized Curves

15.1 Parametric Curves

Definition 15.1.1. A vector function $\vec{\boldsymbol{f}}$ of one variable is a function of one real variable, whose values $\vec{\boldsymbol{f}}(t)$ are vectors.

In other words for any value of t (from a ***domain*** of allowed values, usually an interval) the vector function $\vec{\boldsymbol{f}}$ produces a vector $\vec{\boldsymbol{f}}(t)$. Write $\vec{\boldsymbol{f}}$ in components:

$$\vec{\boldsymbol{f}}(t) = \begin{pmatrix} f_1(t) \\ f_2(t) \end{pmatrix}.$$

The components of a vector function $\vec{\boldsymbol{f}}$ of t are themselves functions of t. They are ordinary functions of a single variable. An example of a vector function is

$$\vec{\boldsymbol{f}}(t) = \begin{pmatrix} t - 2t^2 \\ 1 + \cos^2 \pi t \end{pmatrix}, \quad \text{so } \vec{\boldsymbol{f}}(1) = \begin{pmatrix} 1 - 2(1)^2 \\ 1 + (\cos\pi)^2 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$$

(just to mention one.)

Definition 15.1.2. A ***parametric curve*** is a vector function $\vec{\boldsymbol{x}} = \vec{\boldsymbol{x}}(t)$ of one real variable t. The variable t is called the ***parameter.***

Synonyms: "Parametrized curve," or "parametrization," or "vector function (of one variable)."

Logically speaking a parametrized curve is the same thing as a vector function. The name "parametrized curve" is used to remind you of a very natural and common interpretation of the concept "parametric curve." In this interpretation a vector function, or parametric curve $\vec{\boldsymbol{x}}(t)$ describes the motion of a point in the plane or space. Here t stands for time, and $\vec{\boldsymbol{x}}(t)$ is the position vector at time t of the moving point.

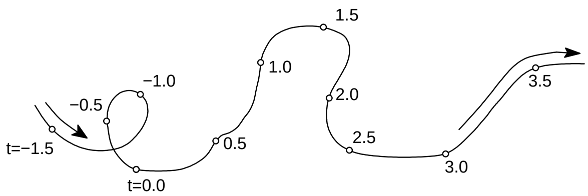


Figure 15.1: A picture of a vector function.

Instead of writing a parametrized curve as a vector function, one sometimes specifies the two (or three) components of the curve. Thus one will say that a parametric curve is given by

$$x_1 = x_1(t), \quad x_2 = x_2(t), \quad \text{(and } x_3 = x_3(t) \text{ if we have a space curve).}$$

15.1.1 Examples of parametrized curves

15.1.2 An example of Rectilinear Motion.

Here's a parametric curve:

$$\vec{\boldsymbol{x}}(t) = \begin{pmatrix} 1+t \\ 2+3t \end{pmatrix}. \tag{15.1}$$

The components of this vector function are

$$x_1(t) = 1 + t, \qquad x_2(t) = 2 + 3t. \tag{15.2}$$

Both components are linear functions of time (i.e. the parameter t), so every time t increases by an amount Δt (every time Δt seconds go by) the first component increases by Δt, and the x_2 component increases by $3\Delta t$. So the point at $\vec{\boldsymbol{x}}(t)$ moves horizontally to the left with speed 1, and it moves vertically upwards with speed 3.

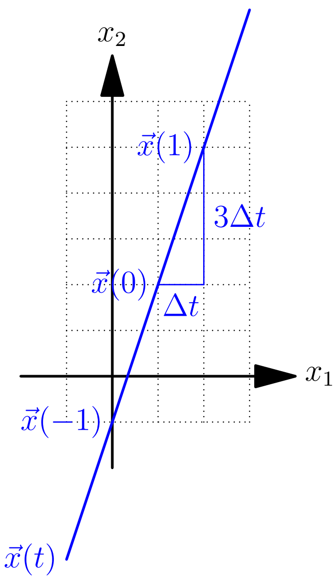


Figure 15.2: Rectilinear motion

Which curve is traced out by this vector function? In this example we can find out by eliminating the parameter, i.e. solve one of the two equations (15.2) for t, and substitute the value of t you find in the other equation. Here you can solve $x_1 = 1 + t$ for t, with result $t = x_1 - 1$. From there you find that

$$x_2 = 2 + 3t = 2 + 3(x_1 - 1) = 3x_1 - 1.$$

So for any t the vector $\vec{\boldsymbol{x}}(t)$ is the position vector of a point on the line $x_2 = 3x_1 - 1$ (or, if you prefer the old fashioned x, y coordinates, $y = 3x - 1$).

Conclusion: This particular parametric curve traces out a straight line with equation $x_2 = 3x_1 - 1$, going from left to right.

15.1.3 Rectilinear Motion in general.

This example generalizes the previous example. The parametric equation for a straight line from the previous chapter

$$\vec{\boldsymbol{x}}(t) = \vec{\boldsymbol{a}} + t\vec{\boldsymbol{v}},$$

is a parametric curve. We had $\vec{\boldsymbol{v}} = \vec{\boldsymbol{b}} - \vec{\boldsymbol{a}}$ in §14.2. At time $t = 0$ the object is at the point with position vector $\vec{\boldsymbol{a}}$, and every second (unit of time) the object translates by $\vec{\boldsymbol{v}}$. The vector $\vec{\boldsymbol{v}}$ is the ***velocity vector*** of this motion.

In the first example we had $\vec{\boldsymbol{a}} = \left(\begin{smallmatrix}1\\2\end{smallmatrix}\right)$, and $\vec{\boldsymbol{v}} = \left(\begin{smallmatrix}1\\3\end{smallmatrix}\right)$.

15.1.4 Going back and forth on a straight line.

Consider

$$\vec{\boldsymbol{x}}(t) = \vec{\boldsymbol{a}} + \sin(t)\vec{\boldsymbol{v}}.$$

At each moment in time the object whose motion is described by this parametric curve finds itself on the straight line ℓ with parametric equation $\vec{\boldsymbol{x}} = \vec{\boldsymbol{a}} + s(\vec{\boldsymbol{b}} - \vec{\boldsymbol{a}})$, where $\vec{\boldsymbol{b}} = \vec{\boldsymbol{a}} + \vec{\boldsymbol{v}}$.

However, instead of moving along the line from one end to the other, the point at $\vec{\boldsymbol{x}}(t)$ keeps moving back and forth along ℓ between $\vec{\boldsymbol{a}} + \vec{\boldsymbol{v}}$ and $\vec{\boldsymbol{a}} - \vec{\boldsymbol{v}}$.

15.1.5 Motion along a graph.

Let $y = f(x)$ be some function of one variable (defined for x in some interval) and consider the parametric curve given by

$$\vec{\boldsymbol{x}}(t) = \begin{pmatrix} t \\ f(t) \end{pmatrix} = t\vec{\boldsymbol{i}} + f(t)\vec{\boldsymbol{j}}.$$

At any moment in time the point at $\vec{\boldsymbol{x}}(t)$ has x_1 coordinate equal to t, and $x_2 = f(t) = f(x_1)$, since $x_1 = t$. So this parametric curve describes motion on the graph of $y = f(x)$ in which the horizontal coordinate increases at a constant rate.

15.1.6 The standard parametrization of a circle.

Consider the parametric curve

$$\vec{\boldsymbol{x}}(\theta) = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}.$$

The two components of this parametrization are

$$x_1(\theta) = \cos\theta, \quad x_2(\theta) = sin\theta,$$

and they satisfy

$$x_1(\theta)^2 + x_2(\theta)^2 = \cos^2\theta + \sin^2\theta = 1,$$

so that $\vec{\boldsymbol{x}}(\theta)$ always points at a point on the unit circle.

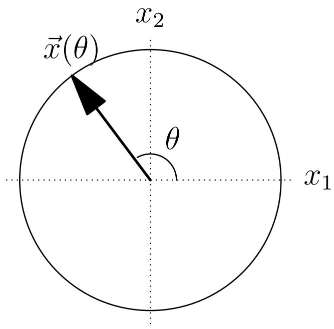


Figure 15.3: parametrization of a circle

As θ increases from $-\infty$ to $+\infty$ the point will rotate through the circle, going around infinitely often. Note that the point runs through the circle in the ***counterclockwise direction,*** which is the mathematician's favorite way of running around in circles.

15.1.7 The Cycloid.

The Free Ferris Wheel Foundation is an organization whose goal is to empower fairground ferris wheels to roam freely and thus realize their potential. With blatant disregard for the public, members of the F^2WF will clandestinely unhinge ferris wheels, thereby setting them free to roll throughout the fairground and surroundings.

Suppose we were to step into the bottom of a ferris wheel at the moment of its liberation: what would happen? Where would the wheel carry us? Let our position be the point X, and let its position vector at time t be $\vec{\boldsymbol{x}}(t)$. The parametric curve $\vec{\boldsymbol{x}}(t)$ which describes our motion is called the cycloid.

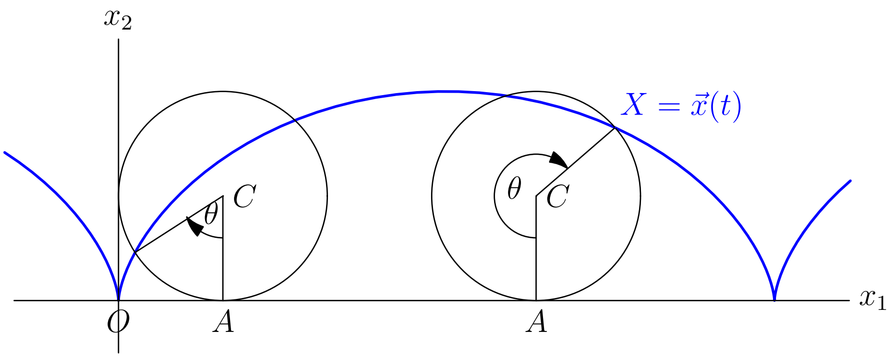


Figure 15.4: constructing a cycloid

In this example we are given a description of a motion, but no formula for the parametrization $\vec{\boldsymbol{x}}(t)$. We will have to derive this formula ourselves.

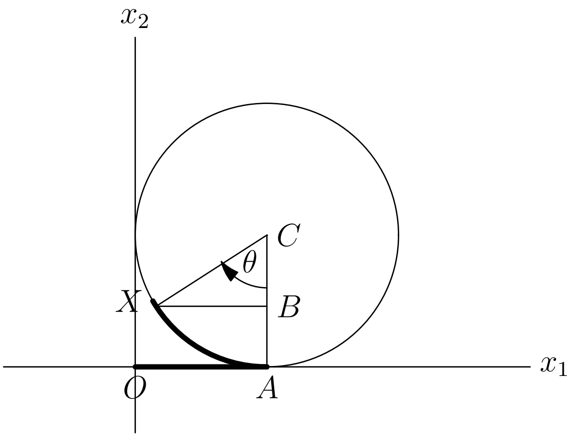


Figure 15.5: derivation of cycloid parameterization

The key to finding $\vec{\boldsymbol{x}}(t)$ is the fact that the arc AX on the wheel is exactly as long as the line segment OA on the ground (i.e. the x_1 axis). The length of the arc AX is exactly the angle θ ("arc = radius times angle in radians"), so the x_1 coordinate of A and hence the center C of the circle is θ. To find X consider the right triangle BCX. Its hypothenuse is the radius of the circle, i.e. CX has length 1. The angle at C is θ, and therefore you get

$$BX = \sin\theta, \qquad BC = \cos\theta,$$

and

$$x_1 = OA - BX = \theta - \sin\theta, \qquad x_2 = AC - BC = 1 - \cos\theta.$$

So the parametric curve defined in the beginning of this example is

$$\vec{\boldsymbol{x}}(\theta) = \begin{pmatrix} \theta - \sin\theta \\ 1 - \cos\theta \end{pmatrix}.$$

Here the angle θ is the parameter, and we can let it run from $\theta = -\infty$ to $\theta = \infty$.

15.1.8 A three dimensional example: the Helix.

Consider the vector function

$$\vec{\boldsymbol{x}}(\theta) = \begin{pmatrix} \cos\theta \\ \sin\theta \\ a\theta \end{pmatrix}$$

where $a > 0$ is some constant.

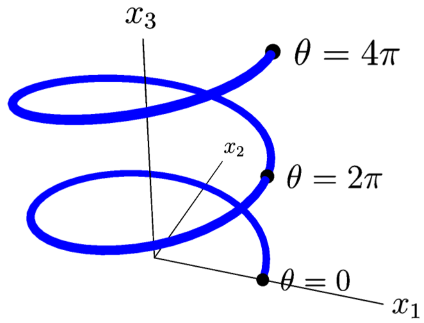


Figure 15.6: A Helix in three dimensions.

If you ignore the x_3 component of this vector function you get the parametrization of the circle from example 15.1.6. So as the parameter θ runs from $-\infty$ to $+\infty$, the x_1, x_2 part of $\vec{\boldsymbol{x}}(\theta)$ runs around on the unit circle infinitely often. While this happens the vertical component, i.e. $x_3(\theta)$ increases steadily from $-\infty$ to ∞ at a rate of a units per second.

15.2 The derivative of a vector function

If $\vec{\boldsymbol{x}}(t)$ is a vector function, then we define its ***derivative*** to be

$$\vec{\boldsymbol{x}}'(t) = \frac{\mathrm{d}\vec{\boldsymbol{x}}}{\mathrm{d}t} = \lim_{h\to 0} \frac{\vec{\boldsymbol{x}}(t+h) - \vec{\boldsymbol{x}}(t)}{h}.$$

This definition looks very much like the definition of the derivative of a function of a single variable, but for it to make sense in the context of vector functions we have to explain what the limit of a vector function is.

By definition, for a vector function $\vec{\boldsymbol{f}}(t) = \begin{pmatrix} f_1(t) \\ f_2(t) \end{pmatrix}$ one has

$$\lim_{t\to a} \vec{\boldsymbol{f}}(t) = \lim_{t\to a} \begin{pmatrix} f_1(t) \\ f_2(t) \end{pmatrix} = \begin{pmatrix} \lim_{t\to a} f_1(t) \\ \lim_{t\to a} f_2(t) \end{pmatrix}$$

In other words, to compute the limit of a vector function you just compute the limits of its components (that will be our definition.)

Let's look at the definition of the velocity vector again. Since

$$\begin{aligned} \frac{\vec{\boldsymbol{x}}(t+h) - \vec{\boldsymbol{x}}(t)}{h} &= \frac{1}{h}\left\{ \begin{pmatrix} x_1(t+h) \\ x_2(t+h) \end{pmatrix} - \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} \right\} \\ &= \begin{pmatrix} \dfrac{x_1(t+h) - x_1(t)}{h} \\ \dfrac{x_2(t+h) - x_2(t)}{h} \end{pmatrix} \end{aligned}$$

we have

$$\begin{aligned}\vec{\boldsymbol{x}}'(t) &= \lim_{h\to 0}\frac{\vec{\boldsymbol{x}}(t+h)-\vec{\boldsymbol{x}}(t)}{h}\\ &= \begin{pmatrix}\lim_{h\to 0}\dfrac{x_1(t+h)-x_1(t)}{h}\\ \lim_{h\to 0}\dfrac{x_2(t+h)-x_2(t)}{h}\end{pmatrix}\\ &= \begin{pmatrix}x_1'(t)\\ x_2'(t)\end{pmatrix}\end{aligned}$$

So: To compute the derivative of a vector function you must differentiate its components.

15.2.1 Example

Compute the derivative of

$$\vec{\boldsymbol{x}}(t) = \begin{pmatrix}\cos t\\ \sin t\end{pmatrix} \quad \text{and of} \quad \vec{\boldsymbol{y}}(t) = \begin{pmatrix}t-\sin t\\ 1-\cos t\end{pmatrix}.$$

Solution:

$$\begin{aligned}\vec{\boldsymbol{x}}'(t) &= \frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}\cos t\\ \sin t\end{pmatrix} = \begin{pmatrix}-\sin t\\ \cos t\end{pmatrix}\\ \vec{\boldsymbol{y}}'(t) &= \frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}t-\sin t\\ 1-\cos t\end{pmatrix} = \begin{pmatrix}1-\cos t\\ \sin t\end{pmatrix}.\end{aligned}$$

15.3 Higher derivatives and product rules

If you differentiate a vector function $\vec{\boldsymbol{x}}(t)$ you get another vector function, namely $\vec{\boldsymbol{x}}'(t)$, and you can try to differentiate that vector function again. If you succeed, the result is called the second derivative of $\vec{\boldsymbol{x}}(t)$. All this is very similar to how the second (and higher) derivative of ordinary functions. One even uses the same notation:[1]

$$\vec{\boldsymbol{x}}''(t) = \frac{\mathrm{d}\vec{\boldsymbol{x}}'(t)}{\mathrm{d}t} = \frac{\mathrm{d}^2\vec{\boldsymbol{x}}}{\mathrm{d}t^2} = \begin{pmatrix}x_1''(t)\\ x_2''(t)\end{pmatrix}.$$

15.3.1 Example

Compute the second derivative of

$$\vec{\boldsymbol{x}}(t) = \begin{pmatrix}\cos t\\ \sin t\end{pmatrix} \quad \text{and of} \quad \vec{\boldsymbol{y}}(t) = \begin{pmatrix}t-\sin t\\ 1-\cos t\end{pmatrix}.$$

Solution: In example 15.2.1 we already found the first derivatives, so you can use those. You find

$$\begin{aligned}\vec{\boldsymbol{x}}''(t) &= \frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}-\sin t\\ \cos t\end{pmatrix} = \begin{pmatrix}-\cos t\\ -\sin t\end{pmatrix}\\ \vec{\boldsymbol{y}}''(t) &= \frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}1-\cos t\\ \sin t\end{pmatrix} = \begin{pmatrix}\sin t\\ \cos t\end{pmatrix}.\end{aligned}$$

[1] Not every function has a derivative, so it may happen that you can find $\vec{\boldsymbol{x}}'(t)$ but not $\vec{\boldsymbol{x}}''(t)$

Note that our standard parametrization $\vec{\boldsymbol{x}}(t)$ of the circle satisfies

$$\vec{\boldsymbol{x}}''(t) = -\vec{\boldsymbol{x}}(t).$$

After defining the derivative for ordinary function of a single variable one quickly introduces the various rules (sum, product, quotient, chain rules) which make it possible to compute derivatives without ever actually having to use the limit-of-difference-quotient-definition. For vector functions there are similar rules which also turn out to be useful.

The ***Sum Rule*** holds. It says that if $\vec{\boldsymbol{x}}(t)$ and $\vec{\boldsymbol{y}}(t)$ are differentiable[2] vector functions, then so is $\vec{\boldsymbol{z}}(t) = \vec{\boldsymbol{x}}(t) \pm \vec{\boldsymbol{y}}(t)$, and one has

$$\frac{\mathrm{d}\vec{\boldsymbol{x}}(t) \pm \vec{\boldsymbol{y}}(t)}{\mathrm{d}t} = \frac{\mathrm{d}\vec{\boldsymbol{x}}(t)}{\mathrm{d}t} \pm \frac{\mathrm{d}\vec{\boldsymbol{y}}(t)}{\mathrm{d}t}.$$

The ***Product Rule*** also holds, but it is more complicated, because there are several different forms of multiplication when you have vector functions. The following three versions all hold:

If $\vec{\boldsymbol{x}}(t)$ and $\vec{\boldsymbol{y}}(t)$ are differentiable vector functions and $f(t)$ is an ordinary differentiable function, then

$$\begin{aligned}
\frac{\mathrm{d}f(t)\vec{\boldsymbol{x}}(t)}{\mathrm{d}t} &= f(t)\frac{\mathrm{d}\vec{\boldsymbol{x}}(t)}{\mathrm{d}t} + \frac{\mathrm{d}f(t)}{\mathrm{d}t}\vec{\boldsymbol{x}}(t) \\
\frac{\mathrm{d}\vec{\boldsymbol{x}}(t)\cdot\vec{\boldsymbol{y}}(t)}{\mathrm{d}t} &= \vec{\boldsymbol{x}}(t)\cdot\frac{\mathrm{d}\vec{\boldsymbol{y}}(t)}{\mathrm{d}t} + \frac{\mathrm{d}\vec{\boldsymbol{x}}(t)}{\mathrm{d}t}\cdot\vec{\boldsymbol{y}}(t) \\
\frac{\mathrm{d}\vec{\boldsymbol{x}}(t) \times \vec{\boldsymbol{y}}(t)}{\mathrm{d}t} &= \vec{\boldsymbol{x}}(t) \times \frac{\mathrm{d}\vec{\boldsymbol{y}}(t)}{\mathrm{d}t} + \frac{\mathrm{d}\vec{\boldsymbol{x}}(t)}{\mathrm{d}t} \times \vec{\boldsymbol{y}}(t)
\end{aligned}$$

I hope these formulae look plausible because they look like the old fashioned product rule, but even if they do, you still have to prove them before you can accept their validity. I will prove one of these in lecture. You will do some more as an exercise.

As an example of how these properties get used, consider this theorem:

Theorem 15.3.1. Let $\vec{\boldsymbol{f}}(t)$ be a vector function of constant length (i.e. $\|\vec{\boldsymbol{f}}(t)\|$ is constant.) Then $\vec{\boldsymbol{f}}'(t) \perp \vec{\boldsymbol{f}}(t)$.

Proof. If $\|\vec{\boldsymbol{f}}\|$ is constant, then so is $\vec{\boldsymbol{f}}(t)\cdot\vec{\boldsymbol{f}}(t) = \|\vec{\boldsymbol{f}}(t)\|^2$. the derivative of a constant function is zero, so

$$0 = \frac{\mathrm{d}}{\mathrm{d}t}\left(\|\vec{\boldsymbol{f}}(t)\|^2\right) = \frac{\mathrm{d}}{\mathrm{d}t}\left(\|\vec{\boldsymbol{f}}(t)\| \cdot \|\vec{\boldsymbol{f}}(t)\|\right) = 2\vec{\boldsymbol{f}}(t)\cdot\frac{\mathrm{d}\vec{\boldsymbol{f}}(t)}{\mathrm{d}t}.$$

So we see that $\vec{\boldsymbol{f}}\cdot\vec{\boldsymbol{f}}' = 0$ which means that $\vec{\boldsymbol{f}}' \perp \vec{\boldsymbol{f}}$. □

15.4 Interpretation of the velocity vector

Let $\vec{\boldsymbol{x}}(t)$ be some vector function and interpret it as describing the motion of some point in the plane (or space). At time t the point has position vector $\vec{\boldsymbol{x}}(t)$; a little later, more precisely, h seconds later the point has position vector $\vec{\boldsymbol{x}}(t+h)$. Its displacement is the difference vector

$$\vec{\boldsymbol{x}}(t+h) - \vec{\boldsymbol{x}}(t).$$

[2] A vector function is differentiable if its derivative actually exists, i.e. if *all* its components are differentiable.

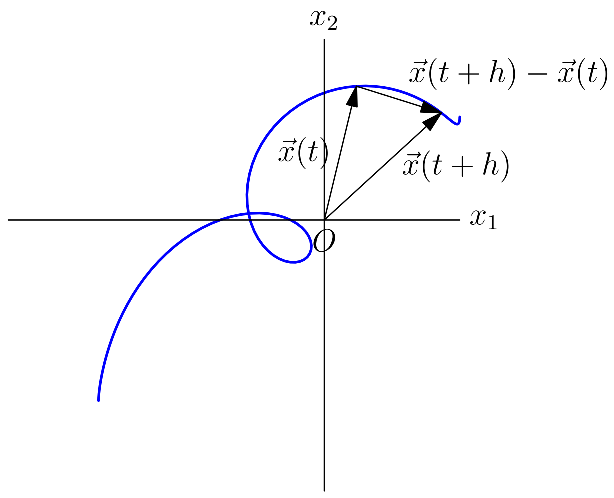


Figure 15.7: The vector velocity of a motion in the plane

Its average velocity vector between times t and $t+h$ is

$$\frac{\text{displacement vector}}{\text{time lapse}} = \frac{\vec{\boldsymbol{x}}(t+h)-\vec{\boldsymbol{x}}(t)}{h}.$$

If the average velocity between times t and $t+h$ converges to one definite vector as $h \to 0$, then this limit is a reasonable candidate for ***the velocity vector at time*** t of the parametric curve $\vec{\boldsymbol{x}}(t)$.

Being a vector, the velocity vector has both *magnitude* and *direction*. The length of the velocity vector is called the ***speed*** of the parametric curve. We use the following notation: we always write

$$\vec{\boldsymbol{v}}(t) = \vec{\boldsymbol{x}}'(t)$$

for the velocity vector, and

$$v(t) = \|\vec{\boldsymbol{v}}(t)\| = \|\vec{\boldsymbol{x}}'(t)\|$$

for its length, i.e. the speed.

The speed v is always a nonnegative number; the velocity is always a vector.

15.4.1 Velocity of linear motion.

If $\vec{\boldsymbol{x}}(t) = \vec{\boldsymbol{a}} + t\vec{\boldsymbol{v}}$, as in examples 15.1.2 and 15.1.3, then

$$\vec{\boldsymbol{x}}(t) = \begin{pmatrix} a_1 + tv_1 \\ a_2 + tv_2 \end{pmatrix}$$

so that

$$\vec{\boldsymbol{x}}'(t) = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \vec{\boldsymbol{v}}.$$

So when you represent a line by a parametric equation $\vec{\boldsymbol{x}}(t) = \vec{\boldsymbol{a}} + t\vec{\boldsymbol{v}}$, the vector $\vec{\boldsymbol{v}}$ is the velocity vector. The length of $\vec{\boldsymbol{v}}$ is the speed of the motion.

In example 15.1.2 we had $\vec{\boldsymbol{v}} = \left(\begin{smallmatrix} 1 \\ 3 \end{smallmatrix}\right)$, so the speed with which the point at $\vec{\boldsymbol{x}}(t) = \left(\begin{smallmatrix} 1+t \\ 1+3t \end{smallmatrix}\right)$ traces out the line is $v = \|\vec{\boldsymbol{v}}\| = \sqrt{1^2+3^2} = \sqrt{10}$.

15.4.2 Motion on a circle.

Consider the parametrization

$$\vec{\boldsymbol{x}}(t) = \begin{pmatrix} R\cos\omega t \\ R\sin\omega t \end{pmatrix}.$$

The point X at $\vec{\boldsymbol{x}}(t)$ is on the circle centered at the origin with radius R. The segment from the origin to X makes an angle ωt with the x_1-axis; this angle clearly increases at a constant rate of ω radians per second.

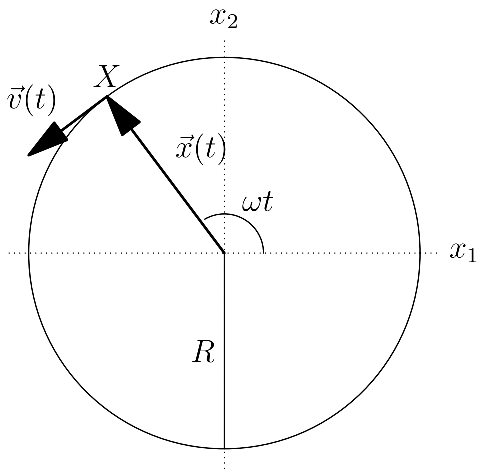


Figure 15.8: motion on a circle

The velocity vector of this motion is

$$\vec{\boldsymbol{v}}(t) = \vec{\boldsymbol{x}}'(t) = \begin{pmatrix} -\omega R\sin\omega t \\ \omega R\cos\omega t \end{pmatrix} = \omega R \begin{pmatrix} -\sin\omega t \\ \cos\omega t \end{pmatrix}.$$

This vector is not constant. however, if you calculate the speed of the point X, you find

$$v = \|\vec{\boldsymbol{v}}(t)\| = \omega R \left\| \begin{pmatrix} \sin\omega t \\ \cos\omega t \end{pmatrix} \right\| = \omega R.$$

So while the direction of the velocity vector $\vec{\boldsymbol{v}}(t)$ is changing all the time, its magnitude is constant. In this parametrization the point X moves along the circle with constant speed $v = \omega R$.

15.4.3 Velocity of the cycloid.

Think of the dot X on the wheel in the cycloid example 15.1.7. We know its position vector and velocity at time t

$$\vec{\boldsymbol{x}}(t) = \begin{pmatrix} t - \sin t \\ 1 - \cos t \end{pmatrix}, \qquad \vec{\boldsymbol{x}}'(t) = \begin{pmatrix} 1 - \cos t \\ \sin t \end{pmatrix}.$$

The speed with which X traces out the cycloid is

$$\begin{aligned} v &= \|\vec{\boldsymbol{x}}'(t)\| \\ &= \sqrt{(1-\cos t)^2 + (\sin t)^2} \\ &= \sqrt{1 - 2\cos t + \cos^2 t + \sin^2 t} \\ &= \sqrt{2(1-\cos t)}. \end{aligned}$$

You can use the double angle formula $\cos 2\alpha = 1 - 2\sin^2\alpha$ with $\alpha = \frac{t}{2}$ to simplify this to

$$v = \sqrt{4\sin^2 \tfrac{t}{2}} = 2\left|\sin\frac{t}{2}\right|.$$

The speed of the point X on the cycloid is therefore always between 0 and 2. At times $t = 0$ and other multiples of 2π we have $\vec{\boldsymbol{x}}'(t) = \vec{\boldsymbol{0}}$. At these times the point X has come to a stop. At times $t = \pi + 2k\pi$ one has $v = 2$ and $\vec{\boldsymbol{x}}'(t) = \left(\begin{smallmatrix}2\\0\end{smallmatrix}\right)$, i.e. the point X is moving horizontally to the right with speed 2.

15.5 Acceleration and Force

Just as the derivative $\vec{\boldsymbol{x}}'(t)$ of a parametric curve can be interpreted as the velocity vector $\vec{\boldsymbol{v}}(t)$, the derivative of the velocity vector measures the rate of change with time of the velocity and is called the ***acceleration*** of the motion. The usual notation is

$$\vec{\boldsymbol{a}}(t) = \vec{\boldsymbol{v}}'(t) = \frac{\mathrm{d}\vec{\boldsymbol{v}}(t)}{\mathrm{d}t} = \frac{\mathrm{d}^2\vec{\boldsymbol{x}}}{\mathrm{d}t^2} = \vec{\boldsymbol{x}}''(t).$$

Sir Isaac Newton's law relating force and acceleration via the formula "$F = ma$" has a vector version. *If an object's motion is given by a parametrized curve $\vec{\boldsymbol{x}}(t)$ then this motion is the result of a force $\vec{\boldsymbol{F}}$ being exerted on the object. The force $\vec{\boldsymbol{F}}$ is given by*

$$\vec{\boldsymbol{F}} = m\vec{\boldsymbol{a}} = m\frac{\mathrm{d}^2\vec{\boldsymbol{x}}}{\mathrm{d}t^2}$$

where m is the mass of the object.

Somehow it is always assumed that the mass m is a positive number.

15.5.1 How does an object move if no forces act on it?

If $\vec{\boldsymbol{F}}(t) = \vec{\boldsymbol{0}}$ at all times, then, assuming $m \neq 0$ it follows from $\vec{\boldsymbol{F}} = m\vec{\boldsymbol{a}}$ that $\vec{\boldsymbol{a}}(t) = \vec{\boldsymbol{0}}$. Since $\vec{\boldsymbol{a}}(t) = \vec{\boldsymbol{v}}'(t)$ you conclude that the velocity vector $\vec{\boldsymbol{v}}(t)$ must be constant, i.e. that there is some fixed vector $\vec{\boldsymbol{v}}$ such that

$$\vec{\boldsymbol{x}}'(t) = \vec{\boldsymbol{v}}(t) = \vec{\boldsymbol{v}} \text{ for all } t.$$

This implies that

$$\vec{\boldsymbol{x}}(t) = \vec{\boldsymbol{x}}(0) + t\vec{\boldsymbol{v}}.$$

So if no force acts on an object, then it will move with constant velocity vector along a straight line (said Newton – Archimedes long before him thought that the object would slow down and come to a complete stop unless there were a force to keep it going.)

15.5.2 Compute the forces acting on a point on a circle.

Consider an object moving with constant angular velocity ω on a circle of radius R, i.e. consider $\vec{\boldsymbol{x}}(t)$ as in example 15.4.2,

$$\vec{\boldsymbol{x}}(t) = \begin{pmatrix} R\cos\omega t \\ R\sin\omega t \end{pmatrix} = R\begin{pmatrix} \cos\omega t \\ \sin\omega t \end{pmatrix}.$$

Then its velocity and acceleration vectors are

$$\vec{\boldsymbol{v}}(t) = \omega R\begin{pmatrix} -\sin\omega t \\ \cos\omega t \end{pmatrix}$$

and

$$\begin{aligned} \vec{\boldsymbol{a}}(t) = \vec{\boldsymbol{v}}'(t) &= \omega^2 R\begin{pmatrix} -\cos\omega t \\ -\sin\omega t \end{pmatrix} \\ &= -\omega^2 R\begin{pmatrix} \cos\omega t \\ \sin\omega t \end{pmatrix} \end{aligned}$$

Since both $\left(\begin{smallmatrix} \cos\theta \\ \sin\theta \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} -\sin\theta \\ \cos\theta \end{smallmatrix}\right)$ are unit vectors, we see that the velocity vector changes its direction but not its size: at all times you have $v = \|\vec{\boldsymbol{v}}\| = \omega R$. The acceleration also keeps changing its direction, but its magnitude is always

$$a = \|\vec{\boldsymbol{a}}\| = \omega^2 R = \left(\frac{v}{R}\right)^2 R = \frac{v^2}{R}.$$

The force which must be acting on the object to make it go through this motion is

$$\vec{\boldsymbol{F}} = m\vec{\boldsymbol{a}} = -m\omega^2 R\begin{pmatrix} \cos\omega t \\ \sin\omega t \end{pmatrix}.$$

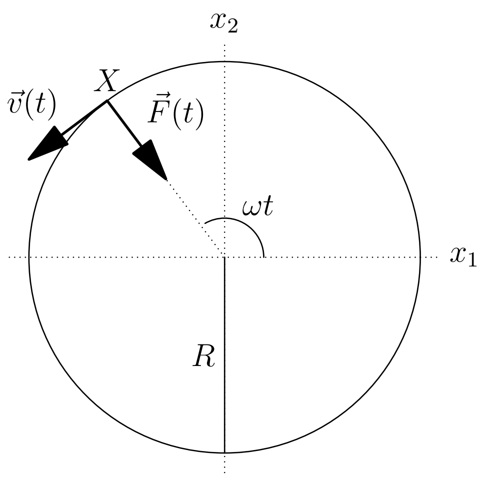


Figure 15.9: force driving motion in a circle

To conclude this example note that you can write this force as

$$\vec{\boldsymbol{F}} = -m\omega^2\vec{\boldsymbol{x}}(t)$$

which tells you which way the force is directed: towards the center of the circle.

15.5.3 How does it feel, to be on the Ferris wheel?

In other words, which force acts on us if we get carried away by a "liberated ferris wheel," as in example 15.1.7?

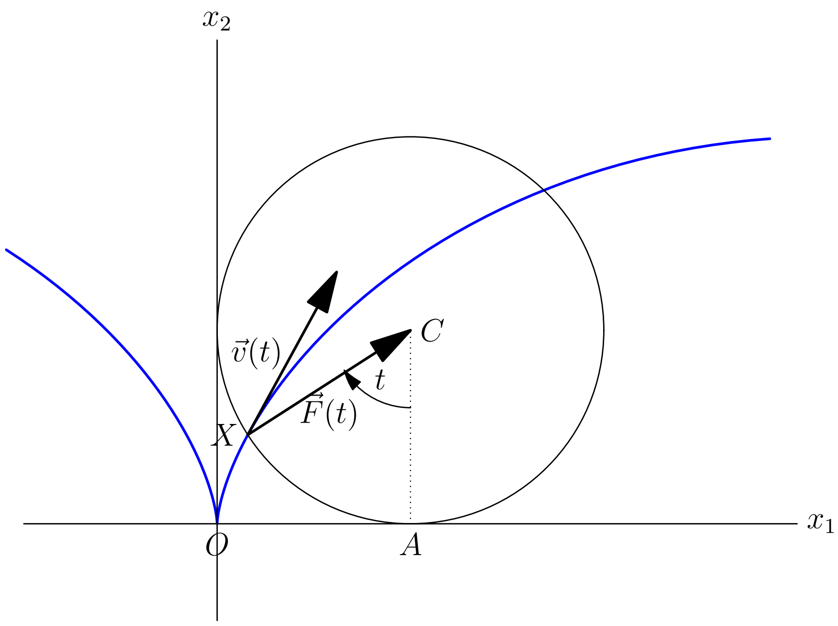


Figure 15.10: force for cycloid motion

Well, you get pushed around by a force $\vec{\boldsymbol{F}}$, which according to Newton is given by $\vec{\boldsymbol{F}} = m\vec{\boldsymbol{a}}$, where m is your mass and $\vec{\boldsymbol{a}}$ is your acceleration, which we now compute:

$$\begin{aligned}\vec{\boldsymbol{a}}(t) &= \vec{\boldsymbol{v}}'(t)\\ &= \frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}1-\cos t\\ \sin t\end{pmatrix}\\ &= \begin{pmatrix}\sin t\\ \cos t\end{pmatrix}.\end{aligned}$$

This is a unit vector: the force that's pushing you around is constantly changing its direction but its strength stays the same. If you remember that t is the angle $\angle ACX$ you see that the force $\vec{\boldsymbol{F}}$ is always pointed at the center of the wheel: its direction is given by the vector $\overrightarrow{XC}$.

15.6 Tangents and the unit tangent vector

Here we address the problem of finding the tangent line at a point on a parametric curve.

Let $\vec{\boldsymbol{x}}(t)$ be a parametric curve, and let's try to find the tangent line at a particular point X_0, with position vector $\vec{\boldsymbol{x}}(t_0)$ on this curve. We follow the same strategy as in the calculus: pick a point X_h on the curve near X_0, draw the line through X_0 and X_h and let $X_h \to X_0$.

The line through two points on a curve is often called a ***secant*** to the curve. So we are going to construct a tangent to the curve as a limit of secants.

The point X_0 has position vector $\vec{\boldsymbol{x}}(t_0)$, the point X_h is at $\vec{\boldsymbol{x}}(t_0+h)$. Consider the line ℓ_h parametrized by

$$\vec{\boldsymbol{y}}(s;h) = \vec{\boldsymbol{x}}(t_0) + s\frac{\vec{\boldsymbol{x}}(t_0+h) - \vec{\boldsymbol{x}}(t_0)}{h}, \tag{15.3}$$

in which s is the parameter we use to parametrize the line.

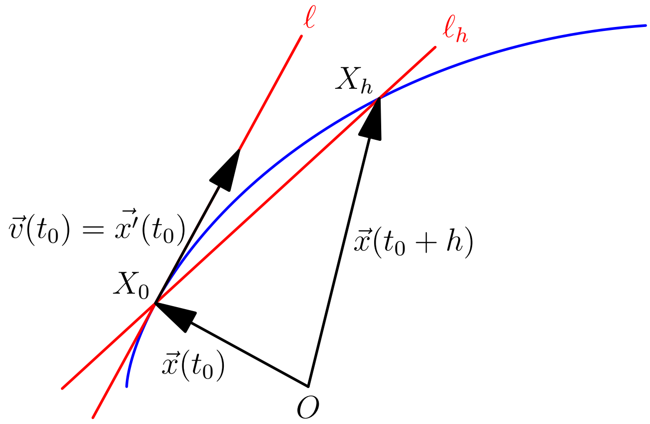


Figure 15.11: secant becomes tangent in the limit as $h \to 0$

The line ℓ_h contains both X_0 (set $s=0$) and X_h (set $s=h$), so it is the line through X_0 and X_h, i.e. a secant to the curve.

Now we let $h \to 0$, which gives

$$\vec{\boldsymbol{y}}(s) \stackrel{\text{def}}{=} \lim_{h\to 0} \vec{\boldsymbol{y}}(s;h) = \vec{\boldsymbol{x}}(t_0) + s\lim_{h\to 0}\frac{\vec{\boldsymbol{x}}(t_0+h) - \vec{\boldsymbol{x}}(t_0)}{h} = \vec{\boldsymbol{x}}(t_0) + s\vec{\boldsymbol{x}}'(t_0),$$

In other words, the tangent line to the curve $\vec{\boldsymbol{x}}(t)$ at the point with position vector $\vec{\boldsymbol{x}}(t_0)$ has parametric equation

$$\vec{\boldsymbol{y}}(s) = \vec{\boldsymbol{x}}(t_0) + s\vec{\boldsymbol{x}}'(t_0),$$

and the vector $\vec{\boldsymbol{x}}'(t_0) = \vec{\boldsymbol{v}}(t_0)$ is parallel to the tangent line ℓ. Because of this one calls the vector $\vec{\boldsymbol{x}}'(t_0)$ a ***tangent vector*** to the curve. Any multiple $\lambda\vec{\boldsymbol{x}}'(t_0)$ with $\lambda \neq 0$ is still parallel to the tangent line ℓ and is therefore also called a tangent vector.

A tangent vector of length 1 is called a ***unit tangent vector.*** If $\vec{\boldsymbol{x}}'(t_0) \neq 0$ then there are exactly two unit tangent vectors. They are

$$\vec{\boldsymbol{T}}(t_0) = \pm\frac{\vec{\boldsymbol{v}}(t_0)}{\|\vec{\boldsymbol{v}}(t_0)\|} = \pm\frac{\vec{\boldsymbol{v}}(t_0)}{v(t_0)}.$$

15.6.1 Example

Find Tangent line, and unit tangent vector at $\vec{\boldsymbol{x}}(1)$, where $\vec{\boldsymbol{x}}(t)$ is the parametric curve given by

$$\vec{\boldsymbol{x}}(t) = \begin{pmatrix} t \\ t^2 \end{pmatrix}, \quad \text{so that } \vec{\boldsymbol{x}}'(t) = \begin{pmatrix} 1 \\ 2t \end{pmatrix}.$$

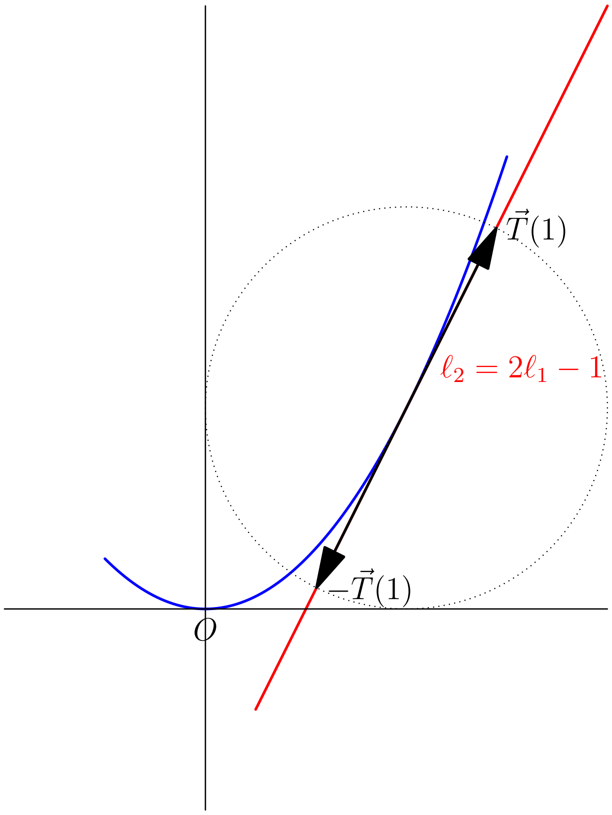


Figure 15.12: example: finding the tangent

Solution: For $t = 1$ we have $\vec{\boldsymbol{x}}'(1) = \binom{1}{2}$, so the tangent line has parametric equation

$$\vec{y}(s) = \vec{\boldsymbol{x}}(1) + s\vec{\boldsymbol{x}}'(1) = \begin{pmatrix}1\\1\end{pmatrix} + s\begin{pmatrix}1\\2\end{pmatrix} = \begin{pmatrix}1+s\\1+2s\end{pmatrix}.$$

In components one could write this as $y_1(s) = 1 + s$, $y_2(s) = 1 + 2s$. After eliminating s you find that on the tangent line one has

$$y_2 = 1 + 2s = 1 + 2(y_1 - 1) = 2y_1 - 1.$$

The vector $\vec{\boldsymbol{x}}'(1) = \binom{1}{2}$ is a tangent vector to the parabola at $\vec{\boldsymbol{x}}(1)$. To get a unit tangent vector we normalize this vector to have length one, i.e. we divide i by its length. Thus

$$\vec{\boldsymbol{T}}(1) = \frac{1}{\sqrt{1^2+2^2}}\begin{pmatrix}1\\2\end{pmatrix} = \begin{pmatrix}\frac{1}{5}\sqrt{5}\\\frac{2}{5}\sqrt{5}\end{pmatrix}$$

is a unit tangent vector. There is another unit tangent vector, namely

$$-\vec{\boldsymbol{T}}(1) = -\begin{pmatrix}\frac{1}{5}\sqrt{5}\\\frac{2}{5}\sqrt{5}\end{pmatrix}.$$

15.6.2 Tangent line and unit tangent vector to Circle.

In example 15.1.6 and 15.2.1 we had parametrized the circle and found the velocity vector of this parametrization,

$$\vec{\boldsymbol{x}}(\theta) = \begin{pmatrix}\cos\theta\\\sin\theta\end{pmatrix}, \qquad \vec{\boldsymbol{x}}'(\theta) = \begin{pmatrix}-\sin\theta\\\cos\theta\end{pmatrix}.$$

If we pick a particular value of θ then the tangent line to the circle at $\vec{\boldsymbol{x}}(\theta_0)$ has parametric equation

$$\vec{\boldsymbol{y}}(s) = \vec{\boldsymbol{x}}(\theta_0) + s\vec{\boldsymbol{x}}'(\theta_0) = \begin{pmatrix} \cos\theta + s\sin\theta \\ \sin\theta - s\cos\theta \end{pmatrix}$$

This equation completely describes the tangent line, but you can try to write it in a more familiar form as a graph

$$y_2 = my_1 + n.$$

To do this you have to eliminate the parameter s from the parametric equations

$$y_1 = \cos\theta + s\sin\theta, \qquad y_2 = \sin\theta - s\cos\theta.$$

When $\sin\theta \neq 0$ you can solve $y_1 = \cos\theta + s\sin\theta$ for s, with result

$$s = \frac{y_1 - \cos\theta}{\sin\theta}.$$

So on the tangent line you have

$$y_2 = \sin\theta - s\cos\theta = \sin\theta - \cos\theta\frac{y_1 - \cos\theta}{\sin\theta}$$

which after a little algebra (add fractions and use $\sin^2\theta + \cos^2\theta = 1$) turns out to be the same as

$$y_2 = -\cot\theta\, y_1 + \frac{1}{\sin\theta}.$$

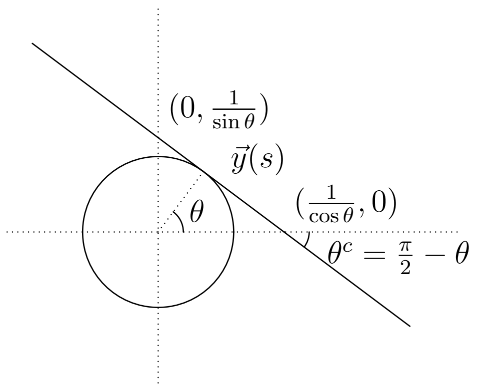


Figure 15.13: tangent to the unit circle at angle θ

The tangent line therefore hits the vertical axis when $y_1 = 0$, at height $n = 1/\sin\theta$, and it has slope $m = -\cot\theta$.

For this example you could have found the tangent line without using any calculus by studying the drawing above carefully.

Finally, let's find a unit tangent vector. A unit tangent is a multiple of $\vec{\boldsymbol{x}}'(\theta)$ whose length is one. But the vector $\vec{\boldsymbol{x}}'(\theta) = \left(\begin{smallmatrix} -\sin\theta \\ \cos\theta \end{smallmatrix}\right)$ already has length one, so the two possible unit vectors are

$$\vec{\boldsymbol{T}}(\theta) = \vec{\boldsymbol{x}}'(\theta) = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix} \quad \text{and} \quad -\vec{\boldsymbol{T}}(\theta) = \begin{pmatrix} \sin\theta \\ -\cos\theta \end{pmatrix}.$$

15.7 Sketching a parametric curve

For a given parametric curve, like

$$\vec{\boldsymbol{x}}(t) = \begin{pmatrix} 1 - t^2 \\ 3t - t^3 \end{pmatrix} \tag{15.4}$$

you might want to know what the curve looks like. The most straightforward way of getting a picture is to compute $x_1(t)$ and $x_2(t)$ for as many values of t as you feel like, and then plotting the computed points. This computation is the kind of repetitive task that computers are very good at, and there are many software packages and graphing calculators that will attempt to do the computation and drawing for you.

If the vector function has a constant whose value is not (completely) known, e.g. if we wanted to graph the parametric curve

$$\vec{\boldsymbol{x}}(t) = \begin{pmatrix} 1 - t^2 \\ 3at - t^3 \end{pmatrix} \qquad (a \text{ is a constant}) \tag{15.5}$$

then plugging parameter values and plotting the points becomes harder, since the unknown constant a shows up in the computed points.

On a graphing calculator you would have to choose different values of a and see what kind of pictures you get (you would expect different pictures for different values of a).

In this section we will use the information stored in the derivative $\vec{\boldsymbol{x}}'(t)$ to create a rough sketch of the graph by hand.

Let's do the specific curve (15.4) first. The derivative (or velocity vector) is

$$\vec{\boldsymbol{x}}'(t) = \begin{pmatrix} -2t \\ 3 - 3t^2 \end{pmatrix}, \qquad \text{so} \ \begin{cases} x_1'(t) &= -2t \\ x_1'(t) &= 3(1 - t^2) \end{cases}$$

We see that $x_1'(t)$ changes its sign at $t = 0$, while $x_2'(t) = 2(1 - t)(1 + t)$ changes its sign twice, at $t = -1$ and then at $t = +1$. You can summarize this in a drawing:

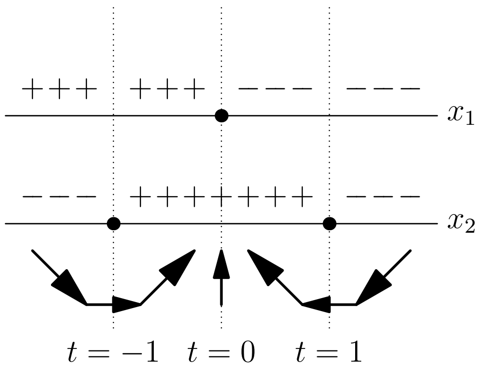


Figure 15.14: a diagram to help you sketch a parametric curve

The arrows indicate the wind direction of the velocity vector $\vec{\boldsymbol{x}}'(t)$ for the various values of t.

For instance, when $t < -1$ you have $x_1'(t) > 0$ and $x_2'(t) < 0$, so that the vector

$$\vec{\boldsymbol{x}}'(t) = \begin{pmatrix} x_1'(t) \\ x_2'(t) \end{pmatrix} = \begin{pmatrix} + \\ - \end{pmatrix}$$

points in the direction "South-East." You see that there are three special t values at which $\vec{\boldsymbol{x}}'(t)$ is either purely horizontal or vertical. Let's compute $\vec{\boldsymbol{x}}(t)$ at those values

$$\begin{array}{lcc} t=-1 & \vec{\boldsymbol{x}}(-1)=\binom{0}{-2} & \vec{\boldsymbol{x}}'(-1)=\binom{2}{0} \\ t=0 & \vec{\boldsymbol{x}}(0)=\binom{1}{0} & \vec{\boldsymbol{x}}'(0)=\binom{0}{3} \\ t=1 & \vec{\boldsymbol{x}}(1)=\binom{0}{2} & \vec{\boldsymbol{x}}'(1)=\binom{-2}{0} \end{array}$$

This leads you to the following sketch:

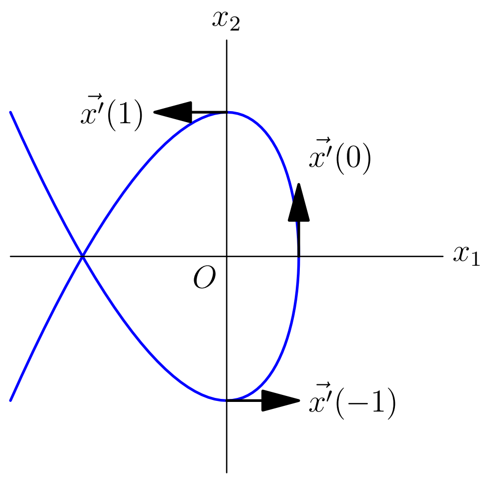


Figure 15.15: example: sketching a parametric curve

15.8 Length of a curve

If you have a parametric curve $\vec{\boldsymbol{x}}(t)$, $a \le t \le b$, then there is a formula for the length of the curve it traces out. We'll go through a brief derivation of this formula before stating it.

To compute the length of the curve $\{\vec{\boldsymbol{x}}(t) : a \le t \le b\}$ we divide it into lots of short pieces. If the pieces are short enough they will be almost straight line segments, and we know how do compute the length of a line segment. After computing the lengths of all the short line segments, you add them to get an approximation to the length of the curve. As you divide the curve into finer & finer pieces this approximation should get better & better. You can smell an integral in this description of what's coming. Here are some more details:

Divide the parameter interval into N pieces,

$$a = t_0 < t_1 < t_2 < \cdots < t_{N-1} < t_N = b.$$

Then we approximate the curve by the polygon with vertices at $\vec{\boldsymbol{x}}(t_0) = \vec{\boldsymbol{x}}(a)$, $\vec{\boldsymbol{x}}(t_1)$, ..., $\vec{\boldsymbol{x}}(t_N)$. The distance between to consecutive points at $\vec{\boldsymbol{x}}(t_{i-1})$ and $\vec{\boldsymbol{x}}(t_i)$ on this polygon is

$$\|\vec{\boldsymbol{x}}(t_i) - \vec{\boldsymbol{x}}(t_{i-1})\|.$$

Since we are going to take $t_{i-1} - t_i$ "very small," we can use the derivative to approximate the distance by

$$\vec{\boldsymbol{x}}(t_i) - \vec{\boldsymbol{x}}(t_{i-1}) = \frac{\vec{\boldsymbol{x}}(t_i) - \vec{\boldsymbol{x}}(t_{i-1})}{t_i - t_{i-1}}\big(t_i - t_{i-1}\big) \approx \vec{\boldsymbol{x}}'(t_i)\big(t_i - t_{i-1}\big),$$

so that

$$\|\vec{x}(t_i) - \vec{x}(t_{i-1})\| \approx \big\|\vec{x}'(t_i)\big\| \big(t_i - t_{i-1}\big).$$

Now add all these distances and you get

$$\text{Length polygon} \approx \sum_{i=1}^{N} \big\|\vec{x}'(t_i)\big\| \big(t_i - t_{i-1}\big) \approx \int_{t=a}^{b} \big\|\vec{x}'(t)\big\| \,\mathrm{d}t.$$

This is our formula for the length of a curve.

Just in case you think this was a proof, *it isn't!* First, we have used the symbol $\approx$ which stands for "approximately equal," and we said "very small" in quotation marks, so there are several places where the preceding discussion is vague. But most of all, we can't *prove* that this integral is the length of the curve, since we don't have a definition of "the length of a curve." This is an opportunity, since it leaves us free to adopt the formula we found as our formal definition of the length of a curve. Here goes:

Definition 15.8.1. If $\{\vec{x}(t) : a \le t \le b\}$ is a parametric curve, then its ***length*** is given by

$$\text{Length} = \int_{a}^{b} \big\|\vec{x}'(t)\big\| \,\mathrm{d}t$$

provided the derivative $\vec{x}'(t)$ exists, and provided $\|\vec{x}'(t)\|$ is a Riemann-integrable function.

In this course we will not worry too much about the two caveats about differentiability and integrability at the end of the definition.

15.8.1 Length of a line segment.

How long is the line segment AB connecting two points $A(a_1, a_2)$ and $B(b_1, b_2)$?

Solution: Parametrize the segment by

$$\vec{x}(t) = \vec{a} + t(\vec{b} - \vec{a}), \quad (0 \le t \le 1).$$

Then

$$\|\vec{x}'(t)\| = \|\vec{b} - \vec{a}\|,$$

and thus

$$\text{Length}(AB) = \int_{0}^{1} \|\vec{x}'(t)\| \,\mathrm{d}t = \int_{0}^{1} \|\vec{b} - \vec{a}\| \,\mathrm{d}t = \|\vec{b} - \vec{a}\|.$$

In other words, the length of the line segment AB is the distance between the two points A and B. It looks like we already knew this, but no, we didn't: what this example shows is that the length of the line segment AB *as defined in definition 15.8.1* is the distance between the points A and B. So definition 15.8.1 gives the right answer in this example. If we had found anything else in this example we would have had to change the definition.

15.8.2 Perimeter of a circle of radius R.

What is the length of the circle of radius R centered at the origin? This is another example where we know the answer in advance. The following computation should give us $2\pi R$ or else there's something wrong with definition 15.8.1.

We parametrize the circle as follows:

$$\vec{\boldsymbol{x}}(t) = R\cos\theta\vec{\boldsymbol{i}} + R\sin\theta\vec{\boldsymbol{j}}, \quad (0 \le \theta \le 2\pi).$$

Then

$$\vec{\boldsymbol{x}}'(\theta) = -R\sin\theta\vec{\boldsymbol{i}} + R\cos\theta\vec{\boldsymbol{j}}, \quad \text{and } \|\vec{\boldsymbol{x}}'(\theta)\| = \sqrt{R^2\sin^2\theta + R^2\cos^2\theta} = R.$$

The length of this circle is therefore

$$\text{Length of circle} = \int_0^{2\pi} R\mathrm{d}\theta = 2\pi R.$$

Fortunately we don't have to fix the definition!

And now the bad news: The integral in the definition of the length looks innocent enough and hasn't caused us any problems in the two examples we have done so far. It is however a reliable source of very difficult integrals. To see why, you must write the integral in terms of the components $x_1(t), x_2(t)$ of $\vec{\boldsymbol{x}}(t)$. Since

$$\vec{\boldsymbol{x}}'(t) = \begin{pmatrix} x_1'(t) \\ x_2'(t) \end{pmatrix} \quad \text{and thus } \|\vec{\boldsymbol{x}}'(t)\| = \sqrt{x_1'(t)^2 + x_2'(t)^2}$$

the length of the curve parametrized by $\{\vec{\boldsymbol{x}}(t) : a \le t \le b\}$ is

$$\text{Length} = \int_a^b \sqrt{x_1'(t)^2 + x_2'(t)^2}\,\mathrm{d}t.$$

For most choices of $x_1(t)$, $x_2(t)$ the sum of squares under the square root cannot be simplified, and, at best, leads to a difficult integral, but more often to an impossible integral.

But, chin up, sometimes, as if by a miracle, the two squares add up to an expression whose square root can be simplified, and the integral is actually not too bad. Here is an example:

15.8.3 Length of the Cycloid.

After getting in at the bottom of a liberated ferris wheel we are propelled through the air along the cycloid whose parametrization is given in example 15.1.7,

$$\vec{\boldsymbol{x}}(\theta) = \begin{pmatrix} \theta - \sin\theta \\ 1 - \cos\theta \end{pmatrix}.$$

How long is one arc of the Cycloid?

Solution: Compute $\vec{\boldsymbol{x}}'(\theta)$ and you find

$$\vec{\boldsymbol{x}}'(\theta) = \begin{pmatrix} 1 - \cos\theta \\ \sin\theta \end{pmatrix}$$

so that

$$\|\vec{\boldsymbol{x}}'(\theta)\| = \sqrt{(1-\cos\theta)^2 + (\sin\theta)^2} = \sqrt{2 - 2\cos\theta}.$$

This doesn't look promising (this is the function we must integrate!), but just as in example 15.4.3 we can put the double angle formula $\cos\theta = 1 - 2\sin^2\frac{\theta}{2}$ to our advantage:

$$\|\vec{\boldsymbol{x}}'(\theta)\| = \sqrt{2 - 2\cos\theta} = \sqrt{4\sin^2\frac{\theta}{2}} = 2\left|\sin\frac{\theta}{2}\right|.$$

We are concerned with only one arc of the Cycloid, so we have $0 \le \theta < 2\pi$, which implies $0 \le \frac{\theta}{2} \le \pi$, which in turn tells us that $\sin\frac{\theta}{2} > 0$ for all θ we are considering. Therefore the length of one arc of the Cycloid is

$$\begin{aligned}\text{Length} &= \int_0^{2\pi} \|\vec{\boldsymbol{x}}'(\theta)\| \, \mathrm{d}\theta \\ &= \int_0^{2\pi} 2\left|\sin\frac{\theta}{2}\right| \, \mathrm{d}\theta \\ &= 2\int_0^{2\pi} \sin\frac{\theta}{2} \, \mathrm{d}\theta \\ &= \left[-4\cos\frac{\theta}{2}\right]_0^{2\pi} \\ &= 8.\end{aligned}$$

To visualize this answer: the height of the cycloid is 2 (twice the radius of the circle), so *the length of one arc of the Cycloid is four times its height* (Look at the drawing on page 333.)

For some light relief the reader should watch this YouTube by Think Twice for a geometric derivation of a formula for the area under a cycloid.

15.9 The arclength function

If you have a parametric curve $\vec{\boldsymbol{x}}(t)$ and you pick a particular point on this curve, say, the point corresponding to parameter value t_0, then one defines the ***arclength function*** (starting at t_0) to be

$$s(t) = \int_{t_0}^{t} \|\vec{\boldsymbol{x}}'(\tau)\| \, \mathrm{d}\tau \tag{15.6}$$

Thus $s(t)$ is the length of the curve segment $\{\vec{\boldsymbol{x}}(\tau) : t_0 \le \tau \le t\}$. ($\tau$ is a dummy variable.)

If you interpret the parametric curve $\vec{\boldsymbol{x}}(t)$ as a description of the motion of some object, then the length $s(t)$ of the curve $\{\vec{\boldsymbol{x}}(\tau) : t_0 \le \tau \le t\}$ is the distance traveled by the object since time t_0.

If you differentiate the distance traveled with respect to time you should get the speed, and indeed, by the Fundamental Theorem of Calculus one has

$$s'(t) = \frac{\mathrm{d}}{\mathrm{d}t}\int_{t_0}^{t} \|\vec{\boldsymbol{x}}'(\tau)\| \, \mathrm{d}\tau = \|\vec{\boldsymbol{x}}'(t)\|,$$

which we had called the speed $v(t)$ in § 15.4.

15.10 Graphs in Cartesian and in Polar Coordinates

Cartesian graphs. Most of calculus deals with a particular kind of curve, namely, the graph of a function, "$y = f(x)$". You can regard such a curve as a special kind of parametric curve, where the parametrization is

$$\vec{\boldsymbol{x}}(t) = \begin{pmatrix} t \\ f(t) \end{pmatrix}$$

and we switch notation from "(x, y)" to "(x_1, x_2)."
For this special case the velocity vector is always given by

$$\vec{\boldsymbol{x}}'(t) = \begin{pmatrix} 1 \\ f'(t) \end{pmatrix},$$

the speed is

$$v(t) = \|\vec{\boldsymbol{x}}'(t)\| = \sqrt{1 + f'(t)^2},$$

and the length of the segment between $t = a$ and $t = b$ is

$$\text{Length} = \int_a^b \sqrt{1 + f'(t)^2}\, dt.$$

Polar graphs. Instead of choosing Cartesian coordinates (x_1, x_2) one can consider so-called ***Polar Coordinates*** in the plane. We have seen these before in the section on complex numbers: to specify the location of a point in the plane you can give its x_1, x_2 coordinates, but you could also give the absolute value and argument of the complex number $x_1 + ix_2$ (see §12.2.) Or, to say it without mentioning complex numbers, you can say where a point P in the plane is by saying (1) how far it is from the origin, and (2) how large the angle between the line segment OP and a fixed half line (usually the positive x-axis) is.

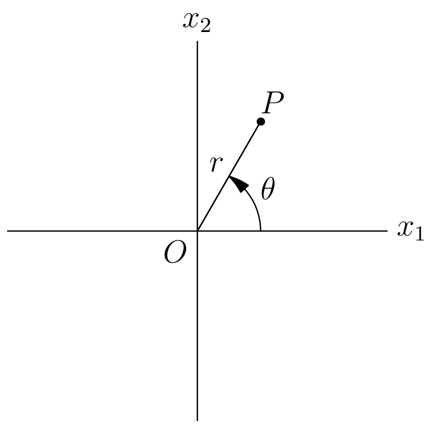


Figure 15.16: polar coordinates

The Cartesian coordinates of a point with polar coordinates (r, θ) are

$$x_1 = r\cos\theta, \qquad x_2 = r\sin\theta, \tag{15.7}$$

or, in our older notation,

$$x = r\cos\theta, \qquad y = r\sin\theta.$$

These are the same formulas as in §12.2, where we had " $r = |z|$ and $\theta = \arg z$."

Often a curve is given as a graph in polar coordinates, i.e. for each angle θ there is one point (X) on the curve, and its distance r to the origin is some function $f(\theta)$ of the angle. In other words, the curve consists of all points whose polar coordinates satisfy the equation $r = f(\theta)$.

You can parametrize such a curve by

$$\vec{\boldsymbol{x}}(\theta) = \begin{pmatrix} r\cos\theta \\ r\sin\theta \end{pmatrix} = \begin{pmatrix} f(\theta)\cos\theta \\ f(\theta)\sin\theta \end{pmatrix}. \tag{15.8}$$

or,

$$\vec{\boldsymbol{x}}(\theta) = f(\theta)\cos\theta\vec{\boldsymbol{i}} + f(\theta)\sin\theta\vec{\boldsymbol{j}}.$$

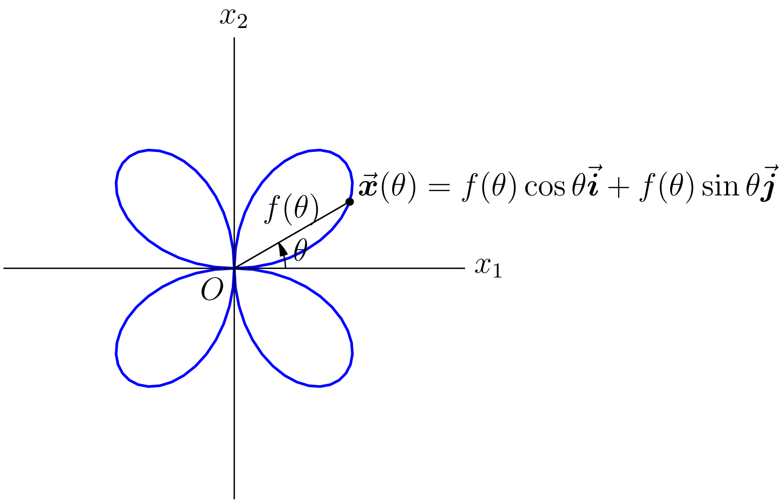


Figure 15.17: the polar curve for $f(\theta) = \sin 2\theta$

You can apply the formulas for velocity, speed and arclength to this parametrization, but instead of doing the straightforward calculation, let's introduce some more notation. For any angle θ we define the vector

$$\vec{\boldsymbol{u}}(\theta) = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} = \cos\theta\vec{\boldsymbol{i}} + \sin\theta\vec{\boldsymbol{j}}.$$

The derivative of $\vec{\boldsymbol{u}}$ is

$$\vec{\boldsymbol{u}}'(\theta) = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix} = -\sin\theta\vec{\boldsymbol{i}} + \cos\theta\vec{\boldsymbol{j}}.$$

The vectors $\vec{\boldsymbol{u}}(\theta)$ and $\vec{\boldsymbol{u}}'(\theta)$ are perpendicular unit vectors.

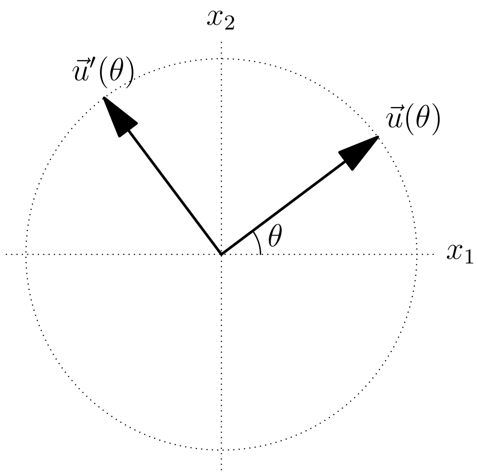


Figure 15.18: derivative of vector moving on a unit circle

Then we have

$$\vec{\boldsymbol{x}}(\theta) = f(\theta)\vec{\boldsymbol{u}}(\theta),$$

so by the product rule one has

$$\vec{\boldsymbol{x}}'(\theta) = f'(\theta)\vec{\boldsymbol{u}}(\theta) + f(\theta)\vec{\boldsymbol{u}}'(\theta).$$

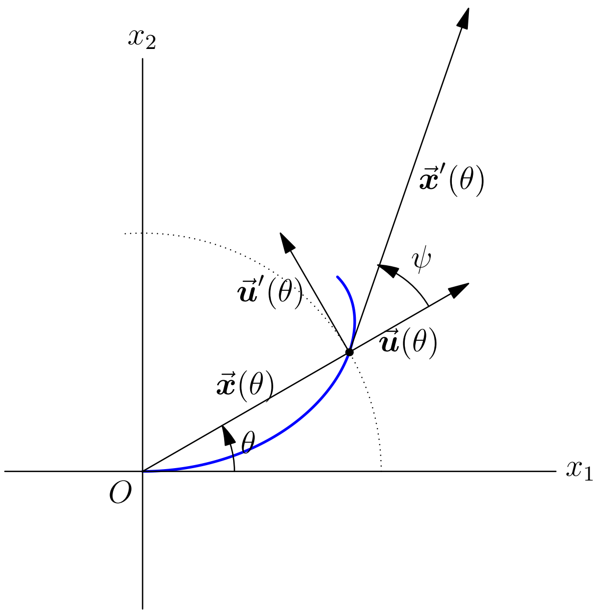


Figure 15.19: derivatives in polar coordinates

Since $\vec{\boldsymbol{u}}(\theta)$ and $\vec{\boldsymbol{u}}'(\theta)$ are perpendicular unit vectors this implies

$$v(\theta) = \|\vec{\boldsymbol{x}}'(\theta)\| = \sqrt{f'(\theta)^2 + f(\theta)^2}.$$

The length of the piece of the curve between polar angles α and β is therefore

$$\text{Length} = \int_{\alpha}^{\beta} \sqrt{f'(\theta)^2 + f(\theta)^2}\, \mathrm{d}\theta. \tag{15.9}$$

You can also read off that the angle ψ between the radius OX and the tangent to the curve satisfies

$$\tan\psi = \frac{f(\theta)}{f'(\theta)}.$$

15.11 PROBLEMS

SKETCHING PARAMETERIZED CURVES

Sketch the curves which are traced out by the following parametrizations. Describe the motion (is the curve you draw traced out once or several times? In which direction?)

In all cases the parameter is allowed to take all values from $-\infty$ to ∞.

If a curve happens to be the graph of some function $x_2 = f(x_1)$ (or $y = f(x)$ if you prefer), then find the function $f(\cdots)$.

Is there a geometric interpretation of the parameter as an angle, or a distance, etc?

971. $\vec{x}(t) = \begin{pmatrix} 1-t \\ 2-t \end{pmatrix}$ †412

972. $\vec{x}(t) = \begin{pmatrix} 3t+2 \\ 3t+2 \end{pmatrix}$ †412

973. $\vec{x}(t) = \begin{pmatrix} e^t \\ e^t \end{pmatrix}$ †412

974. $\vec{x}(t) = \begin{pmatrix} e^t \\ t \end{pmatrix}$ †412

975. $\vec{x}(t) = \begin{pmatrix} e^t \\ e^{-t} \end{pmatrix}$ †412

976. $\vec{x}(t) = \begin{pmatrix} t \\ t^2 \end{pmatrix}$ †412

977. $\vec{x}(t) = \begin{pmatrix} \sin t \\ t \end{pmatrix}$ †412

978. $\vec{x}(t) = \begin{pmatrix} \sin t \\ \cos 2t \end{pmatrix}$ †413

979. $\vec{x}(t) = \begin{pmatrix} \sin 25t \\ \cos 25t \end{pmatrix}$ †413

980. $\vec{x}(t) = \begin{pmatrix} 1+\cos t \\ 1+\sin t \end{pmatrix}$ †413

981. $\vec{x}(t) = \begin{pmatrix} 2\cos t \\ \sin t \end{pmatrix}$ †413

982. $\vec{x}(t) = \begin{pmatrix} t^2 \\ t^3 \end{pmatrix}$ †414

* * *

Find parametric equations for the curve traced out by the X in each of the following descriptions.

983. A circle of radius 1 rolls over the x_1 axis, and X is a point on a spoke of the circle at a distance $a > 0$ from the center of the circle (the case $a = 1$ gives the cycloid.) †414

984. A circle of radius $r > 0$ rolls on the outside of the unit circle. X is a point on the rolling circle (These curves are called *epicycloids.*)

985. A circle of radius $0 < r < 1$ rolls on the *inside* of the unit circle. X is a point on the rolling circle.

986. Let O be the origin, A the point $(1, 0)$, and B the point on the unit circle for which the angle $\angle AOB = \theta$. Then X is the point on the tangent to the unit circle through B for which the distance BX equals the length of the circle arc AB. †414

987. X is the point where the tangent line at $\vec{x}(\theta)$ to the helix of example 15.1.8 intersects the x_1x_2 plane.

CURVE SKETCHING, USING THE TANGENT VECTOR

988. Consider a triangle ABC and let $\vec{a}, \vec{b}$ and $\vec{c}$ be the position vectors of A, B and C.

(*i*) Show that the parametric curve given by

$$\vec{x}(t) = (1-t)^2\vec{a} + 2t(1-t)\vec{b} + t^2\vec{c},$$

goes through the points A and C, and that at these points it is tangent to the sides of the triangle. Make a drawing. †414

(*ii*) At which point on this curve is the tangent parallel to the side AC of the triangle? †414

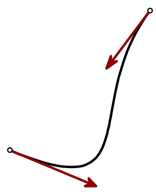
989. Let $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ be four given vectors. Consider the parametric curve (known as a *Bezier curve*)

$$\vec{x}(t) = (1-t)^3\vec{a} + 3t(1-t)^2\vec{b} + 3t^2(1-t)\vec{c} + t^3\vec{d}$$

where $0 \le t \le 1$.

Compute $\vec{x}(0), \vec{x}(1), \vec{x}'(0)$, and $\vec{x}'(1)$.

The characters in most fonts (like the fonts used for these notes) are made up of lots of Bezier curves.



990. Sketch the following curves by finding all points at which the tangent is either horizontal or vertical (in these problems, a is a positive constant.)

(*i*) $\vec{x}(t) = \begin{pmatrix} 1-t^2 \\ t+2t^2 \end{pmatrix}$ †415

(*ii*) $\vec{x}(t) = \begin{pmatrix} \sin t \\ \sin 2t \end{pmatrix}$ †415

(*iii*) $\vec{x}(t) = \begin{pmatrix} \cos t \\ \sin 2t \end{pmatrix}$ †415

(*iv*) $\vec{x}(t) = \begin{pmatrix} 1-t^2 \\ 3at-t^3 \end{pmatrix}$ †415

(*v*) $\vec{x}(t) = \begin{pmatrix} 1-t^2 \\ 3at+t^3 \end{pmatrix}$ †415

(*vi*) $\vec{x}(t) = \begin{pmatrix} \cos 2t \\ \sin 3t \end{pmatrix}$ †415

(*vii*) $\vec{x}(t) = \begin{pmatrix} t/(1+t^2) \\ t^2 \end{pmatrix}$ †415

(*viii*) $\vec{x}(t) = \begin{pmatrix} t^2 \\ \sin t \end{pmatrix}$

(*ix*) $\vec{x}(t) = \begin{pmatrix} 1+t^2 \\ 2t^4 \end{pmatrix}$ †415

PRODUCT RULES

991. If a moving object has position vector $\vec{\boldsymbol{x}}(t)$ at time t, and *if it's speed is constant,* then show that the acceleration vector is always perpendicular to the velocity vector. [Hint: differentiate $v^2 = \vec{\boldsymbol{v}} \cdot \vec{\boldsymbol{v}}$ with respect to time and use some of the product rules from §15.3.]

992. If a charged particle moves in a magnetic field $\vec{\boldsymbol{B}}$, then the laws of electromagnetism say that the magnetic field exerts a force on the particle and that this force is given by the following miraculous formula:

$$\vec{\boldsymbol{F}} = q\vec{\boldsymbol{v}} \times \vec{\boldsymbol{B}}.$$

where q is the charge of the particle, and $\vec{\boldsymbol{v}}$ is its velocity.

Not only does the particle know calculus (since Newton found $\vec{\boldsymbol{F}} = m\vec{\boldsymbol{a}}$), it also knows vector geometry!

Show that even though the magnetic field is pushing the particle around, and even though its velocity vector may be changing with time, its speed $v = \|\vec{\boldsymbol{v}}\|$ remains constant.

993. Newton's law of gravitation states that the Earth pulls any object of mass m towards its center with a force inversely proportional to the squared distance of the object to the Earth's center.

(***i***) Show that if the Earth's center is the origin, and $\vec{\boldsymbol{r}}$ is the position vector of the object of mass m, then the gravitational force is given by

$$\vec{\boldsymbol{F}} = -C\frac{\vec{\boldsymbol{r}}}{\|\vec{\boldsymbol{r}}\|^3} \qquad (C \text{ is a positive constant.})$$

[No calculus required. You are supposed to check that this vector satisfies the description in the beginning of the problem, i.e. that it has the right length and direction.]

(***ii***) If the object is moving, then its ***angular momentum*** is defined in physics books by the formula $\vec{\boldsymbol{L}} = m\vec{\boldsymbol{r}} \times \vec{\boldsymbol{v}}$. Show that, if the Earth's gravitational field is the only force acting on the object, then its angular momentum remains constant. [Hint: you should differentiate $\vec{\boldsymbol{L}}$ with respect to time, and use a product rule.]

LENGTHS OF CURVES

994. Find the length of each of the following curve segments. An "$\int$" indicates a difficult but possible integral which you should do; "$\int\int$" indicates that the resulting integral cannot reasonably be done with the methods explained in this course – you may leave an integral in your answer after simplifying it as much as you can. All other problems lead to integrals that shouldn't be too hard.

(***i***) The *cycloid* $\vec{x}(\theta) = \binom{R(\theta-\sin\theta)}{R(1-\cos\theta)}$, with $0 \le \theta \le 2\pi$.

(***ii***) [$\int\int$] The *ellipse* $\vec{x}(t) = \begin{pmatrix}\cos t\\ A\sin t\end{pmatrix}$ with $0 \le t \le 2\pi$.

(***iii***) [$\int$] The *parabola* $\vec{x}(t) = \begin{pmatrix}t\\ t^2\end{pmatrix}$ with $0 \le t \le 1$.

(***iv***) [$\int\int$] The *Sine graph* $\vec{x}(t) = \begin{pmatrix}t\\ \sin t\end{pmatrix}$ with $0 \le t \le \pi$.

(***v***) The *evolute of the circle* $\vec{x} = \begin{pmatrix}\cos t + t\sin t\\ \sin t - t\cos t\end{pmatrix}$ (with $0 \le t \le L$).

(***vi***) The *Catenary*, i.e. the graph of $y = \cosh x = \dfrac{e^x + e^{-x}}{2}$ for $-a \leqslant x \leqslant a$.

(***vii***) The *Cardioid*, which in polar coordinates is given by $r = 1 + \cos\theta$, $(|\theta| < \pi)$, so $\vec{x}(\theta) = \begin{pmatrix}(1+\cos\theta)\cos\theta\\ (1+\cos\theta)\sin\theta\end{pmatrix}$.

(***viii***) The *Helix* from example 15.1.8, $\vec{x}(\theta) = \begin{pmatrix}\cos\theta\\ \sin\theta\\ a\theta\end{pmatrix}$, $0 \le \theta \le 2\pi$.

995. Below are a number of parametrized curves. For each of these curves find all points with horizontal or vertical tangents; also find all points for which the tangent is parallel to the diagonal. Finally, find the length of the piece of these curves corresponding to the indicated parameter interval (I tried hard to find examples where the integral can be done).

(***i***) $\vec{x}(t) = \begin{pmatrix}t^{1/3} - \frac{9}{20}t^{5/3}\\ t\end{pmatrix}$ $\quad 0 \le t \le 1$

(***ii***) $\vec{x}(t) = \begin{pmatrix}t^2\\ t^2\sqrt{t}\end{pmatrix}$ $\quad 1 \le t \le 2$

(***iii***) $\vec{x}(t) = \begin{pmatrix}t^2\\ t - t^3/3\end{pmatrix}$ $\quad 0 \le t \le \sqrt{3}$

(***iv***) $\vec{x}(t) = \begin{pmatrix}8\sin t\\ 7t - \sin t\cos t\end{pmatrix}$ $\quad |t| \le \dfrac{\pi}{2}$

(***v***) $\vec{x}(t) = \begin{pmatrix}t\\ \sqrt{1+t}\end{pmatrix}$ $\quad 0 \le t \le 1$

(The last problem is harder, but it can be done. In all the other ones the quantity under the square root that appears when you set up the integral for the length of the curve *is a perfect square.*)

996. Consider the polar graph $r = e^{k\theta}$, with $-\infty < \theta < \infty$, where k is a positive constant. This curve is called the *logarithmic spiral.*

(***i***) Find a parametrization for the polar graph of $r = e^{k\theta}$.

(***ii***) Compute the arclength function $s(\theta)$ starting at $\theta_0 = 0$.

(***iii***) Show that the angle between the radius and the tangent is the same at all points on the logarithmic spiral.

(***iv***) Which points on this curve have horizontal tangents?

997. The *Archimedean spiral* is the polar graph of $r = \theta$, where $\theta \ge 0$.

(***i***) Which points on the part of the spiral with $0 < \theta < \pi$ have a horizontal tangent? Which have a vertical tangent?

(***ii***) Find all points on the whole spiral (allowing all $\theta > 0$) which have a horizontal tangent.

(***iii***) Show that the part of the spiral with $0 < \theta < \pi$ is exactly as long as the piece of the parabola $y = \frac{1}{2}x^2$ between $x = 0$ and $x = \pi$. (It is not impossible to compute the lengths of both curves, but you don't have to to answer this problem!)

KEPLER's LAW's

Kepler's first law: Planets move in a plane in an ellipse with the sun at one focus.

Kepler's second law: The position vector from the sun to a planet sweeps out area at a constant rate.

Kepler's third law: The square of the period of a planet is proportional to the cube of its mean distance from the sun. The mean distance is the average of the closest distance and the furthest distance. The period is the time required to go once around the sun.

Let $\vec{p} = x\vec{i} + y\vec{j} + z\vec{k}$ be the position of a planet in space where x, y and z are all function of time t. Assume the sun is at the origin. Newton's law of gravity implies that

$$\frac{d^2\vec{p}}{dt^2} = \alpha\frac{\vec{p}}{||\vec{p}||^3} \tag{1}$$

where α is $-GM$, G is a universal gravitational constant and M is the mass of the sun. It does not depend on the mass of the planet.
First let us show that planets move in a plane. By the product rule

$$\frac{d}{dt}(\vec{p}\times\frac{d\vec{p}}{dt}) = (\frac{d\vec{p}}{dt}\times\frac{d\vec{p}}{dt}) + (\vec{p}\times\frac{d^2\vec{p}}{dt^2}) \tag{2}$$

By (1) and the fact that the cross product of parallel vectors is $\vec{0}$ the right hand side of (2) is $\vec{0}$. It follows that there is a constant vector $\vec{c}$ such that at all times

$$\vec{p}\times\frac{d\vec{p}}{dt} = \vec{c} \tag{3}$$

Thus we can conclude that both the position and velocity vector lie in the plane with normal vector $\vec{c}$. Without loss of generality we assume that $\vec{c} = \beta\vec{k}$ for some scaler β and $\vec{p} = x\vec{i} + y\vec{j}$. Let $x = r\cos(\theta)$ and $y = r\sin(\theta)$ where we consider r and θ as functions of t. If we calculate the derivative of $\vec{p}$ we get

$$\frac{d\vec{p}}{dt} = [\frac{dr}{dt}\cos(\theta) - r\sin(\theta)\frac{d\theta}{dt}]\,\vec{i} + [\frac{dr}{dt}\sin(\theta) + r\cos(\theta)\frac{d\theta}{dt}]\,\vec{j} \tag{4}$$

Since $\vec{p}\times\frac{d\vec{p}}{dt} = \beta\vec{k}$ we have

$$r\cos(\theta)(\frac{dr}{dt}\sin(\theta) + r\cos(\theta)\frac{d\theta}{dt}) - r\sin(\theta)(\frac{dr}{dt}\cos(\theta) - r\sin(\theta)\frac{d\theta}{dt}) = \beta \tag{5}$$

After multiplying out and simplifying this reduces to

$$r^2\frac{d\theta}{dt} = \beta \tag{6}$$

The area swept out from time t_0 to time t_1 by a curve in polar coordinates is

$$A = \frac{1}{2}\int_{t_0}^{t_1} r^2\frac{d\theta}{dt}dt \tag{7}$$

By (6) A is proportional to $t_1 - t_0$. This is Kepler's second law.

We will now prove Kepler's third law for the special case of a circle. So let T be the time it takes the planet to go around the sun one time and let r be its distance from the sun. We will show that

$$\frac{T^2}{r^3} = -\frac{(2\pi)^2}{\alpha} \tag{8}$$

The second law implies that $\theta(t)$ is a linear function of t and so in fact

$$\frac{d\theta}{dt} = \frac{2\pi}{T} \tag{9}$$

Since r is constant we have that $\frac{dr}{dt} = 0$ and so (4) simplifies to

$$\frac{d\vec{p}}{dt} = \left[-r\sin(\theta)\frac{2\pi}{T}\right]\vec{i} + \left[r\cos(\theta)\frac{2\pi}{T}\right]\vec{j} \tag{10}$$

Differentiating once more we get

$$\frac{d^2\vec{p}}{dt^2} = \left[-r\cos(\theta)\left(\frac{2\pi}{T}\right)^2\right]\vec{i} + \left[-r\sin(\theta)\left(\frac{2\pi}{T}\right)^2\right]\vec{j} = -\left(\frac{2\pi}{T}\right)^2\vec{p} \tag{11}$$

Noting that $r = ||\vec{p}||$ and using (1) we get

$$\frac{\alpha}{r^3} = -\left(\frac{2\pi}{T}\right)^2 \tag{12}$$

from which (8) immediately follows.

Complete derivations of the three laws from Newton's law of gravity can be found in T.M.Apostal, Calculus vol I , Blaisdel(1967), p.545-548. Newton deduced the law of gravity from Kepler's laws. The argument can be found in L.Bers, Calculus vol II , Holt,Rinhart,and Winston(1969), p.748-754.

The planet earth is 93 million miles from the sun. The year has 365 days. The moon is 250,000 miles from the earth and circles the earth once every 28 days. The earth's diameter is 7850 miles. In the first four problems you may assume orbits are circular. Use only the data in this paragraph.

998. The former planet Pluto takes 248 years to orbit the sun. How far is Pluto from the sun? Mercury is 36 million miles from the sun. How many (Earth) days does it take for Mercury to complete one revolution of the sun? †415

999. Russia launched the first orbital satelite in 1957. Sputnik orbited the earth every 96 minutes. How high off the surface of the earth was this satelite? †416

1000. A communication satellite is to orbit the earth around the equator at such a distance so as to remain above the same spot on the earth's surface at all times. What is the distance from the center of the earth such a satellite should orbit? †416

1001. Find the ratio of the masses of the sun and the earth. †416

1002. The Kmart7 satellite is to be launched into polar earth orbit by firing it from a large cannon. This is possible since the satellite is very small, consisting of a single blinking blue light. Polar orbit means that the orbit passes over both the north and south poles. Let p(t) be the point on the earth's surface at which the blinking blue light is directly overhead at time t. Find the largest orbit that the Kmart7 can have so that every person on earth will be within 1000 miles of p(t) at least once a day. You may assume that the satellite orbits the earth exactly n times per day for some integer n. †416

1003. Let A be the total area swept out by an elliptical orbit. Show that $\beta = \frac{2A}{T}$. †416

1004. Let E be an ellipse with one of the focal points f. Let d be the minimum distance from some point of the ellipse to f and let D be the maximum distance. In terms of d and D only what is the area of the ellipse E?

Hint: The area of an ellipse is πab where a is its minimum radius and b its maximum radius (both from the center of the ellipse). If f_1 and f_2 are the focal points of E then the sum of the distances from f_1 to p and f_2 to p is constant for all points p on E. †416

1005. Halley's comet orbits the sun every 77 years. Its closest approach is 53 million miles. What is its furthest distance from the sun? What is the maximum speed of the comet and what is the minimum speed? †416

Chapter 16

Miscellaneous exercises

For graphing problems you may be asked to determine
(a) where $f(x)$ is defined,
(b) where $f(x)$ is continuous,
(c) where $f(x)$ is differentiable,
(d) where $f(x)$ is increasing and where it is decreasing,
(e) where $f(x)$ is concave up and where it is concave down,
(f) what the critical points of $f(x)$ are,
(g) where the points of inflection are,
(h) what (if any) the horizontal asymptotes to $f(x)$ are, and
(i) what (if any) the vertical asymptotes to $f(x)$ are.

(A horizontal line $y = b$ is called a *horizontal asymptote* if $\lim_{x\to\infty} f(x) = b$ or $\lim_{x\to-\infty} f(x) = b$. A vertical line $x = a$ is called a *vertical asymptote* if $\lim_{x\to a+} f(x) = \pm\infty$ or $\lim_{x\to a-} f(x) = \pm\infty$.)

For proofs the question will be carefully worded to indicate what you may assume in your proof. (See Problem 1015 for example.) In this document you may use without proof any previously asserted fact. For example, you may use the fact that $\sin'(\theta) = \cos(\theta)$ to prove that $\cos'(\theta) = -\sin(\theta)$ since the former question precedes the latter below. (See Problems 1017 and 1018.) You may always use high school algebra (like $\cos(\theta) = \sin(\pi/2 - \theta)$) in your proofs.

1006. State and prove the Sum Rule for derivatives. You may use (without proof) the Limit Laws.

1007. State and prove the Product Rule for derivatives. You may use (without proof) the Limit Laws.

1008. State and prove the Quotient Rule for derivatives. You may use (without proof) the Limit Laws.

1009. State and prove the Chain Rule for derivatives. You may use (without proof) the Limit Laws. You may assume (as the proof in the Stewart text does) that the inner function has a nonzero derivative.

1010. State the Sandwich[1] Theorem.

1011. Prove that $\dfrac{dx^n}{dx} = nx^{n-1}$, for all positive integers n.

[1] Also called the Squeeze Theorem

1012. Prove that $\dfrac{dx^n}{dx} = nx^{n-1}$, for $n = 0$.

1013. Prove that $\dfrac{dx^n}{dx} = nx^{n-1}$, for all negative integers n.

1014. Prove that $\dfrac{de^x}{dx} = e^x$.

1015. Prove that $\lim_{\theta\to 0} \frac{\sin(\theta)}{\theta} = 1$.

You may assume without proof the Sandwich Theorem, the Limit Laws, and that the sin and cos are continuous. Hint: See Problem 1101.

1016. Prove that $\lim_{\theta\to 0} \frac{1-\cos(\theta)}{\theta} = 0$.

1017. Prove that $\dfrac{d\sin x}{dx} = \cos x$.

1018. Prove that $\dfrac{d\cos x}{dx} = -\sin x$.

1019. Prove that $\dfrac{d\tan x}{dx} = \sec^2 x$.

1020. Prove that $\dfrac{d\cot x}{dx} = -\csc^2 x$.

1021. Prove that $\dfrac{d\ln x}{dx} = \dfrac{1}{x}$.

1022. Prove that $\dfrac{d\arcsin x}{dx} = \dfrac{1}{\sqrt{1-x^2}}$.

1023. Prove that $\dfrac{d\arccos x}{dx} = -\dfrac{1}{\sqrt{1-x^2}}$.

1024. Prove that $\dfrac{d\arctan x}{dx} = \dfrac{1}{1+x^2}$.

1025. True or false? A differentiable function must be continuous. If true, give a proof; if false, illustrate with an example.

1026. True or false? A continuous function must be differentiable. If true, give a proof; if false, illustrate with an example.

1027. Explain why $\lim\limits_{x\to 0} 1/x$ does not exist.

1028. Explain why $\lim\limits_{\theta\to\pi/2} \tan\theta$ does not exist.

1029. Explain why $\lim\limits_{x\to 0} \sin(1/x)$ does not exist.

1030. Explain why $\lim\limits_{\theta\to\infty} \cos\theta$ does not exist.

1031. Let $\text{sign}(x)$ be the sign function (see example 3.1) Explain why $\lim\limits_{x\to 0} \text{sign}(x)$ does not exist.

1032. Explain why $\lim\limits_{y\to 0} 2^{1/y}$ does not exist.

1033. Explain why $\lim\limits_{x\to 1} 2^{1/(x-1)}$ does not exist.

1034. Calculate $\lim\limits_{\Delta x\to 0} \dfrac{f(x+\Delta x) - f(x)}{(x+\Delta x) - x}$ when $f(x) = \sin 2x$.

1035. Calculate $\lim\limits_{h\to 0} \dfrac{f(x+h)-f(x)}{(x+h)-x}$ when $f(x)=\cos 2x$.

1036. Calculate $\lim\limits_{x\to a} \dfrac{f(x)-f(a)}{x-a}$ when $f(x)=\sin(x^2)$.

1037. Calculate $\lim\limits_{x\to x_0} \dfrac{f(x)-f(x_0)}{x-x_0}$ when $f(x)=\cos(x^2)$.

1038. Calculate $\lim\limits_{\Delta x\to 0} \dfrac{f(x+\Delta x)-f(x)}{(x+\Delta x)-x}$ when $f(x)=\sqrt{\sin x}$.

1039. Calculate $\lim\limits_{\Delta x\to 0} \dfrac{f(x+\Delta x)-f(x)}{(x+\Delta x)-x}$ when $f(x)=x\sin x$.

1040. Calculate $\lim\limits_{\Delta x\to 0} \dfrac{f(x+\Delta x)-f(x)}{(x+\Delta x)-x}$ when $f(x)=e^{\sqrt{x}}$.

1041. Calculate $\lim\limits_{\Delta x\to 0} \dfrac{f(x+\Delta x)-f(x)}{(x+\Delta x)-x}$ when $f(x)=e^{\sin x}$.

1042. Calculate $\lim\limits_{\Delta x\to 0} \dfrac{f(x+\Delta x)-f(x)}{(x+\Delta x)-x}$ when $f(x)=\ln(ax+b)$.

1043. Calculate $\lim\limits_{\Delta x\to 0} \dfrac{f(x+\Delta x)-f(x)}{(x+\Delta x)-x}$ when $f(x)=e^{\cos x}$.

1044. Calculate $\lim\limits_{\Delta x\to 0} \dfrac{f(x+\Delta x)-f(x)}{(x+\Delta x)-x}$ when $f(x)=x^x$.

1045. Calculate $\lim\limits_{\Delta x\to 0} \dfrac{f(x+\Delta x)-f(x)}{(x+\Delta x)-x}$ when $f(x)=\dfrac{\sin x}{x}$.

1046. Calculate $\lim\limits_{\Delta x\to 0} \dfrac{f(x+\Delta x)-f(x)}{(x+\Delta x)-x}$ when $f(x)=\sqrt{ax+b}$.

1047. Calculate $\lim\limits_{\Delta x\to 0} \dfrac{f(x+\Delta x)-f(x)}{(x+\Delta x)-x}$ when $f(x)=(mx+c)^n$.

1048. Use differentiation to estimate the number $\dfrac{127^{4/3}-125^{4/3}}{2}$ approximately without a calculator. Your answer should have the form p/q where p and q are integers. Hint: $5^3=125$.

1049. What is the derivative of the area of a circle with respect to its radius?

1050. What is the derivative of the volume of a sphere with respect to its radius?

1051. Find the slope of the tangent to the curve $y=x^3-x$ at $x=2$.

1052. Find the equations of the tangent and normal to the curve $y=x^3-2x+7$ at the point $(1,6)$.

1053. Find the equation of the tangent line to the curve $3xy^2-2x^2y=1$ at the point $(1,1)$. Find d^2y/dx^2 at this point.

1054. Find the equations of the tangent and normal to the curve $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ at the point $(a\cos\theta, b\sin\theta)$.

1055. Find the equations of the tangent and normal to the curve $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1$ at the point $(a\sec\theta, b\tan\theta)$.

1056. Find the equations of the tangent and normal to the curve $c^2(x^2 + y^2) = x^2y^2$ at the point $(c/\cos\theta, c/\sin\theta)$.

1057. Find the equations of the tangent and normal to the parabola $y^2 = 4ax$ at the point $(at^2, 2at)$.

1058. Show that the equation of the tangent to the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ at the point (p, q) is $\dfrac{xp}{a^2} - \dfrac{yq}{b^2} = 1$

1059. Find the equations of the tangent and normal to the curve $y = x^4 - 6x^3 + 13x^2 - 10x + 5$ at the point where $x = 1$.

1060. Find the linear approximations to $f(x) = \dfrac{1}{\sqrt{4+x}}$ at $x = 0$.

1061. Find the linear approximations to $f(x) = \sqrt{1+x}$ at $x = 0$.

1062. Find the linear approximations to $f(x) = \dfrac{1}{(1+2x)^4}$ at $x = 0$.

1063. Find the linear approximations to $f(x) = (1+x)^3$ at $x = 0$.

1064. Find the linear approximations to $f(x) = \sec x$ at $x = 0$.

1065. Find the linear approximations to $f(x) = x\sin x$ at $x = 0$.

1066. Find the linear approximations to $f(x) = x^3$ at $x = 1$.

1067. Find the linear approximations to $f(x) = x^{1/3}$ at $x = -8$.

1068. Find the linear approximations to $f(\theta) = \sin\theta$ at $\theta = \pi/6$.

1069. Find the linear approximations to $f(x) = x^{-1}$ at $x = 4$.

1070. Find the linear approximations to $f(x) = x^3 - x$ at $x = 1$.

1071. Find the linear approximations to $f(x) = \sqrt{x}$ at $x = 4$.

1072. Find the linear approximations to $f(x) = \sqrt{x^2+9}$ at $x = -4$.

1073. Use quadratic approximation to find the approximate value of $\sqrt{401}$ without a calculator. Hint: $\sqrt{400} = 20$.

1074. Use quadratic approximation to find the approximate value of $(255)^{1/4}$ without a calculator. Hint: $256^{1/4} = 4$.

1075. Use quadratic approximation to find the approximate value of $\dfrac{1}{(2.002)^2}$ without a calculator.

1076. Approximate $(1.97)^6$ without a calculator. (Leave arithmetic undone.)

1077. Let f be a function such that $f(1) = 2$ and whose derivative is known to be $f'(x) = \sqrt{x^3+1}$. Use a linear approximation to estimate the value of $f(1.1)$. Use a quadratic approximation to estimate the value of $f(1.1)$.

1078. Find the second derivative of x^7 with respect to x.

1079. Find the second derivative of $\ln x$ with respect to x.

1080. Find the second derivative of 5^x with respect to x.

1081. Find the second derivative of $\tan\theta$ with respect to θ.

1082. Find the second derivative of x^2e^{3x} with respect to x.

1083. Find the second derivative of $\sin 3x \cos 5x$ with respect to x.

1084. Find the third derivative of u^4 with respect to u.

1085. Find the third derivative of $\ln x$ with respect to x.

1086. Find the second derivative of $\tan x$ with respect to x.

1087. If $\theta = \arcsin y$ show that $\dfrac{d^2\theta}{dy^2} = \dfrac{y}{(1-y^2)^{3/2}}$.

1088. If $y = e^{-t}\cos t$ show that $\dfrac{d^2y}{dt^2} = 2e^{-t}\sin t$.

1089. If $u = t + \cot t$ show that $\sin^2 t \cdot \dfrac{d^2u}{dt^2} - 2u + 2t = 0$.

1090. If $y = e^{\tan x}$ show that $\cos^2 x \cdot \dfrac{d^2y}{dx^2} - (1+\sin 2x)\dfrac{dy}{dx} = 0$.

1091. State L'Hôpital's rule and give an example which illustrates how it is used.

1092. Explain why L'Hôpital's rule works. Hint: Expand the numerator and the denominator in terms of Δx.

1093. Give three examples to illustrate that a limit problem that looks like it is coming out to $0/0$ could be really getting closer and closer to almost anything and must be looked at a different way.

1094. Give three examples to illustrate that a limit problem that looks like it is coming out to 1^∞ could be really getting closer and closer to almost anything and must be looked at a different way.

1095. Give three examples to illustrate that a limit problem that looks like it is coming out to 0^0 could be really getting closer and closer to almost anything and must be looked at a different way.

1096. Give three examples to illustrate that a limit problem that looks like it is coming out to $\infty - \infty$ could be really getting closer and closer to almost anything and must be looked at a different way.

1097. Explain how limit problems that come out to ∞/∞ can always be converted into limit problems that come out to $0/0$ and why doing such a conversion is useful.

1098. Explain how limit problems that come out to $\infty - \infty$ can be converted into limit problems that come out to $0/0$ and why doing such a conversion is useful.

1099. Explain how limit problems that come out to 0^0 can be converted into limit problems that come out to $0/0$ and why doing such a conversion is useful.

1100. Explain how limit problems that come out to 1^∞ can be converted into limit problems that come out to $0/0$ and why doing such a conversion is useful.

1101. Use calculus to show that the area A of a sector of a circle with central angle θ is $A = (\theta/2)R^2$ where R is the radius and θ is measured in radians. Hint: Divide the sector into n equal sectors of central angle $\Delta\theta = \theta/n$ and area ΔA. As in the proof (see Problem 1015) that

$$\lim_{\Delta\theta \to 0} \frac{\sin(\Delta\theta)}{\Delta\theta} = 1,$$

the area ΔA lies between the areas of two right triangles whose areas can be expressed in terms of R and trig functions of $\Delta\theta$. Apply the Sandwich Theorem to $A = n\Delta A$ and use l'Hôpital's rule or Problem 1015.

1102. Use calculus to show that the area of a circle of radius R is πR^2. Hint: The area of a sector is a more general problem. (See problem 1101.)

1103. For which values of x is the function $f(x) = x^2 + 3x + 4$ continuous? Justify your answer with limits if necessary and draw a graph of the function to illustrate your answer.

1104. For which values of x is the function $f(x) = \begin{cases} \frac{x^2-x-6}{x-3}, & \text{if } x \neq 3, \\ 5, & \text{if } x = 3, \end{cases}$ continuous? Justify your answer with limits if necessary and draw a graph of the function to illustrate your answer.

1105. For which values of x is the function $f(x) = \begin{cases} \frac{\sin 3x}{x}, & \text{if } x \neq 0, \\ 1, & \text{if } x = 0, \end{cases}$ continuous? Justify your answer with limits if necessary and draw a graph of the function to illustrate your answer.

1106. Determine the value of k for which the function

$$f(x) = \begin{cases} \dfrac{\sin 2x}{5x}, & \text{if } x \neq 0, \\ k, & \text{if } x = 0, \end{cases}$$

is continuous at $x = 0$. Justify your answer with limits if necessary and draw a graph of the function to illustrate your answer.

1107. What does it mean for a function $f(x)$ to be continuous at $x = a$?

1108. What does it mean for a function $f(x)$ to be differentiable at $x = a$?

1109. What does $f'(a)$ indicate you about the graph of $y = f(x)$? Explain why this is true.

1110. What does it mean for a function to be increasing? Explain how to use calculus to tell if a function is increasing. Explain why this works.

1111. What does it mean for a function to be concave up? Explain how to use calculus to tell if a function is concave up. Explain why this works.

1112. What is a horizontal asymptote of a function $f(x)$? Explain how to justify that a given line $y = b$ is a horizontal asymptote of $f(x)$.

1113. What is a vertical asymptote of a function $f(x)$? Explain how to justify that a given line $x = a$ is an vertical asymptote of $f(x)$.

1114. If $f(x) = |x|$, what is $f'(-2)$?

1115. Find the values of a and b so that the function

$$f(x) = \begin{cases} x^2 + 3x + a, & \text{if } x \leq 1, \\ bx + 2, & \text{if } x > 1, \end{cases}$$

is differentiable for all values of x.

1116. Graph $f(x) = \begin{cases} 2 - x, & \text{if } x \geq 1, \\ x, & \text{if } 0 \leq x \leq 1. \end{cases}$

1117. Graph $f(x) = \begin{cases} 2 + x, & \text{if } x \geq 0, \\ 2 - x, & \text{if } x < 0. \end{cases}$

1118. Graph $f(x) = \begin{cases} 1-x, & \text{if } x < 1, \\ x^2-1, & \text{if } x \geq 1. \end{cases}$

1119. Graph $f(x) = x + 1/x$.

1120. Graph $f(x) = \dfrac{x^2+2x-20}{x-4}$ for $5 < x < 9$.

1121. Graph $f(x) = \dfrac{1}{x^2+1}$.

1122. Graph $f(x) = xe^x$.

1123. State Rolle's theorem and draw a picture which illustrates the statement of the theorem.

1124. State the Mean Value Theorem and draw a picture which illustrates the statement of the theorem.

1125. Explain why Rolle's theorem is a *special case* of the Mean Value Theorem.

1126. Let $f(x) = 1 - x^{2/3}$. Show that $f(-1) = f(1)$ but that there is no number c in the interval $(-1, 1)$ such that $f'(c) = 0$. Why does this not contradict Rolle's theorem?

1127. Let $f(x) = (x-1)^{-2}$. Show that $f(0) = f(2)$ but that there is no number c in the interval $(0, 2)$ such that $f'(c) = 0$. Why does this not contradict Rolle's theorem?

1128. Show that the Mean Value Theorem is not applicable to the function $f(x) = |x|$ in the interval $[-1, 1]$.

1129. Show that the Mean Value Theorem is not applicable to the function $f(x) = 1/x$ in the interval $[-1, 1]$.

1130. Find a point on the curve $y = x^3$ where the tangent is parallel to the chord joining $(1, 1)$ and $(3, 27)$.

1131. Show that the equation $x^5 + 10x + 3 = 0$ has exactly one real root.

1132. Find the local maxima and minima of $f(x) = (5x-1)^2 + 4$ without using derivatives.

1133. Find the local maxima and minima of $f(x) = -(x-3)^2 + 9$ without using derivatives.

1134. Find the local maxima and minima of $f(x) = -|x+4| + 6$ without using derivatives.

1135. Find the local maxima and minima of $f(x) = \sin 2x + 5$ without using derivatives.

1136. Find the local maxima and minima of $f(x) = |\sin 4x + 3|$ without using derivatives.

1137. Find the local maxima and minima of $f(x) = x^4 - 62x^2 + 120x + 9$.

1138. Find the local maxima and minima of $f(x) = (x-1)(x+2)^2$.

1139. Find the local maxima and minima of $f(x) = -(x-1)^3(x+1)^2$.

1140. Find the local maxima and minima of $f(x) = x/2 + 2/x$ for $x > 0$.

1141. Find the local maxima and minima of $f(x) = 2x^3 - 24x + 107$ in the interval $[1, 3]$.

1142. Find the local maxima and minima of $f(x) = \sin x + (1/2)\cos x$ in $0 \leq x \leq \pi/2$.

1143. Show that the maximum value of $\left(\dfrac{1}{x}\right)^x$ is $e^{1/e}$.

1144. Show that $f(x) = x + 1/x$ has a local maximum and a local minimum, but the value at the local maximum is less than the value at the local minimum.

1145. Find the maximum profit that a company can make if the profit function is given by $p(x) = 41 + 24x - 18x^2$.

1146. A train is moving along the curve $y = x^2 + 2$. A girl is at the point $(3, 2)$. At what point will the train be at when the girl and the train are closest? Hint: You will have to solve a cubic equation, but the numbers have been chosen so there is an obvious root.

1147. Find the local maxima and minima of $f(x) = -x + 2\sin x$ in $[0, 2\pi]$.

1148. Divide 15 into two parts such that the square of one times the cube of the other is maximum.

1149. Suppose the sum of two numbers is fixed. Show that their product is maximum exactly when each one of them is half of the total sum.

1150. Divide a into two parts such that the pth power of one times the qth power of the other is maximum.

1151. Which number between 0 and 1 exceeds its pth power by the maximum amount?

1152. Find the dimensions of the rectangle of area 96 cm^2 which has minimum perimeter. What is this minimum perimeter?

1153. Show that the right circular cone with a given volume and minimum surface area has altitude equal to $\sqrt{2}$ times the radius of the base.

1154. Show that the altitude of the right circular cone with maximum volume that can be inscribed in a sphere of radius R is $4R/3$.

1155. Show that the height of a right circular cylinder with maximum volume that can be inscribed in a given right circular cone of height h is $h/3$.

1156. A cylindrical can is to be made to hold 1 liter of oil. Find the dimensions of the can which will minimize the cost of the metal to make the can.

1157. An open box is to be made out of a given quantity of cardboard of area p^2. Find the maximum volume of the box if its base is square.

1158. Find the dimensions of the maximum rectangular area that can be fenced with a fence 300 yards long.

1159. Show that the triangle of the greatest area with given base and vertical angle is isosceles.

1160. Show that a right triangle with a given perimeter has greatest area when it is isosceles.

1161. What do distance, speed and acceleration have to do with calculus? Explain thoroughly.

1162. A particle, starting from a fixed point P, moves in a straight line. Its position relative to P after t seconds is $s = 11 + 5t + t^3$ meters. Find the distance, velocity and acceleration of the particle after 4 seconds, and find the distance it travels during the 4th second.

1163. The displacement of a particle at time t is given by $x = 2t^3 - 5t^2 + 4t + 3$. Find (i) the time when the acceleration is 8cm/s^2, and (ii) the velocity and displacement at that instant.

1164. A particle moves along a straight line according to the law $s = t^3 - 6t^2 + 19t - 4$. Find (i) its displacement and acceleration when its velocity is 7m/s, and (ii) its displacement and velocity when its acceleration is 6m/s^2.

1165. A particle moves along a straight line so that after t seconds its position relative to a fixed point P on the line is s meters, where $s = t^3 - 4t^2 + 3t$. Find (i) when the particle is at P, and (ii) its velocity and acceleration at these times t.

1166. A particle moves along a straight line according to the law $s = at^2 - 2bt + c$, where a, b, c are constants. Prove that the acceleration of the particle is constant.

1167. The displacement of a particle moving in a straight line is $x = 2t^3 - 9t^2 + 12t + 1$ meters at time t. Find (i) the velocity and acceleration at $t = 1$ second, (ii) the time when the particle stops momentarily, and (iii) the distance between two stops.

1168. The distance s in meters travelled by a particle in t seconds is given by $s = ae^t + be^{-t}$. Show that the acceleration of the particle at time t is equal to the distance the particle travels in t seconds.

1169. A ladder 10 feet long rests against a vertical wall. If the bottom of the ladder slides away from the wall at a speed of 2 ft/s, how fast is the angle between the top of the ladder and the wall changing when the angle is $\pi/4$ radians?

1170. A ladder 13 meters long is leaning against a wall. The bottom of the ladder is pulled along the ground away from the wall at the rate of 2 m/s. How fast is its height on the wall decreasing when the foot of the ladder is 5 m away from the wall?

1171. A television camera is positioned 4000 ft from the base of a rocket launching pad. A rocket rises vertically and its speed is 600 ft/s when it has risen 3000 feet. (a) How fast is the distance from the television camera to the rocket changing at that moment? (b) How fast is the camera's angle of elevation changing at that same moment? (Assume that the telivision camera points toward the rocket.)

1172. Explain why exponential functions arise in computing radioactive decay.

1173. Explain why exponential functions are used as models for population growth.

1174. Radiocarbon dating works on the principle that ^{14}C decays according to radioactive decay with a half life of 5730 years. A parchment fragment was discovered that had about 74% as much ^{14}C as does plant material on earth today. Estimate the age of the parchment.

1175. If $f'(x) = x - 1/x^2$ and $f(1) = 1/2$ find $f(x)$.

1176. $\int (6x^5 - 2x^{-4} - 7x + 3/x - 5 + 4e^x + 7^x)\, dx$

1177. $\int (x/a + a/x + x^a + a^x + ax)\, dx$

1178. $\int \left(\sqrt{x} - \sqrt[3]{x^4} + \frac{7}{\sqrt[3]{x^2}} - 6e^x + 1\right) dx$

1179. $\int 2^x\, dx$

1180. $\int_{-2}^{4} (3x - 5)\, dx$

1181. $\int_{1}^{2} x^{-2}\, dx$

1182. $\int_{0}^{1} (1 - 2x - 3x^2)\, dx$

1183. $\int_{1}^{2} (5x^2 - 4x + 3)\, dx$

1184. $\int_{-3}^{0} (5y^4 - 6y^2 + 14)\, dy$

1185. $\int_{0}^{1} (y^9 - 2y^5 + 3y)\, dy$

1186. $\int_{0}^{4} \sqrt{x}\, dx$

1187. $\int_{0}^{1} x^{3/7}\, dx$

1188. $\int_{1}^{3} \left(\frac{1}{t^2} - \frac{1}{t^4}\right) dt$

1189. $\int_{1}^{2} \frac{t^6 - t^2}{t^4}\, dt$

1190. $\int_{1}^{2} \frac{x^2 + 1}{\sqrt{x}}\, dx$

1191. $\int_{0}^{2} (x^3 - 1)^2\, dx$

1192. $\displaystyle\int_0^1 u(\sqrt{u}+\sqrt[3]{u})\,du$

1193. $\displaystyle\int_1^2 (x+1/x)^2\,dx$

1194. $\displaystyle\int_3^3 \sqrt{x^5+2}\,dx$

1195. $\displaystyle\int_1^{-1} (x-1)(3x+2)\,dx$

1196. $\displaystyle\int_1^4 (\sqrt{t}-2/\sqrt{t})\,dt$

1197. $\displaystyle\int_1^8 \left(\sqrt[3]{r}+\frac{1}{\sqrt[3]{r}}\right)\,dr$

1198. $\displaystyle\int_{-1}^0 (x+1)^3\,dx$

1199. $\displaystyle\int_{-5}^{-2} \frac{x^4-1}{x^2+1}\,dx$

1200. $\displaystyle\int_1^e \frac{x^2+x+1}{x}\,dx$

1201. $\displaystyle\int_4^9 \left(\sqrt{x}+\frac{1}{\sqrt{x}}\right)^2\,dx$

1202. $\displaystyle\int_0^1 \left(\sqrt[4]{x^5}+\sqrt[5]{x^4}\right)\,dx$

1203. $\displaystyle\int_1^8 \frac{x-1}{\sqrt[3]{x^2}}\,dx$

1204. $\displaystyle\int_{\pi/4}^{\pi/3} \sin t\,dt$

1205. $\displaystyle\int_0^{\pi/2} (\cos\theta+2\sin\theta)\,d\theta$

1206. $\displaystyle\int_0^{\pi/2} (\cos\theta+\sin 2\theta)\,d\theta$

1207. $\displaystyle\int_{2\pi/3}^{\pi} \sec x\tan x\,dx$

1208. $\displaystyle\int_{\pi/3}^{\pi/2} \csc x\cot x\,dx$

1209. $\displaystyle\int_{\pi/6}^{\pi/3} \csc^2\theta\,d\theta$

1210. $\displaystyle\int_{\pi/4}^{\pi/3} \sec^2\theta\,d\theta$

1211. $\displaystyle\int_1^{\sqrt{3}} \frac{6}{1+x^2}\,dx$

1212. $\displaystyle\int_0^{0.5} \frac{dx}{\sqrt{1-x^2}}$

1213. $\displaystyle\int_4^8 (1/x)\,dx$

1214. $\displaystyle\int_{\ln 3}^{\ln 6} 8e^x\,dx$

1215. $\displaystyle\int_8^9 2^t\,dt$

1216. $\displaystyle\int_{-e^2}^{-e} \frac{3}{x}\,dx$

1217. $\displaystyle\int_{-2}^3 |x^2-1|\,dx$

1218. $\displaystyle\int_{-1}^2 |x-x^2|\,dx$

1219. $\displaystyle\int_{-1}^2 (x-2|x|)\,dx$

1220. $\displaystyle\int_0^2 (x^2-|x-1|)\,dx$

1221. $\displaystyle\int_0^2 f(x)\,dx$ where $f(x)=\begin{cases} x^4, & \text{if } 0\le x<1,\\ x^5, & \text{if } 1\le x\le 2\end{cases}.$

1222. $\displaystyle\int_{-\pi}^{\pi} f(x)\,dx$ where $f(x)=\begin{cases} x, & \text{if } -\pi\le x\le 0,\\ \sin x, & \text{if } 0<x\le \pi.\end{cases}$

1223. True or false? $\displaystyle\int_{-1}^1 \frac{3}{t^4}\,dt = \left.\frac{-1}{t^3}\right|_{-1}^{1} = -1+1=0.$

1224. Explain what a Riemann sum is and write the definition of $\displaystyle\int_a^b f(x)dx$ as a limit of Riemann sums.

1225. State the Fundamental Theorem of Calculus.

1226. (1) Water flows into a container at a rate of three gallons per minute for two minutes, five gallons per minute for seven minutes and eleven gallons per minute for two minutes. How much water is in the container? (2) Water flows into a container at a rate of t^2 gallons per minute for $0 \leq t \leq 5$. How much water is in the container?

1227. Let $f(x)$ be a function which is continuous and let $A(x)$ be the area under $f(x)$ from a to x. Compute the derivative of $A(x)$ by using limits.

1228. Why is the Fundamental Theorem of Calculus true? Explain carefully and thoroughly.

1229. Give an example which illustrates the Fundamental Theorem of Calculus. In order to do this compute an area by summing up the areas of narrow rectangles and then show that applying the Fundamental Theorem of Calculus gives the same answer.

1230. Sketch the graph of the curve $y = \sqrt{x+1}$ and determine the area of the region enclosed by the curve, the x-axis and the lines $x = 0$, $x = 4$.

1231. Make a sketch of the graph of the function $y = 4 - x^2$ and determine the area enclosed by the curve, the x-axis and the lines $x = 0$, $x = 2$.

1232. Find the area under the curve $y = \sqrt{6x+4}$ and above the x-axis between $x = 0$ and $x = 2$. Draw a sketch of the curve.

1233. Graph the curve $y = x^3$ and determine the area enclosed by the curve and the lines $y = 0$, $x = 2$ and $x = 4$.

1234. Graph the function $f(x) = 9 - x^2$, $0 \leq x \leq 3$, and determine the area enclosed between the curve and the x-axis.

1235. Graph the curve $y = 2\sqrt{1-x^2}$, $x \in [0, 1]$, and find the area enclosed between the curve and the x-axis.

1236. Determine the area under the curve $y = \sqrt{a^2 - x^2}$ and between the lines $x = 0$ and $x = a$.

1237. Graph the curve $y = 2\sqrt{9 - x^2}$ and determine the area enclosed between the curve and the x-axis.

1238. Graph the area between the curve $y^2 = 4x$ and the line $x = 3$. Find the area of this region.

1239. Find the area bounded by the curve $y = 4 - x^2$ and the lines $y = 0$ and $y = 3$.

1240. Find the area bounded by the curve $y = x(x-3)(x-5)$, the x-axis and the lines $x = 0$ and $x = 5$.

1241. Find the area enclosed between the curve $y = \sin 2x$, $0 \leq x \leq \pi/4$ and the axes.

1242. Find the area enclosed between the curve $y = \cos 2x$, $0 \leq x \leq \pi/4$ and the axes.

1243. Find the area enclosed between the curve $y = 3\cos x$, $0 \leq x \leq \pi/2$ and the axes.

1244. Find the area enclosed between the curve $y = \cos 3x$, $0 \leq x \leq \pi/6$ and the axes.

1245. Find the area enclosed between the curve $y = \tan^2 x$, $0 \leq x \leq \pi/4$ and the axes.

1246. Find the area enclosed between the curve $y = \csc^2 x$, $\pi/4 \leq x \leq \pi/2$ and the axes.

1247. Find the area of the region bounded by $y = -1$, $y = 2$, $x = y^3$, and $x = 0$.

1248. Find the area of the region bounded by the parabola $y = 4x^2$, $x \geq 0$, the y-axis, and the lines $y = 1$ and $y = 4$.

1249. Find the area of the region bounded by the curve $y = 4 - x^2$ and the lines $y = 0$ and $y = 3$.

1250. Graph $y^2 + 1 = x$, $x \le 2$ and find the area enclosed by the curve and the line $x = 2$.

1251. Graph the curve $y = x/\pi + 2\sin^2 x$ and write a definite integral whose value is the area between the x-axis, the curve and the lines $x = 0$ and $x = \pi$. *Do not* evaluate the integral. *Do* specify the limits of integration.

1252. Find the area bounded by $y = \sin x$ and the x-axis between $x = 0$ and $x = 2\pi$. Hint: Make a careful drawing to decide what area is intended.

1253. Find the area bounded by the curve $y = \cos x$ and the x-axis between $x = 0$ and $x = 2\pi$.

1254. Give an example which shows that $\int_a^b f(x)\,dx$ is not always the true area bounded by the curves $y = f(x)$, $y = 0$, $x = a$, and $x = b$ even though $f(x)$ is continuous between a and b.

1255. Find the area of the region bounded by the parabola $y^2 = 4x$ and the line $y = 2x$.

1256. Find the area bounded by the curve $y = x(2 - x)$ and the line $x = 2y$.

1257. Find the area bounded by the curve $x^2 = 4y$ and the line $x = 4y - 2$.

1258. Calculate the area of the region bounded by the parabolas $y = x^2$ and $x = y^2$.

1259. Find the area of the region included between the parabola $y^2 = x$ and the line $x + y = 2$.

1260. Find the area of the region bounded by the curves $y = \sqrt{x}$ and $y = x$.

1261. Use integration to find the area of the triangular region bounded by the lines $y = 2x + 1$, $y = 3x + 1$ and $x = 4$.

1262. Find the area bounded by the parabola $x^2 - 2 = y$ and the line $x + y = 0$.

1263. Graph the curve $y = (1/2)x^2 + 1$ and the straight line $y = x + 1$ and find the area between the curve and the line.

1264. Find the area of the region between the parabolas $y^2 = x$ and $x^2 = 16y$.

1265. Find the area of the region enclosed by the parabola $y^2 = 4ax$ and the line $y = mx$.

1266. Find a so that the curves $y = x^2$ and $y = a\cos x$ intersect at the points $(x, y) = (\frac{\pi}{4}, \frac{\pi^2}{16})$. Then find the area between these curves.

1267. Write a definite integral whose value is the area of the region between the two circles $x^2 + y^2 = 1$ and $(x - 1)^2 + y^2 = 1$. Find this area. If you cannot evaluate the integral by calculus you may use geometry to find the area. Hint: The part of a circle cut off by a line is a circular sector with a triangle removed.

1268. Write a definite integral whose value is the area of the region between the circles $x^2 + y^2 = 4$ and $(x - 2)^2 + y^2 = 4$. *Do not* evaluate the integral. *Do* specify the limits of integration.

1269. Write a definite integral whose value is the area of the region between the curves $x^2 + y^2 = 2$ and $x = y^2$ *Do not* evaluate the integral. *Do* specify the limits of integration.

1270. Write a definite integral whose value is the area of the region between the curves $x^2 + y^2 = 2$ and $x = y^2$. Find this area. If you cannot evaluate the integral by calculus you may use geometry to find the area. Hint: Divide the region into two parts.

1271. Write a definite integral whose value is the area of the part of the first quadrant which is between the parabola $y^2 = x$ and the circle $x^2 + y^2 - 2x = 0$. Find this area. If you cannot evaluate the integral by calculus you may use geometry to find the area. Hint: Draw a careful graph. Divide a semicircle in two.

1272. Find the area bounded by the curves $y = x$ and $y = x^3$.

1273. Graph $y = \sin x$ and $y = \cos x$ for $0 \le x \le \pi/2$ and find the area enclosed by them and the x-axis.

1274. Write a definite integral whose value is the area inside the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$. Evaluate this area. Hint: After a suitable change of variable, the definite integral becomes the definite integral whose value is the area of a circle.

1275. Using integration find the area of the triangle with vertices $(-1, 1)$, $(0, 5)$ and $(3, 2)$.

1276. Find the volume that results by rotating the triangle $1 \le x \le 2$, $0 \le y \le 3x - 3$ around the x axis.

1277. Find the volume that results by rotating the semicircle $y = \sqrt{R^2 - x^2}$ about the x-axis.

1278. A triangle is formed by drawing lines from the two endpoints of a line segment of length b to a vertex V which is at a height h above the line of the line segment. Its area is then $A = \int_{y=0}^{h} dA$ where dA is the area of the strip cut out by two parallel lines separated by a distance of dz and at a height of z above the line containing the line segment. Find a formula for dA in terms of b, z, and dz and evaluate the definite integral.

1279. A pyramid is formed by drawing lines from the four vertices of a rectangle of area A to an apex P which is at a height h above the plane of the rectangle. Its volume is then $V = \int_{z=0}^{h} dV$ where dV is the volume of the slice cut out by two planes parallel to the plane of the rectangle and separated by a distance of dz and at a height of z above the plane of the rectangle. Find a formula for dV in terms of A, z, and dz and evaluate the definite integral.

1280. A tetrahedron is formed by drawing lines from the three vertices of a triangle of area A to an apex P which is at a height h above the plane of the triangle. Its volume is then $V = \int_{z=0}^{h} dV$ where dV is the volume of the slice cut out by two planes parallel to the plane of the triangle and separated by a distance of dz and at a height of z above the plane of the rectangle. Find a formula for dV in terms of A, z, and dz and evaluate the definite integral.

1281. A cone is formed by drawing lines from the perimeter of a circle of area A to an apex V which is at a height h above the plane of the circle. Its volume is then $V = \int_{z=0}^{h} dV$ where dV is the volume of the slice cut out by two planes parallel to the plane of the circle and separated by a distance of dz and at a height of z above the plane of the rectangle. Find a formula for dV in terms of A, z, and dz and evaluate the definite integral.

1282. (a) A hemispherical bowl of radius a contains water to a depth h. Find the volume of the water in the bowl. (b) Water runs into a hemispherical bowl of radius 5 ft at the rate of 0.2 ft^3/sec. How fast is the water level rising when the water is 4 ft deep?

1283. (Alternate wording for previous problem.) A hemispherical bowl is obtained by rotating the semicircle $x^2 + (y - a)^2 = a^2$, $y \le a$ about the y-axis. It is filled with water to a depth of h, i.e. the water level is the line $y = h$. (a) Find the volume of the water in the bowl as a function of h. (b) Water runs into a hemispherical bowl of radius 5 ft at the rate of 0.2 ft^3/sec. How fast is the water level rising when the water is 4 ft deep? (Hint: Use the method of related rates and the Fundamental Theorem.)

1284. A vase is constructed by rotating the curve $x^3 - y^3 = 1$ for $0 \le y \le 8$ around the y axis. It is filled with water to a height $y = h$ where $h < 8$. (a) Find the volume of the water in terms of h. (Express your answer as a definite integral. Do not try to evaluate the integral.) (b) If the vase is filling with water at the rate of 2 cubic units per second, how fast is the height of the water increasing when this height is 2 units?

1285. STEP 3, 1987 specimen paper, Q11

Two identical snowploughs plough the same stretch of road in the same direction. The first starts at time $t = 0$ when the depth of snow is h metres, and the second starts from the same point T seconds later. Snow falls so that the depth of snow increases at a constant rate of k ms^{-1}. It may be assumed that each snowplough moves at a speed equal to $k/(\alpha z)$ ms^{-1} with α a constant, where z is the depth of snow it is ploughing, and that it clears all the snow.

- Show that the time taken for the first snowplough to travel x metres is

$$(\mathrm{e}^{\alpha x} - 1)\frac{h}{k} \text{ seconds.}$$

- Show that at time $t > T$, the second snowplough has moved y metres, where t satisfies

$$\frac{1}{\alpha}\frac{\mathrm{d}t}{\mathrm{d}y} = t - (\mathrm{e}^{\alpha y} - 1)\,\frac{h}{k}.$$

  Verify that the required solution of this equation is

$$t = (\mathrm{e}^{\alpha y} - 1)\,\frac{h}{k} + \left(T - \frac{\alpha h y}{k}\right)\mathrm{e}^{\alpha y}$$

  and deduce that the snowploughs collide when they have moved a distance $kT/(\alpha h)$ metres.

1286. STEP 3, 1987 specimen paper, Q12

One end A of a uniform straight rod AB of mass M and length L rests against a smooth vertical wall. The other end B is attached to a light inextensible string BC of length αL which is fixed to the wall at a point C vertically above A. The rod is in equilibrium with the points A, B and C not collinear. Determine the inclination of the rod to the vertical and the set of possible values of α.

Show that the tension in the string is

$$\frac{Mg\alpha}{2}\left(\frac{3}{\alpha^2 - 1}\right)^{\frac{1}{2}}.$$

1287. STEP 3, 1987 specimen paper, Q13

A particle of mass m is attached to a light circular hoop of radius a which is free to roll in a vertical plane on a rough horizontal table. Initially the hoop stands with the particle at its highest point and is then displaced slightly. Show that while the hoop is rolling on the table, the speed v of the particle when the radius to the particle makes an angle 2θ with the upward vertical is given by

$$v = 2(ga)^{\frac{1}{2}} \sin\theta.$$

Write down expressions in terms of θ for x, the horizontal displacement of the particle from its initial position, and y, its height above the table, and use them to show that

$$\dot\theta = \tfrac{1}{2}(g/a)^{\frac{1}{2}} \tan\theta$$

and

$$\ddot{y} = -2g\sin^2\theta.$$

By considering the reaction of the table on the hoop, or otherwise, describe what happens to prevent the hoop rolling beyond the position for which $\theta = \pi/4$.

1288. STEP 3, 1987 specimen paper, Q14

A uniform straight rod of mass m and length $4a$ can rotate freely about its midpoint on a smooth horizontal table. Initially the rod is at rest. A particle of mass m travelling on the table with speed u at right angles to the rod collides perfectly elastically with the rod at a distance a from the centre of the rod. Show that the angular speed, ω, of the rod after the collision is given by

$$a\omega = 6u/7.$$

Show also that the particle and rod undergo a subsequent collision.

1289. STEP 3, 2018, Q9

A particle P of mass m is projected with speed u_0 along a smooth horizontal floor directly towards a wall. It collides with a particle Q of mass km which is moving directly away from the wall with speed v_0. In the subsequent motion, Q collides alternately with the wall and with P. The coefficient of restitution between Q and P is e, and the coefficient of restitution between Q and the wall is 1.

Let u_n and v_n be the velocities of P and Q, respectively, towards the wall after the nth collision between P and Q.

- Show that, for $n \geq 2$,

$$(1+k)u_n - (1-k)(1+e)u_{n-1} + e(1+k)u_{n-2} = 0\,. \qquad (*)$$

- You are now given that $e = \frac{1}{2}$ and $k = \frac{1}{34}$, and that the solution of $(*)$ is of the form

$$u_n = A\left(\tfrac{7}{10}\right)^n + B\left(\tfrac{5}{7}\right)^n \qquad (n \geq 0)\,,$$

  where A and B are independent of n. Find expressions for A and B in terms of u_0 and v_0.

  Show that, if $0 < 6u_0 < v_0$, then u_n will be negative for large n.

1290. STEP 3, 2018, Q10

A uniform disc with centre O and radius a is suspended from a point A on its circumference, so that it can swing freely about a horizontal axis L through A. The plane of the disc is perpendicular to L. A particle P is attached to a point on the circumference of the disc. The mass of the disc is M and the mass of the particle is m.

In equilibrium, the disc hangs with OP horizontal, and the angle between AO and the downward vertical through A is β. Find $\sin\beta$ in terms of M and m and show that

$$\frac{AP}{a} = \sqrt{\frac{2M}{M+m}}\,.$$

The disc is rotated about L and then released. At later time t, the angle between OP and the horizontal is θ; when P is higher than O, θ is positive and when P is lower than O, θ is negative. Show that

$$\tfrac{1}{2}I\dot{\theta}^2 + (1-\sin\beta)ma^2\dot{\theta}^2 + (m+M)ga\cos\beta\,(1-\cos\theta)$$

is constant during the motion, where I is the moment of inertia of the disc about L.

Given that $m = \frac{3}{2}M$ and that $I = \frac{3}{2}Ma^2$, show that the period of small oscillations is

$$3\pi\sqrt{\frac{3a}{5g}}\,.$$

1291. STEP 3, 2018, Q11

A particle is attached to one end of a light inextensible string of length b. The other end of the string is attached to a fixed point O. Initially the particle hangs vertically below O. The particle then receives a horizontal impulse.

The particle moves in a circular arc with the string taut until the acute angle between the string and the upward vertical is α, at which time it becomes slack. Express V, the speed of the particle when the string becomes slack, in terms of b, g and α.

Show that the string becomes taut again a time T later, where

$$gT = 4V\sin\alpha\,,$$

and that just before this time the trajectory of the particle makes an angle β with the horizontal where $\tan\beta = 3\tan\alpha$.

When the string becomes taut, the momentum of the particle in the direction of the string is destroyed. Show that the particle comes instantaneously to rest at this time if and only if

$$\sin^2\alpha = \frac{1+\sqrt{3}}{4}\,.$$

Appendix A

Answers to selected exercises

(1) The decimal expansion of

$$1/7 = 0.\overline{142857}\,142857\,142857\,\cdots$$

repeats after 6 digits. Since $2007 = 334 \times 6 + 3$ the 2007^{th} digit is the same as the 3^{rd}, which happens to be a 2.

(6) $100x = 31.313131\cdots = 31 + x \implies 99x = 31 \implies x = \frac{31}{99}$.

Similarly, $y = \frac{273}{999}$.

In z the initial "2" is not part of the repeating pattern, so subtract it: $z = 0.2 + 0.0154154154\cdots$. Now

$$\begin{aligned}
1000\times 0.0154154154\cdots &= 15.4154154154\cdots \\
&= 15.4 + 0.0154154154\cdots \\
&= 15\tfrac{2}{5} + 0.0154154154\cdots \\
\implies 0.0154154\cdots &= \frac{15\frac{2}{5}}{999}.
\end{aligned}$$

From this you get

$$z = \tfrac{1}{5} + \frac{15\frac{2}{5}}{999} = \frac{1076}{4995}.$$

(8) They are the same function. Both are defined for all real numbers, and both will square whatever number you give them, so they are the same function.

(14) Both are false:

(a) Since $\arcsin x$ is only defined if $-1 \le x \le 1$ and hence not for ***all*** x, it is not true that $\sin\bigl(\arcsin x\bigr) = x$ for all x. However, it is true that $\sin(\arcsin x) = x$ for all x in the interval $[-1, 1]$.

(b) $\arcsin(\sin x)$ is defined for all x since $\sin x$ is defined for all x, and $\sin x$ is always between -1 and 1. However the arcsine function always returns a number (angle) between $-\pi/2$ and $\pi/2$, so $\arcsin(\sin x) = x$ can't be true when $x > \pi/2$ or $x < -\pi/2$.

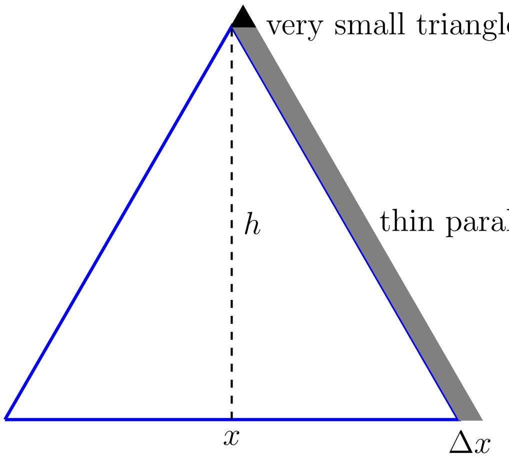
(30) **(a)**

$$\begin{aligned}\Delta y &= (x+\Delta x)^2 - 2(x+\Delta x) + 1 - [x^2 - 2x + 1]\\ &= (2x-2)\Delta x + (\Delta x)^2 \text{ so that}\\ \frac{\Delta y}{\Delta x} &= 2x - 2 + \Delta x\end{aligned}$$

(31) Δx : feet. Δy pounds. $\frac{\Delta y}{\Delta x}$ and $\frac{dy}{dx}$ are measured in pounds per feet.

(32) Gallons per second.

(33) (a) $A(x)$ is an area so it has units square inch and x is measured in inches, so $\dfrac{dA}{dx}$ is measured in $\dfrac{\text{inch}^2}{\text{inch}} = \text{inch}$.



(b) Hint: The extra area ΔA that you get when the side of an equilateral triangle grows from x to $x + \Delta x$ can be split into a thin parallelogram and a very tiny triangle. Ignore the area of the tiny triangle since the area of the parallelogram will be much larger. What is the area of this parallelogram?

The area of a parallelogram is "base time height" so here it is $h \times \Delta x$, where h is the height of the triangle.

Conclusion: $\dfrac{\Delta A}{\Delta x} \approx \dfrac{h\Delta x}{\Delta x} = h$.

The derivative is therefore the height of the triangle.

Note that an equilateral triangle with side of length x has height $h = \frac{\sqrt{3}}{2}x$ and area $A = \frac{1}{2}xh = \frac{\sqrt{3}}{4}x^2$. and so $\frac{dA}{dx} = \frac{\sqrt{3}}{2}x = h$ as the geometry demonstrates.

(38) $\delta = \varepsilon/2$.

(39) $\delta = \min\{1, \frac{1}{6}\varepsilon\}$

(40) $|f(x) - (-7)| = |x^2 - 7x + 10| = |x-2| \cdot |x-5|$. If you choose $\delta \le 1$ then $|x-2| < \delta$ implies $1 < x < 3$, so that $|x-5|$ is at most $|1-5| = 4$.

So, choosing $\delta \le 1$ we always have $|f(x) - L| < 4|x-2|$ and $|f(x) - L| < \varepsilon$ will follow from $|x-2| < \frac{1}{4}\varepsilon$.

Our choice is then: $\delta = \min\{1, \frac{1}{4}\varepsilon\}$.

(41) $f(x) = x^3$, $a = 3$, $L = 27$.

When $x = 3$ one has $x^3 = 27$, so $x^3 - 27 = 0$ for $x = 3$. Therefore you can factor out $x - 3$ from $x^3 - 27$ by doing a long division. You get $x^3 - 27 = (x - 3)(x^2 + 3x + 9)$, and thus

$$|f(x) - L| = |x^3 - 27| = |x^2 + 3x + 9| \cdot |x - 3|.$$

Never choose $\delta > 1$. Then $|x - 3| < \delta$ will imply $2 < x < 4$ and therefore

$$|x^2 + 3x + 9| \leq 4^2 + 3 \cdot 4 + 9 = 37.$$

So if we always choose $\delta \leq 1$, then we will always have

$$|x^3 - 27| \leq 37\delta \quad \text{for} \quad |x - 3| < \delta.$$

Hence, if we choose $\delta = \min\left\{1, \frac{1}{37}\varepsilon\right\}$ then $|x - 3| < \delta$ guarantees $|x^3 - 27| < \varepsilon$.

(43) $f(x) = \sqrt{x}$, $a = 4$, $L = 2$.

You have

$$\sqrt{x} - 2 = \frac{(\sqrt{x} - 2)(\sqrt{x} + 2)}{\sqrt{x} + 2} = \frac{x - 4}{\sqrt{x} + 2}$$

and therefore

$$|f(x) - L| = \frac{1}{\sqrt{x} + 2}|x - 4|. \tag{A.1}$$

Once again it would be nice if we could replace $1/(\sqrt{x} + 2)$ by a constant, and we achieve this by always choosing $\delta \leq 1$. If we do that then for $|x - 4| < \delta$ we always have $3 < x < 5$ and hence

$$\frac{1}{\sqrt{x} + 2} < \frac{1}{\sqrt{3} + 2},$$

since $1/(\sqrt{x} + 2)$ increases as you decrease x.

So, if we always choose $\delta \leq 1$ then $|x - 4| < \delta$ guarantees

$$|f(x) - 2| < \frac{1}{\sqrt{3} + 2}|x - 4|,$$

which prompts us to choose $\delta = \min\left\{1, (\sqrt{3} + 2)\varepsilon\right\}$.

A smarter solution: We ***can*** replace $1/(\sqrt{x} + 2)$ by a constant in (A.1), because for all x in the domain of f we have $\sqrt{x} \geq 0$, which implies

$$\frac{1}{\sqrt{x} + 2} \leq \frac{1}{2}.$$

Therefore $|\sqrt{x} - 2| \leq \frac{1}{2}|x - 4|$, and we could choose $\delta = 2\varepsilon$.

(44) Hints:

$$\sqrt{x + 6} - 3 = \frac{x + 6 - 9}{\sqrt{x + 6} + 3} = \frac{x - 3}{\sqrt{x + 6} + 3}$$

so

$$|\sqrt{x + 6} - 3| \leq \tfrac{1}{3}|x - 3|.$$

(45) We have

$$\left|\frac{1+x}{4+x} - \frac{1}{2}\right| = \left|\frac{x-2}{4+x}\right|.$$

If we choose $\delta \le 1$ then $|x-2| < \delta$ implies $1 < x < 3$ so that

$$\frac{1}{7} < \text{we don't care} \frac{1}{4+x} < \frac{1}{5}.$$

Therefore

$$\left|\frac{x-2}{4+x}\right| < \tfrac{1}{5}|x-2|,$$

so if we want $|f(x) - \frac{1}{2}| < \varepsilon$ then we must require $|x-2| < 5\varepsilon$. This leads us to choose

$$\delta = \min\{1, 5\varepsilon\}.$$

(50) The equation (2.5) already contains a function f, but that is not the right function. In (2.5) Δx is the variable, and $g(\Delta x) = (f(x+\Delta x) - f(x))/\Delta x$ is the function; we want $\lim_{\Delta x \to 0} g(\Delta x)$.

(66) False! The limit must not only exist *but also be equal to* $f(a)$!

(67) There are of course many examples. Here are two: $f(x) = 1/x$ and $f(x) = \sin(\pi/x)$ (see §3.6.3)

(68) False! Here's an example: $f(x) = \frac{1}{x}$ and $g(x) = x - \frac{1}{x}$. Then f and g don't have limits at $x = 0$, but $f(x) + g(x) = x$ *does* have a limit as $x \to 0$.

(69) False again, as shown by the example $f(x) = g(x) = \frac{1}{x}$.

(77) $\sin 2\alpha = 2\sin\alpha\cos\alpha$ so the limit is $\lim_{\alpha\to 0} \frac{2\sin\alpha\cos\alpha}{\sin\alpha} = \lim_{\alpha\to 0} 2\cos\alpha = 2$.

Other approach: $\dfrac{\sin 2\alpha}{\sin\alpha} = \dfrac{\frac{\sin 2\alpha}{2\alpha}}{\frac{\sin\alpha}{\alpha}} \cdot \dfrac{2\alpha}{\alpha}$. Take the limit and you get 2.

(78) $\frac{3}{2}$.

(79) Hint: $\tan\theta = \frac{\sin\theta}{\cos\theta}$. Answer: the limit is 1.

(80) $\frac{\tan 4\alpha}{\sin 2\alpha} = \frac{\tan 4\alpha}{4\alpha} \cdot \frac{2\alpha}{\sin 2\alpha} \cdot \frac{4\alpha}{2\alpha} = 1 \cdot 1 \cdot 2 = 2$

(81) Hint: multiply top and bottom with $1 + \cos x$.

(82) Hint: substitute $\theta = \frac{\pi}{2} - \varphi$, and let $\varphi \to 0$. Answer: -1.

(88) Substitute $\theta = x - \pi/2$ and remember that $\cos x = \cos(\theta + \frac{\pi}{2}) = -\sin\theta$. You get

$$\lim_{x\to\pi/2} \frac{x - \frac{\pi}{2}}{\cos x} = \lim_{\theta\to 0} \frac{\theta}{-\sin\theta} = -1.$$

(89) Similar to the previous problem, once you use $\tan x = \frac{\sin x}{\cos x}$. The answer is again -1.

(91) Substitute $\theta = x - \pi$. Then $\lim_{x\to\pi} \theta = 0$, so

$$\lim_{x\to\pi} \frac{\sin x}{x - \pi} = \lim_{\theta\to 0} \frac{\sin(\pi+\theta)}{\theta} = -\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = -1.$$

Here you have to remember from trigonometry that $\sin(\pi + \theta) = -\sin\theta$.

(93) Note that the limit is for $x \to \infty$! As x goes to infinity $\sin x$ oscillates up and down between -1 and $+1$. Dividing by x then gives you a quantity which goes to zero. To give a good proof you use the Sandwich Theorem like this:

Since $-1 \le \sin x \le 1$ for all x you have

$$\frac{-1}{x} \le \frac{\sin x}{x} \le \frac{1}{x}.$$

Since both $-1/x$ and $1/x$ go to zero as $x \to \infty$ the function in the middle must also go to zero. Hence

$$\lim_{x\to\infty} \frac{\sin x}{x} = 0.$$

(95) No. As $x \to 0$ the quantity $\sin \frac{1}{x}$ oscillates between -1 and $+1$ and does not converge to any particular value. Therefore, no matter how you choose k, it will never be true that $\lim_{x\to 0} \sin \frac{1}{x} = k$, because the limit doesn't exist.

(96) The function $f(x) = (\sin x)/x$ is continuous at all $x \neq 0$, so we only have to check that $\lim_{x\to 0} f(x) = f(0)$, i.e. $\lim_{x\to 0} \frac{\sin x}{2x} = A$. This only happens if you choose $A = \frac{1}{2}$.

(151) $f'(x) = 2\tan x/\cos^2 x$ and $f''(x) = 2/\cos^4 x + 4\tan x \sin x/\cos^3 x$.

Since $\tan^2 x = \frac{1}{\cos^2 x} - 1$ one has $g'(x) = f'(x)$ and $g''(x) = f''(x)$.

(157) $f'(x) = 2\cos 2x + 3\sin 3x$.

(158) $f'(x) = -\dfrac{\pi}{x^2} \cos \dfrac{\pi}{x}$

(159) $f'(x) = \cos(\cos 3x) \cdot (-\sin 3x) \cdot 3 = -3\sin 3x \cos\cos 3x$.

(160) $f'(x) = \dfrac{x^2 \cdot 2x \sin x^2 - 2x \sin x^2}{x^4}$

(161) $f'(x) = \dfrac{1}{\left(\cos\sqrt{1+x^2}\right)^2} \dfrac{1}{2\sqrt{1+x^2}} \cdot 2x$

(162) $f'(x) = 2(\cos x)(-\sin x) - 2(\cos x) \cdot 2x$

(164) $f'(-\frac{2}{3}) = -\frac{3}{2}\pi$.

(165) $v(x) = f(g(x)) = (x+5)^2 + 1 = x^2 + 10\,x + 26$
$w(x) = g(f(x)) = (x^2+1) + 5 = x^2 + 6$
$p(x) = f(x)g(x) = (x^2+1)(x+5) = x^3 + 5x^2 + x + 5$
$q(x) = g(x)f(x) = f(x)g(x) = p(x)$.

(168) **(a)** If $f(x) = \sin ax$, then $f''(x) = -a^2 \sin ax$, so $f''(x) = -64 f(x)$ holds if $a^2 = 64$, i.e. $a = \pm 8$. So $\sin 8x$ and $\sin(-8x) = -\sin 8x$ are the two solutions you find this way.

(b) $a = \pm 8$, but A and b can have any value. All functions of the form $f(x) = A\sin(8x+b)$ satisfy (†).

(169) **(a)** $V = S^3$, so the function f for which $V(t) = f(S(t))$ is the function $f(x) = x^3$.

(b) $S'(t)$ is the rate with which Bob's side grows with time. $V'(t)$ is the rate with which the Bob's volume grows with time.

Quantity	Units
t	minutes
$S(t)$	inch
$V(t)$	inch^3
$S'(t)$	inch/minute
$V'(t)$	inch^3/minute

(c) Three versions of the same answer:

$V(t) = f(S(t))$ so the chain rule says $V'(t) = f'(S(t))S'(t)$

$V(t) = S(t)^3$ so the chain rule says $V'(t) = 3S(t)^2 S'(t)$

$V = S^3$ so the chain rule says $\dfrac{dV}{dt} = 3S^2\dfrac{dS}{dt}$.

(d) We are given $V(t) = 8$, and $V'(t) = 2$. Since $V = S^3$ we get $S = 2$. From (c) we know $V'(t) = 3S(t)^2 S'(t)$, so $2 = 3 \cdot 2^2 \cdot S'(t)$, whence $S'(t) = \frac{1}{6}$ inch per minute.

(211) At $x = 3$.

(212) At $x = a/2$.

(213) At $x = a + 2a^3$.

(214) At $x = a + \frac{1}{2}$.

(218) False. If you try to solve $f(x) = 0$, then you get the equation $\frac{x^2+|x|}{x} = 0$. If $x \neq 0$ then this is the same as $x^2 + |x| = 0$, which has no solutions (both terms are positive when $x \neq 0$). If $x = 0$ then $f(x)$ isn't even defined. So there is no solution to $f(x) = 0$.

This doesn't contradict the IVT, because the function isn't continuous, in fact it isn't even defined at $x = 0$, so the IVT doesn't have to apply.

(224) Not necessarily true, and therefore false. Consider the example $f(x) = x^4$, and see the next problem.

(225) An inflection point is a point on the graph of a function where the second derivative changes its sign. At such a point you must have $f''(x) = 0$, but by itself that it is no enough.

(228) The first is possible, e.g. $f(x) = x$ satisfies $f'(x) > 0$ and $f''(x) = 0$ for all x.

The second is impossible, since f'' is the derivative of f', so $f'(x) = 0$ for all x implies that $f''(x) = 0$ for all x.

(229) $y = 0$ at $x = -1, 0, 0$. Only sign change at $x = -1$, not at $x = 0$.

$x = 0$ loc min; $x = -\frac{4}{3}$ loc max; $x = -2/3$ inflection point. No global max or min.

(230) zero at $x = 0, 4$; sign change at $x = 4$; loc min at $x = \frac{8}{3}$; loc max at $x = 0$; inflection point at $x = 4/3$. No global max or min.

(231) sign changes at $x = 0, -3$; global min at $x = -3/4^{1/3}$; no inflection poitns, the graph is convex.

(232) mirror image of previous problem.

(233) $x^4+2x^2-3=\big(x^2-1\big)\big(x^2+3\big)$ so sign changes at $x=\pm 1$. Global min at $x=0$; graph is convex, no inflection points.

(234) Sign changes at $\pm 2, \pm 1$; ***two*** global minima, at $\pm\sqrt{5/2}$; one local max at x=0; two inflection points, at $x=\pm\sqrt{5/6}$.

(235) Sign change at $x=0$; function is always increasing so no stationary points; inflection point at $x=0$.

(236) sign change at $x=0,\pm 2$; loc max at $x=2/5^{1/4}$; loc min at $x=-2/5^{1/4}$. inflection point at $x=0$.

(237) Function not defined at $x=-1$. For $x>-1$ sign change at $x=0$, no stationary points, no inflection points (graph is concave). Horizontal asymptote $\lim_{x\to\infty} f(x)=1$.

For $x<-1$ no sign change , function is increasing and convex, horizontal asymptote with $\lim_{x\to-\infty} f(x)=1$.

(238) global max (min) at $x=1$ (x=-1), inflection points at $x=\pm\sqrt{3}$; horizontal asymptotes $\lim_{x\to\pm\infty} f(x)=0$.

(239) $y=0$ at $x=0$ but no sign changes anywhere; $x=0$ is a global min; there's no local or global max; two inflection points at $x=\pm\frac{1}{3}\sqrt{3}$; horizontal asymptotes at height $y=1$.

(240) Not defined at $x=-1$. For $x>-1$ the graph is convex and has a minimum at $x=-1+\sqrt{2}$; for $x<-1$ the graph is concave with a maximum at $x=-1-\sqrt{2}$. No horizontal aymptotes.

(241) Not def'd at $x=0$. No sign changes (except at $x=0$). For $x>0$ convex with minimum at $x=1$, for $x<0$ concave with maximum at $x=-1$.

(242) Not def'd at $x=0$. Sign changes at $x=\pm 1$ and also at $x=0$. No stationary points. Both branches ($x>0$ and $x<0$) are increasing. Non inflection points, no horizontal asymptotes.

(243) Zero at $x=0,-1$ sign only changes at -1; loc min at $x=-\frac{1}{3}$; loc max at $x=-1$. Inflection point at $x=-2/3$.

(244) Changes sign at $x=-1\pm\sqrt{2}$ and $x=0$; loc min at $(-2+\sqrt{7})/3$, loc max at $(-2-\sqrt{7})/3$; inflection point at $x=-\frac{2}{3}$.

(245) Factor $y=x^4-x^3-x=x(x^3-x^2-1)$. One zero is obvious, namely at $x=0$. For the other(s) you must solve $x^3-x^2-1=0$ which is beyond what's expected in this course.
The derivative is $y'=4x^3-3x^2-1$. A cubic function whose coefficients add up to 0 so $x=1$ is a root, and you can factor $y'=4x^3-3x^2-1=(x-1)(4x^2+x+1)$ from which you see that $x=1$ is the only root. So: one stationary point at $x=1$, which is a global minimum
The second derivative is $y''=12x^2-6x$; there are two inflection points, at $x=\frac{1}{2}$ and at $x=0$.

(246) Again one obvious solution to $y=0$, namely $x=0$. The other require solving a cubic equation.
The derivative is $y'=4x^3-6x^2+2$ which is also cubic, but the coefficients add up to 0, so $x=1$ is a root. You can then factor $y'=4x^3-6x^2+2=(x-1)(4x^2-2x-2)$. There are three stationary points: local minima at $x=1$, $x=-\frac{1}{4}-\frac{1}{2}\sqrt{3}$, local max at $x=-\frac{1}{4}+\frac{1}{2}\sqrt{3}$. one of the two loc min is a global minimum.

(247) Global min at $x = 0$, no other stationary points; function is convex, no inflection points. No horizontal asymptotes.

(248) The graph is the upper half of the unit circle.

(249) Always positive, so no sign changes; global minimum at $x = 0$, no other stationary points; two inflection points at $\pm\sqrt{2}$. No horizontal asymptotes since $\lim_{x\to\pm\infty} \sqrt[4]{1+x^2} = \infty$(DNE).

(250) Always positive hence no sign changes; global max at $x = 0$, no other stationary points; two inflection points at $x = \pm\sqrt[4]{3/5}$; second derivative also vanishes at $x = 0$ but this is not an inflection point.

(252) Zeroes at $x = 3\pi/4$, $7\pi/4$. Absolute max at $x = \pi/4$, abs min at $x = 5\pi/4$. Inflection points and zeroes coincide. Note that $\sin x + \cos x = \sqrt{2}\sin(x + \frac{\pi}{4})$.

(253) Zeroes at $x = 0, \pi, 3\pi/2$ but no sign change at $3\pi/2$. Global max at $x = \pi/2$, local max at $x = 3\pi/2$, global min at $x = 7\pi/6, 11\pi/6$.

(271) If the length of one side is x and the other y, then the perimeter is $2x + 2y = 1$, so $y = \frac{1}{2} - x$. Thus the area enclosed is $A(x) = x(\frac{1}{2} - x)$, and we're only interested in values of x between 0 and $\frac{1}{2}$.

The maximal area occurs when $x = \frac{1}{4}$ (and it is $A(1/4) = 1/16$.) The minimal area occurs when either $x = 0$ or $x = 1/2$. In either case the "rectangle" is a line segment of length $\frac{1}{2}$ and width 0, or the other way around. So the minimal area is 0.

(272) If the sides are x and y, then the area is $xy = 100$, so $y = 1/x$. Therefore the height plus twice the width is $f(x) = x + 2y = x + 2/x$. This is extremal when $f'(x) = 0$, i.e. when $f'(x) = 1 - 2/x^2 = 0$. This happens for $x = \sqrt{2}$.

(273) Perimeter is $2R + R\theta = 1$ (given), so if you choose the angle to be θ then the radius is $R = 1/(2+\theta)$. The area is then $A(\theta) = \theta R^2 = \theta/(2+\theta)^2$, which is maximal when $\theta = 2$ (radians). The smallest area arises when you choose $\theta = 0$. Choosing $\theta \geq 2\pi$ doesn't make sense (why? Draw the corresponding wedge!)

You could also say that for any given radius $R > 0$ "perimeter = 1" implies that one has $\theta = (1/R) - 2$. Hence the area will be $A(R) = \theta R^2 = R^2\big((1/R) - 2)\big) = R - 2R^2$. Thus the area is maximal when $R = \frac{1}{4}$, and hence $\theta =$ 2radians. Again we note that this answer is reasonable because values of $\theta > 2\pi$ don't make sense, but $\theta = 2$ does.

(274) **(a)** The intensity at x is a function of x. Let's call it $I(x)$. Then at x the distance to the big light is x, and the distance to the smaller light is $1000 - x$. Therefore

$$I(x) = \frac{1000}{x^2} + \frac{125}{(1000 - x)^2}$$

(b) Find the minimum of $I(x)$ for $0 < x < 1000$.

$$I'(x) = -2000x^{-3} + 250(1000 - x)^{-3}.$$

$I'(x) = 0$ has one solution, namely, $x = \frac{1000}{3}$. By looking at the signs of $I'(x)$ you see that $I(x)$ must have a minimum. If you don't like looking at signs, you could instead look at the second derivative

$$I''(x) = 6000x^{-4} + 750(1000 - x)^{-4}$$

which is always positive.

(275) $r = \sqrt{50/3\pi}$, $h = 100/(3\pi r) = 100/\sqrt{150\pi}$.

(309) $dy/dx = e^x - 2e^{-2x}$. Local min at $x = \frac{1}{3}\ln 2$.

$d^2y/dx^2 = e^x + 4e^{-2x} > 0$ always, so the function is convex.

$\lim_{x\to\pm\infty} y = \infty$, no asymptotes.

(310) $dy/dx = 3e^{3x} - 4e^x$. Local min at $x = \frac{1}{2}\ln\frac{4}{3}$.

$d^2y/dx^2 = 9e^{3x} - 4e^x$ changes sign when $e^{2x} = \frac{4}{9}$, i.e. at $x = \frac{1}{2}\ln\frac{4}{9} = \ln\frac{2}{3} = \ln 2 - \ln 3$. Inflection point at $x = \ln 2 - \ln 3$.

$\lim_{x\to-\infty} f(x) = 0$ so negative x axis is a horizontal asymptote.

$\lim_{x\to\infty} f(x) = \infty$… no asymptote there.

(356) Choosing left endpoints for the c's gives you

$$f(0)\frac{1}{3} + f(\frac{1}{3})\frac{2}{3} + f(1)\frac{1}{2} + f(\frac{3}{2})\frac{1}{2} = \cdots$$

Choosing right endpoints gives

$$f(\frac{1}{3})\frac{1}{3} + f(1)\frac{2}{3} + f(\frac{3}{2})\frac{1}{2} + f(2)\frac{1}{2} = \cdots$$

(357) The Riemann-sum is the total area of the rectangles, so to get the smallest Riemann-sum you must make the rectangles as small as possible. You can't change their widths, but you can change their heights by changing the c_i. To get the smallest area we make the heights as small as possible. Since f appears to be decreasing, the heights $f(c_i)$ will be smallest when c_i is as large as possible. So we choose the intermediate points c_i all the way to the right of the interval $x_{i-1} \le c_i \le x_i$, i.e. $c_1 = x_1$, $c_2 = x_2$, $c_3 = x_3$, $c_4 = x_4$, $c_5 = x_5$, $c_6 = b$,

To get the *largest* Riemann-sums you choose $c_1 = a$, $c_2 = x_1$, …, $c_6 = x_5$.

(398) **(a)** The first derivative of $\operatorname{erf}(x)$ is, by definition

$$\operatorname{erf}'(x) = \frac{2}{\sqrt{\pi}}e^{-x^2},$$

so you get the second derivative by differentiating this:

$$\operatorname{erf}''(x) = \frac{-4x}{\sqrt{\pi}}e^{-x^2}.$$

This is negative when $x > 0$, and positive when $x < 0$ so the graph of $\operatorname{erf}(x)$ has an inflection point at $x = 0$.

(b) Wikipedia is not wrong. Let's figure out what sign $\operatorname{erf}(-1)$ has (for instance). By definition you have

$$\operatorname{erf}(-1) = \frac{2}{\sqrt{\pi}}\int_0^{-1} e^{-t^2}\,dt.$$

Note that in this integral the upper bound (-1) is less than the lower bound (0). To fix that we switch the upper and lower integration bounds, which introduces a minus sign:

$$\operatorname{erf}(-1) = -\frac{2}{\sqrt{\pi}}\int_{-1}^{0} e^{-t^2}\,dt.$$

The integral we have here is positive because it's an integral of a positive function from a smaller number to a larger number, i.e. it is of the form $\int_a^b f(x)dx$ with $f(x) \geq 0$ and with $a < b$.

With the minus sign that makes $\operatorname{erf}(-1)$ negative.

(507) The answer is $\pi/6$.

To get this using the integral you use formula (9.9) with $f(x) = \sqrt{1-x^2}$. You get $f'(x) = -x/\sqrt{1-x^2}$, so

$$\sqrt{1+f'(x)^2} = \frac{1}{\sqrt{1-x^2}}.$$

The integral of that is $\arcsin x (+C)$, so the answer is $\arcsin \frac{1}{2} - \arcsin 0 = \pi/6$.

(543) $\frac{11}{6}$

(545) $\frac{2}{3}$

(556) $-\ln(1+\cos^2(x)) + C$

(560) $\frac{1}{4}(\ln(2x^2))^2 + C$

(570) (a) $\frac{1}{2}(x\sqrt{1+x^2} + \ln|x+\sqrt{1+x^2}|\,) + C$

(b) $\frac{1}{2}\,\frac{x}{1+x^2} + \frac{1}{2}\,\arctan(x) + C$

(c) $\ln|x+\sqrt{1+x^2}| + C$

(573) $\arcsin(x-1) + C$

(581) $\frac{1}{3}\arctan(x+1) + C$

(584) $$\int x^n \ln x \, dx = \frac{x^{n+1}\ln x}{n+1} - \frac{x^{n+1}}{(n+1)^2} + C.$$

(585) $$\int e^{ax} \sin bx \, dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx) + C.$$

(586) $$\int e^{ax} \cos bx \, dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx) + C.$$

(590) $\int_0^\pi \sin^{14} x dx = \frac{13\cdot 11\cdot 9\cdot 7\cdot 5\cdot 3\cdot 1}{14\cdot 12\cdot 10\cdot 8\cdot 6\cdot 4\cdot 2}\pi$

(591) $\int \cos^n x dx = \frac{1}{n}\sin x \cos^{n-1} x + \frac{n-1}{n}\int \cos^{n-2} x dx$; $\int_0^{\pi/4} \cos^4 x dx = \frac{7}{16} + \frac{3}{32}\pi$

(592) Hint: first integrate x^m.

(593) $x\ln x - x + C$

(594) $x(\ln x)^2 - 2x\ln x + 2x + C$

(596) Substitute $u = \ln x$.

(597) $\int_0^{\pi/4} \tan^5 x dx = \frac{1}{4}(1)^4 - \frac{1}{2}(1)^2 + \int_0^{\pi/4} \tan x dx = -\frac{1}{4} + \ln\frac{1}{2}\sqrt{2}$

(604) $1 + \frac{4}{x^3-4}$

(605) $1 + \frac{2x+4}{x^3-4}$

(606) $1 - \frac{x^2+x+1}{x^3-4}$

(607) $\frac{x^3-1}{x^2-1} = x + \frac{x-1}{x^2-1}$. You can simplify this further: $\frac{x^3-1}{x^2-1} = x + \frac{x-1}{x^2-1} = x + \frac{1}{x+1}$.

(608) $x^2 + 6x + 8 = (x+3)^2 - 1 = (x+4)(x+2)$ so $\frac{1}{x^2+6x+8} = \frac{1/2}{x+2} + \frac{-1/2}{x+4}$ and $\int \frac{dx}{x^2+6x+8} = \frac{1}{2}\ln(x+2) - \frac{1}{2}\ln(x+4) + C$.

(609) $\int \frac{dx}{x^2+6x+10} = \arctan(x+3) + C$.

(610) $\frac{1}{5}\int \frac{dx}{x^2+4x+5} = \frac{1}{5}\arctan(x+2) + C$

(611) We add

$$\begin{aligned}\frac{A}{x} + \frac{B}{x+1} + \frac{C}{x-1} &= \frac{A(x+1)(x-1) + Bx(x-1) + Cx(x+1)}{x(x+1)(x-1)} \\ &= \frac{(A+B+C)x^2 + (C-B)x - A}{x(x+1)(x-1)}.\end{aligned}$$

The numerators must be equal, i.e.

$$x^2 + 3 = (A+B+C)x^2 + (C-B)x - A$$

for all x, so equating coefficients gives a system of three linear equations in three unknowns A, B, C:

$$\begin{cases} A+B+C = 1 \\ C-B = 0 \\ -A = 3 \end{cases}$$

so $A = -3$ and $B = C = 2$, i.e.

$$\frac{x^2+3}{x(x+1)(x-1)} = -\frac{3}{x} + \frac{2}{x+1} + \frac{2}{x-1}$$

and hence

$$\int \frac{x^2+3}{x(x+1)(x-1)}\,\mathrm{d}x = -3\ln|x| + 2\ln|x+1| + 2\ln|x-1| + \text{constant}.$$

(612) To solve

$$\frac{x^2+3}{x(x+1)(x-1)} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{x-1},$$

multiply by x:

$$\frac{x^2+3}{(x+1)(x-1)} = A + \frac{Bx}{x+1} + \frac{Cx}{x-1}$$

and plug in $x = 0$ to get $A = -3$; then multiply by $x+1$:

$$\frac{x^2+3}{x(x-1)} = \frac{A(x+1)}{x} + B + \frac{C(x+1)}{x-1}$$

and plug in $x = -1$ to get $B = 2$; finally multiply by $x-1$:

$$\frac{x^2+3}{x(x+1)} = \frac{A(x-1)}{x} + \frac{B(x-1)}{x+1} + C,$$

and plug in $x = 1$ to get $C = 2$.

(613) Apply the method of equating coefficients to the form

$$\frac{x^2+3}{x^2(x-1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1}.$$

In this problem, the Heaviside trick can still be used to find C and B; we get $B = -3$ and $C = 4$. Then

$$\frac{A}{x} - \frac{3}{x^2} + \frac{4}{x-1} = \frac{Ax(x-1)+3(x-1)+4x^2}{x^2(x-1)}$$

so $A = -3$. Hence

$$\int \frac{x^2+3}{x^2(x-1)}\,\mathrm{d}x = -3\ln|x| + \frac{3}{x} + 4\ln|x-1| + \text{constant}.$$

(618) $\frac{1}{2}(x^2 + \ln|x^2-1|) + C$

(619) $\frac{1}{4}\ln|e^x-1| - \frac{1}{4}\ln|e^x+1| + \frac{1}{2}\arctan(e^x) + C$

(621) $\arctan(e^x+1) + C$

(622) $x - \ln(1+e^x) + C$

(625) $-\ln|x| + \frac{1}{x} + \ln|x-1| + C$

(633) $\int_0^a x\sin x dx = \sin a - a\cos a$

(634) $\int_0^a x^2\cos x dx = (a^2+2)\sin a + 2a\cos a$

(635) $\int_3^4 \frac{x\,dx}{\sqrt{x^2-1}} = \left[\sqrt{x^2-1}\right]_3^4 = \sqrt{15} - \sqrt{8}$

(636) $\int_{1/4}^{1/3} \frac{x\,dx}{\sqrt{1-x^2}} = \left[-\sqrt{1-x^2}\right]_{1/4}^{1/3} = \frac{1}{4}\sqrt{15} - \frac{1}{3}\sqrt{8}$

(637) same as previous problem after substituting $x = 1/t$

(638) $\frac{1}{2}\ln|x^2+2x+17| + \frac{1}{4}\arctan(\frac{x+1}{4}) + C$

(641) $\ln|x| + \frac{1}{x} + \ln|x-1| - \ln|x+1| + C$

(651) $x^2\ln(x+1) - \frac{x^2}{2} + x - \ln(x+1) + C$

(654) $x^2\arctan(\sqrt{x}) - \sqrt{x} + \arctan(\sqrt{x}) + C$

(657) $\tan(x) - \sec(x) + C$

(659) $\frac{1}{4}\ln(\frac{(x+1)^2}{x^2+1}) + \frac{1}{2}\arctan(x) + C$

(669) $\displaystyle\int \sqrt{1+x^2}\,\mathrm{d}x = \int \frac{(t^2+1)^2}{4t^3}\,\mathrm{d}t$

(673) $t = \sqrt{y^2-1} + y$

(674)

$$\int \frac{\mathrm{d}y}{(y^2-1)^{1/2}} = \int \frac{1}{t}\,dt$$

(678)

$$z = \frac{1}{y}$$

$$t = \sqrt{y^2-1} + y = \sqrt{(\frac{1}{z})^2 - 1} + \frac{1}{z}$$

(679)

$$\int \sqrt{1-z^2}\ \mathrm{d}z = \int \frac{2(t^2-1)^2}{(t^2+1)^3}\ dt$$

(680)

$$\int \frac{\mathrm{d}z}{\sqrt{1-z^2}} = \int \frac{-2\ \mathrm{d}t}{1+t^2}$$

(684) $\cos(\theta) = \sqrt{1-z^2}$ and $d\theta = \frac{dz}{\sqrt{1-z^2}}$, therefore

$$\int r(sin(\theta), cos(\theta))\ d\theta = \int \frac{r(z, \sqrt{1-z^2})}{\sqrt{1-z^2}}\ dz$$

(685)

$$\int \frac{r(\frac{1}{y}, \frac{x}{y})}{\frac{x}{y}}\ (\frac{-1}{y^2})(\frac{dy}{dt})\ dt$$

where

$$x = \frac{1}{2}(t - \frac{1}{t}) \qquad y = \frac{1}{2}(t + \frac{1}{t}) \qquad \frac{dy}{dt} = \frac{1}{2}(1 - \frac{1}{t^2})$$

(686)

$$t = \sqrt{(\frac{1}{z})^2 - 1} + \frac{1}{z} \qquad \text{where } z = \sin(\theta)$$

(687) Use Taylor's formula : $Q(x) = 43 + 19(x-7) + \frac{11}{2}(x-7)^2$.

A different, correct, but more laborious (clumsy) solution is to say that $Q(x) = Ax^2 + Bx + C$,, compute $Q'(x) = 2Ax + B$ and $Q''(x) = 2A$. Then

$$Q(7) = 49A + 7B + C = 43, \qquad Q'(7) = 14A + B = 19, \qquad Q''(7) = 2A = 11.$$

This implies $A = 11/2$, $B = 19 - 14A = 19 - 77 = -58$, and $C = 43 - 7B - 49A = 179\frac{1}{2}$.

(688) $p(x) = 3 + 8(x-2) - \frac{1}{2}(x-2)^2$

(703) $T_\infty e^t = 1 + t + \frac{1}{2!}t^2 + \cdots + \frac{1}{n!}t^n + \cdots$

(704) $T_\infty e^{\alpha t} = 1 + \alpha t + \frac{\alpha^2}{2!}t^2 + \cdots + \frac{\alpha^n}{n!}t^n + \cdots$

(705) $T_\infty \sin(3t) = 3t - \frac{3^3}{3!}t^3 + \frac{3^5}{5!}t^5 + \cdots + \frac{(-1)^k 3^{2k+1}}{(2k+1)!}t^{2k+1} + \cdots$

(706) $T_\infty \sinh t = t + \frac{1}{3!}t^3 + \cdots + \frac{1}{(2k+1)!}t^{2k+1} + \cdots$

(707) $T_\infty \cosh t = 1 + \frac{1}{2!}t^2 + \cdots + \frac{1}{(2k)!}t^{2k} + \cdots$

(708) $T_\infty \frac{1}{1+2t} = 1 - 2t + 2^2t^2 - \cdots + (-1)^n 2^n t^n + \cdots$

(709) $T_\infty \frac{3}{(2-t)^2} = \frac{3}{2^2} + \frac{3\cdot 2}{2^3}t + \frac{3\cdot 3}{2^4}t^2 + \frac{3\cdot 4}{2^5}t^3 + \cdots + \frac{3\cdot(n+1)}{2^{n+2}}t^n + \cdots$ (note the cancellation of factorials)

(710) $T_\infty \ln(1+t) = t - \frac{1}{2}t^2 + \frac{1}{3}t^3 + \cdots + \frac{(-1)^{n+1}}{n}t^n + \cdots$

(711) $T_\infty \ln(2+2t) = T_\infty \ln[2\cdot(1+t)] = \ln 2 + \ln(1+t) = \ln 2 + t - \frac{1}{2}t^2 + \frac{1}{3}t^3 + \cdots + \frac{(-1)^{n+1}}{n}t^n + \cdots$

(712) $T_\infty \ln\sqrt{1+t} = T_\infty \frac{1}{2}\ln(1+t) = \frac{1}{2}t - \frac{1}{4}t^2 + \frac{1}{6}t^3 + \cdots + \frac{(-1)^{n+1}}{2n}t^n + \cdots$

(713) $T_\infty \ln(1+2t) = 2t - \frac{2^2}{2}t^2 + \frac{2^3}{3}t^3 + \cdots + \frac{(-1)^{n+1}2^n}{n}t^n + \cdots$

(714) $T_\infty \ln\sqrt{\left(\frac{1+t}{1-t}\right)} = T_\infty\left[\frac{1}{2}\ln(1+t) - \frac{1}{2}\ln(1-t)\right] = t + \frac{1}{3}t^3 + \frac{1}{5}t^5 + \cdots + \frac{1}{2k+1}t^{2k+1} + \cdots$

(715) $T_\infty \frac{1}{1-t^2} = T_\infty\left[\frac{1/2}{1-t} + \frac{1/2}{1+t}\right] = 1 + t^2 + t^4 + \cdots + t^{2k} + \cdots$ (you could also substitute $x = -t^2$ in the geometric series $1/(1+x) = 1 - x + x^2 + \cdots$, later in this chapter we will use "little-oh" to justify this point of view.)

(716) $T_\infty \frac{t}{1-t^2} = T_\infty\left[\frac{1/2}{1-t} - \frac{1/2}{1+t}\right] = t + t^3 + t^5 + \cdots + t^{2k+1} + \cdots$ (note that this function is t times the previous function so you would think its Taylor series is just t times the taylor series of the previous function. Again, "little-oh" justifies this.)

(717) The pattern for the n^{th} derivative repeats every time you increase n by 4. So we indicate the the general terms for $n = 4m, 4m+1, 4m+2$ and $4m+3$:

$$T_\infty(\sin t + \cos t) = 1 + t - \frac{1}{2!}t^2 - \frac{1}{3!}t^3 + \frac{1}{4!}t^4 + \cdots + \frac{t^{4m}}{(4m)!} + \frac{t^{4m+1}}{(4m+1)!} - \frac{t^{4m+2}}{(4m+2)!} - \frac{t^{4m+3}}{(4m+3)!} + \cdots$$

(718) Use a double angle formula

$$T_\infty(2\sin t\cos t) = \sin 2t = 2t - \frac{2^3}{3!}t^3 + \cdots + \frac{2^{4m+1}}{(4m+1)!}t^{4m+1} - \frac{2^{4m+3}}{(4m+3)!}t^{4m+3} + \cdots$$

(719) $T_3 \tan t = t + \frac{1}{3}t^3$. There is no simple general formula for the n^{th} term in the Taylor series for $\tan x$.

(720) $T_\infty\left[1 + t^2 - \frac{2}{3}t^4\right] = 1 + t^2 - \frac{2}{3}t^4$

(721) $T_\infty[(1+t)^5] = 1 + 5t + 10t^2 + 10t^3 + 5t^4 + t^5$

(722) $T_\infty \sqrt[3]{1+t} = 1 + \frac{1/3}{1!}t + \frac{(1/3)(1/3-1)}{2!}t^2 + \cdots + \frac{(1/3)(1/3-1)(1/3-2)\cdots(1/3-n+1)}{n!}t^n + \cdots$

(723) $10!\cdot 2^6$

(724) Because of the addition formula

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \sin\beta\cos\alpha$$

you should get the same answer for f and g, since they are the same function!

The solution is

$$T_\infty \sin(x+a) = \sin a + \cos(a)x - \frac{\sin a}{2!}x^2 - \frac{\cos a}{3!}x^3 + \cdots$$
$$\cdots + \frac{\sin a}{(4n)!}x^{4n} + \frac{\cos a}{(4n+1)!}x^{4n+1} - \frac{\sin a}{(4n+2)!}x^{4n+2} - \frac{\cos a}{(4n+3)!}x^{4n+3} + \cdots$$

(727)

$$f(x) = f^{(4)}(x) = \cos x, \qquad f^{(1)}(x) = f^{(5)}(x) = -\sin x,$$
$$f^{(2)}(x) = -\cos x, \qquad f^{(3)}(x) = \sin x,$$

so

$$f(0) = f^{(4)}(0) = 1, \qquad f^{(1)}(0) = f^{(3)}(0) = 0, \qquad f^{(2)}(0) = -1.$$

and hence the fourth degree Taylor polynomial is

$$T_4\{\cos x\} = \sum_{k=0}^{4} \frac{f^{(k)}(0)}{k!}x^k = 1 - \frac{x^2}{2!} + \frac{x^4}{4!}.$$

The error is

$$R_4\{\cos x\} = \frac{f^{(5)}(\xi)}{5!}x^5 = \frac{(-\sin\xi)}{5!}x^5$$

for some ξ between 0 and x. As $|\sin\xi| \le 1$ we have

$$\left|\cos x - \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!}\right)\right| = |R_4(x)| \le \frac{|x^5|}{5!} < \frac{1}{5!}$$

for $|x| < 1$.

Remark: Since the fourth and fifth order Taylor polynomial for the cosine are the same, it must be that $R_4(x) = R_5(x)$. It follows that $\frac{1}{6!}$ is also an upperbound.

(729)

(a) The polynomial is $p(x) = 2 + \frac{1}{12}x - \frac{1}{9\cdot 32}x^2$. Then

$$p(1) \approx 2.07986111$$

and the error satisfies:

$$|\sqrt[3]{9} - p(1)| \le \frac{10}{27} \cdot 8^{-\frac{8}{3}} \cdot \frac{1}{3!} \approx 0.00024112654321$$

The $\sqrt[3]{9}$ according to a computer is:

$$\sqrt[3]{9} \approx 2.08008382305$$

(746) The PFD of g is $g(x) = \dfrac{1}{x-2} - \dfrac{1}{x-1}$.
$g(x) = \frac{1}{2} + \big(1 - \frac{1}{2^2}\big)x + \big(1 - \frac{1}{2^3}\big)x^2 + \cdots + \big(1 - \frac{1}{2^{n+1}}\big)x^n + \cdots$.
So $g_n = 1 - 1/2^{n+1}$ and $g^{(n)}(0)$ is $n!$ times that.

(747) You could repeat the computations from problem 746, and this would get you the right answer with the same amount of work. In this case you could instead note that $h(x) = xg(x)$ so that

$$h(x) = \tfrac{1}{2}x + \big(1 - \tfrac{1}{2^2}\big)x^2 + \big(1 - \tfrac{1}{2^3}\big)x^3 + \cdots + \big(1 - \tfrac{1}{2^{n+1}}\big)x^{n+1} + \cdots$$

Therefore $h_n = 1 - 1/2^n$.
The PFD of $k(x)$ is

$$k(x) = \frac{2-x}{(x-2)(x-1)} \overset{\text{cancel!}}{=} \frac{1}{1-x},$$

the Taylor series of k is just the Geometric series.

(749) $T_\infty e^{at} = 1 + at + \dfrac{a^2}{2!}t^2 + \cdots + \dfrac{a^n}{n!}t^n + \cdots$.

(750) $e^{1+t} = e \cdot e^t$ so $T_\infty e^{1+t} = e + et + \frac{e}{2!}t^2 + \cdots + \frac{e}{n!}t^n + \cdots$

(751) Substitute $u = -t^2$ in the Taylor series for e^u.

$$T_\infty e^{-t^2} = 1 - t^2 + \frac{1}{2!}t^4 - \frac{1}{3!}t^6 + \cdots + \frac{(-1)^n}{n!}t^{2n} + \cdots$$

(752) PFD! The PFD of $\frac{1+t}{1-t}$ is $\frac{1+t}{1-t} = -1 + \frac{2}{1-t}$. Remembering the Geometric Series you get

$$T_\infty \frac{1+t}{1-t} = 1 + 2t + 2t^2 + 2t^3 + \cdots + 2t^n + \cdots$$

(753) Substitute $u = -2t$ in the Geometric Series $1/(1-u)$. You get

$$T_\infty \frac{1}{1+2t} = 1 - 2t + 2^2t^2 - 2^3t^3 + \cdots + \cdots + (-1)^n 2^n t^n + \cdots$$

(754) $f(x) = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\, x^{2n}$

(755)

$$T_\infty \frac{\ln(1+x)}{x} = \frac{x - \frac12 x^2 + \frac13 x^3 + \cdots + (-1)^{n-1}\frac1n x^n + \cdots}{x}$$
$$= 1 - \frac12 x + \frac13 x^2 + \cdots + (-1)^{n-1}\frac1n x^{n-1} + \cdots$$

(756)

$$T_\infty \frac{e^t}{1-t} = 1 + 2t + \big(1+1+\tfrac{1}{2!}\big)t^2 + \big(1+1+\tfrac{1}{2!}+\tfrac{1}{3!}\big)t^3 + \cdots + \big(1+1+\tfrac{1}{2!}+\cdots+\tfrac{1}{n!}\big)t^n + \cdots$$

(757) $1/\sqrt{1-t} = (1-t)^{-1/2}$ so

$$T_\infty \frac{1}{\sqrt{1-t}} = 1 + \tfrac12 t + \frac{\frac12\frac32}{1\cdot 2}t^2 + \frac{\frac12\frac32\frac52}{1\cdot 2\cdot 3}t^3 + \cdots$$

(be careful with minus signs when you compute the derivatives of $(1-t)^{-1/2}$.)

You can make this look nicer if you multiply top and bottom in the n^{th} term with 2^n:

$$T_\infty \frac{1}{\sqrt{1-t}} = 1 + \frac12 t + \frac{1\cdot 3}{2\cdot 4}t^2 + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}t^3 + \cdots + \frac{1\cdot 3\cdots(2n-1)}{2\cdot 4\cdots 2n}t^n + \cdots$$

(758)

$$T_\infty \frac{1}{\sqrt{1-t^2}} = 1 + \frac12 t^2 + \frac{1\cdot 3}{2\cdot 4}t^4 + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}t^6 + \cdots + \frac{1\cdot 3\cdots(2n-1)}{2\cdot 4\cdots 2n}t^{2n} + \cdots$$

(759)

$$T_\infty \arcsin t = t + \frac12\frac{t^3}{3} + \frac{1\cdot 3}{2\cdot 4}\frac{t^5}{5} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\frac{t^7}{7} + \cdots + \frac{1\cdot 3\cdots(2n-1)}{2\cdot 4\cdots 2n}\frac{t^{2n+1}}{2n+1} + \cdots$$

(760) $T_4[e^{-t}\cos t] = 1 - t + \frac13 t^3 - \frac16 t^4$.

(761) $T_4[e^{-t}\sin 2t] = t - t^2 + \dfrac13 t^3 + o(t^4)$ (the t^4 terms cancel).

(762) PFD of $1/(2-t-t^2) = \frac{1}{(2+t)(1-t)} = \frac{-\frac13}{2+t} + \frac{\frac13}{1-t}$. Use the geometric series.

(763) $\sqrt[3]{1+2t+t^2} = \sqrt[3]{(1+t)^2} = (1+t)^{2/3}$. This is very similar to problem 757. The answer follows from Newton's binomial formula.

(766) 1/2

(767) Does not exist (or "$+\infty$")

(768) 1/2

(769) -1

(770) 0

(771) Does not exist (or "$-\infty$") because $e > 2$.

(772) 0.

(773) 0.

(774) 0 (write the limit as $\lim_{n\to\infty} \frac{n!+1}{(n+1)!} = \lim_{n\to\infty} \frac{n!}{(n+1)!} + \lim_{n\to\infty} \frac{1}{(n+1)!} = \lim_{n\to\infty} \frac{1}{n+1} + \lim_{n\to\infty} \frac{1}{(n+1)!}$).

(776) Use the explicit formula (11.8) from Example 11.6.10. The answer is the Golden Ratio ϕ.

(777) The remainder term $R_n(x)$ is equal to $\frac{f^{(n)}(\zeta_n)}{n!}x^n$ for some ζ_n. For either the cosine or sine and any n and ζ we have $|f^{(n)}(\zeta)| \le 1$. So $|R_n(x)| \le \frac{|x|^n}{n!}$. But we know $\lim_{n\to\infty} \frac{|x|^n}{n!} = 0$ and hence $\lim_{n\to\infty} R_n(x) = 0$.

(778) The k^{th} derivative of $g(x) = \sin(2x)$ is $g^{(k)}(x) = \pm 2^k \text{soc}(2x)$. Here $\text{soc}(\theta)$ is either $\sin\theta$ or $\cos\theta$, depending on k. Therefore k^{th} remainder term is bounded by

$$|R_k[\sin 2x]| \le \frac{|g^{(k+1)}(c)|}{(k+1)!}|x|^{k+1} = \frac{2^{k+1}|x|^{k+1}}{(k+1)!}|\text{soc}(2x)| \le \frac{|2x|^{k+1}}{(k+1)!}.$$

Since $\lim_{k\to\infty} \frac{|2x|^{k+1}}{(k+1)!} = 0$ we can use the Sandwich Theorem and conclude that $\lim_{k\to\infty} R_k[g(x)] = 0$, so the Taylor series of g converges for every x.

(782) Read the example in §11.8.3.

(783) $-1 < x < 1$.

(784) $-1 < x < 1$.

(785) $-1 < x < 1$.

(786) $-\frac{3}{2} < x < \frac{3}{2}$. Write $f(x)$ as $f(x) = \frac{1}{3}\frac{1}{1-(-\frac{2}{3}x)}$ and use the Geometric Series.

(787) $|x| < 2/5$

(796) The Taylor series is

$$\sin(t) = t - t^3/6 + \cdots$$

and the order one and two Taylor polynomial is the same $p(t) = t$. For any t there is a ζ between 0 and t with

$$\sin(t) - p(t) = \frac{f^{(3)}(\zeta)}{3!}t^3$$

When $f(t) = \sin(t)$, $|f^{(n)}(\zeta)| \le 1$ for any n and ζ. Consequently,

$$|\sin(t) - p(t)| \le \frac{t^3}{3!}$$

for nonnegative t. Hence

$$|\int_0^{\frac{1}{2}} \sin(x^2)\,dx - \int_0^{\frac{1}{2}} p(x^2)\,dx| \le \int_0^{\frac{1}{2}} |\sin(x^2) - p(x^2)|\,dx \le \int_0^{\frac{1}{2}} \frac{x^6}{3!}\,dx = \frac{(\frac{1}{2})^7}{3!\,7} = \epsilon$$

Since $\int_0^{\frac{1}{2}} p(x^2)\,dx = \frac{(\frac{1}{2})^3}{3} = A$ (the approximate value) we have that

$$A - \epsilon \le \int_0^{\frac{1}{2}} \sin(x^2)\,dx \le A + \epsilon$$

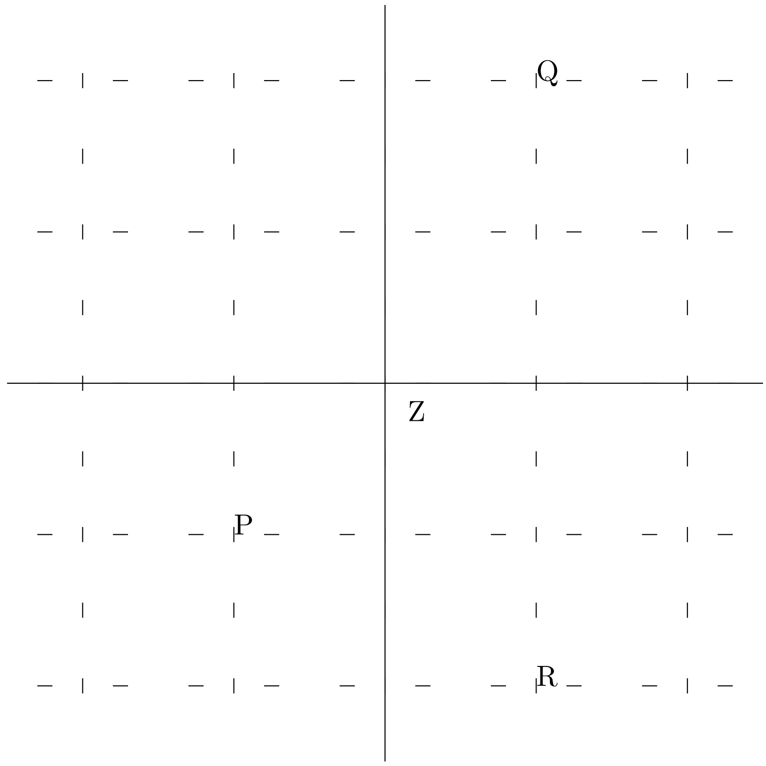
(797) (b) $\frac{43}{30}$ (c) $\frac{3}{6!\cdot 13}$

(803) (1) $-5+12i$ (2) $2-3i$ (3) $\sqrt{13}$ (4) $\frac{2}{13}-\frac{3}{13}i$

(1) 3 (2) 2 (3) $4e^{6i}$ (4) $\frac{1}{2}e^{-3i}$

(1) π (2) $\frac{3}{2}\pi$

(804) The dotted lines are one unit apart.



(805) (a) $\arg(1+i\tan\theta)=\theta+2k\pi$, with k any integer.

(b) $zw=1-\tan\theta\tan\phi+i(\tan\theta+\tan\phi)$

(c) $\arg(zw)=\arg z+\arg w=\theta+\phi$ (+ a multiple of 2π.)

(d) $\tan(\arg zw)=\tan(\theta+\phi)$ on one hand, and $\tan(\arg zw)=\dfrac{\tan\theta+\tan\phi}{1-\tan\theta\tan\phi}$ on the other hand. The conclusion is that

$$\tan(\theta+\phi)=\frac{\tan\theta+\tan\phi}{1-\tan\theta\tan\phi}$$

(806) $\cos 4\theta=$ real part of $(\cos\theta+i\sin\theta)^4$. Expand, using Pascal's triangle to get

$\cos 4\theta=\cos^4\theta-6\cos^2\theta\sin^2\theta+\sin^4\theta.$

$\sin 4\theta=4\cos^3\theta\sin\theta-4\cos\theta\sin^3\theta.$

$\cos 5\theta=\cos^5\theta-10\cos^3\theta\sin^2\theta+5\cos\theta\sin^4\theta$

$\sin 6\theta=6\cos^5\theta\sin\theta-20\cos^3\theta\sin^3\theta+6\cos\theta\sin^5\theta.$

(809) To prove or disprove the statements set $z=a+bi$, $w=c+di$ and substitute in the equation. Then compare left and right hand sides.

(a) $\mathfrak{Re}(z) + \mathfrak{Re}(w) = \mathfrak{Re}(z + w)$ TRUE, *because:*
$\mathfrak{Re}(z + w) = \mathfrak{Re}(a + bi + c + di) = \mathfrak{Re}[(a + c) + (b + d)i] = a + c$ and
$\mathfrak{Re}(z) + \mathfrak{Re}(w) = \mathfrak{Re}(a + bi) + \mathfrak{Re}(c + di) = a + c.$

The other proofs go along the same lines.

(b) $\overline{z + w} = \bar{z} + \bar{w}$ TRUE. *Proof:* if $z = a + bi$ and $w = c + di$ with a, b, c, d real numbers, then

$$\begin{aligned} \mathfrak{Re}(z) = a, \quad \mathfrak{Re}(w) = c &\implies \mathfrak{Re}(z) + \mathfrak{Re}(w) = a + c \\ z + w = a + c + (b + d)i &\implies \mathfrak{Re}(z + w) = a + c. \end{aligned}$$

So you see that $\mathfrak{Re}(z) + \mathfrak{Re}(w)$ and $\mathfrak{Re}(z + w)$ are equal.

(c) $\mathfrak{Im}(z) + \mathfrak{Im}(w) = \mathfrak{Im}(z + w)$ TRUE. *Proof:* if $z = a + bi$ and $w = c + di$ with a, b, c, d real numbers, then

$$\begin{aligned} \mathfrak{Im}(z) = b, \quad \mathfrak{Im}(w) = d &\implies \mathfrak{Im}(z) + \mathfrak{Im}(w) = b + d \\ z + w = a + c + (b + d)i &\implies \mathfrak{Im}(z + w) = b + d. \end{aligned}$$

So you see that $\mathfrak{Im}(z) + \mathfrak{Im}(w)$ and $\mathfrak{Im}(z + w)$ are equal.

(d) $\overline{zw} = (\bar{z})(\bar{w})$ TRUE

(e) $\mathfrak{Re}(z)\mathfrak{Re}(w) = \mathfrak{Re}(zw)$ FALSE. *Counterexample:* Let $z = i$ and $w = i$. Then $\mathfrak{Re}(z)\mathfrak{Re}(w) = 0 \cdot 0 = 0$, but $\mathfrak{Re}(zw) = \mathfrak{Re}(i \cdot i) = \mathfrak{Re}(-1) = -1$.

(f) $\overline{z/w} = (\bar{z})/(\bar{w})$ TRUE

(g) $\mathfrak{Re}(iz) = \mathfrak{Im}(z)$ FALSE (almost true though, only off by a minus sign)

(h) $\mathfrak{Re}(iz) = i\mathfrak{Re}(z)$ FALSE. The left hand side is a real number, the right hand side is an imaginary number: they can never be equal (except when $z = 0$.)

(i) $\mathfrak{Re}(iz) = \mathfrak{Im}(z)$ same as (g), sorry.

(j) $\mathfrak{Re}(iz) = i\mathfrak{Im}(z)$ FALSE

(k) $\mathfrak{Im}(iz) = \mathfrak{Re}(z)$ TRUE

(l) $\mathfrak{Re}(\bar{z}) = \mathfrak{Re}(z)$ TRUE

(810) The number is either $\frac{1}{5}\sqrt{5} + \frac{2}{5}i\sqrt{5}$ or $-\frac{1}{5}\sqrt{5} - \frac{2}{5}i\sqrt{5}$.

(811) It is $\frac{1}{3}\sqrt{3} + i$.

(813) The absolute value is 2 and the argument is $\ln 2$.

(814) e^z can be negative, or any other complex number except zero.

If $z = x + iy$ then $e^z = e^x(\cos y + i \sin y)$, so the absolute value and argument of e^z are $|z| = e^x$ and $\arg e^z = y$. Therefore the argument can be anything, and the absolute value can be any *positive* real number, but not 0.

(815) $\dfrac{1}{e^{it}} = \dfrac{1}{\cos t + i \sin t} = \dfrac{1}{\cos t + i \sin t}\dfrac{\cos t - i \sin t}{\cos t - i \sin t} = \dfrac{\cos t - i \sin t}{\cos^2 t + \sin^2 t} = \cos t - i \sin t = e^{-it}.$

(818) $Ae^{i\beta t} + Be^{-i\beta t} = A(\cos \beta t + i \sin \beta t) + B(\cos \beta t - i \sin \beta t) = (A + B) \cos \beta t + i(A - B) \sin \beta t.$

So $Ae^{i\beta t} + Be^{-i\beta t} = 2 \cos \beta t + 3 \sin \beta t$ holds if $A + B = 2, i(A - B) = 3$. Solving these two equations for A and B we get $A = 1 - \frac{3}{2}i$, $B = 1 + \frac{3}{2}i$.

(824) (a) $z^2+6z+10=(z+3)^2+1=0$ has solutions $z=-3\pm i$.

(b) $z^3+8=0 \iff z^3=-8$. Since $-8=8e^{\pi i+2k\pi}$ we find that $z=8^{1/3}e^{\frac{\pi}{3}i+\frac{2}{3}k\pi i}$ (k any integer). Setting $k=0,1,2$ gives you all solutions, namely

$$\begin{aligned} k=0 &\ : \ z=2e^{\frac{\pi}{3}i}=1+i\sqrt{3}\\ k=1 &\ : \ z=2e^{\frac{\pi}{3}i+2\pi i/3}=-2\\ k=2 &\ : \ z=2e^{\frac{\pi}{3}i+4\pi i/3}=1-i\sqrt{3} \end{aligned}$$

(c) $z^3-125=0$: $z_0=5$, $z_1=-\frac{5}{2}+\frac{5}{2}i\sqrt{3}$, $z_2=-\frac{5}{2}-\frac{5}{2}i\sqrt{3}$

(d) $2z^2+4z+4=0$: $z=-1\pm i$.

(e) $z^4+2z^2-3=0$: $z^2=1$ or $z^2=-3$, so the ***four*** solutions are $\pm 1, \pm i\sqrt{3}$.

(f) $3z^6=z^3+2$: $z^3=1$ or $z^3=-\frac{2}{3}$. The ***six*** solutions are therefore

$$\begin{aligned} -\tfrac{1}{2}\pm\tfrac{i}{2}\sqrt{3}, 1 &\quad \text{(from } z^3=1)\\ -\sqrt[3]{\tfrac{2}{3}}, \sqrt[3]{\tfrac{2}{3}}\big(\tfrac{1}{2}\pm\tfrac{i}{2}\sqrt{3}\big), &\quad \text{(from } z^3=-\tfrac{2}{3}) \end{aligned}$$

(g) $z^5-32=0$: The ***five*** solutions are

$$2,\quad 2\cos\tfrac{2}{5}\pi\pm 2i\sin\tfrac{2}{5}\pi,\quad 2\cos\tfrac{4}{5}\pi\pm 2i\sin\tfrac{4}{5}\pi.$$

Note that $2\cos\frac{6}{5}\pi+2i\sin\frac{6}{5}\pi=2\cos\frac{4}{5}\pi-2i\sin\frac{4}{5}\pi$, and likewise, $2\cos\frac{8}{5}\pi+2i\sin\frac{8}{5}\pi=2\cos\frac{2}{5}\pi-2i\sin\frac{2}{5}\pi$. (Make a drawing of these numbers to see why).

(h) $z^5-16z=0$: Clearly $z=0$ is a solution. Factor out z to find the equation $z^4-16=0$ whose solutions are ± 2, $\pm 2i$. So the ***five*** solutions are 0, ± 2, and $\pm 2i$

(i) $\sqrt{3}$, $2i$, $-\sqrt{3}$, $-2i$

(825) $f'(x)=\frac{-1}{(x+i)^2}$. In this computation you use the quotient rule, which is valid for complex valued functions.

$g'(x)=\frac{1}{x}+\frac{i}{1+x^2}$

$h'(x)=2ixe^{ix^2}$. Here we are allowed to use the Chain Rule because $h(x)$ is of the form $h_1(h_2(x))$, where $h_1(y)=e^{iy}$ is a complex valued function of a real variable, and $h_2(x)=x^2$ is a real valued function of a real variable (a "221 function").

(826) (a) Use the hint:

$$\begin{aligned} \int(\cos 2x)^4\,dx &= \int\left(\frac{e^{2ix}+e^{-2ix}}{2}\right)^4 dx\\ &= \tfrac{1}{16}\int\big(e^{2ix}+e^{-2ix}\big)^4\,dx \end{aligned}$$

The fourth line of Pascal's triangle says $(a+b)^4=a^4+4a^3b+6a^2b^2+4ab^3+b^4$. Apply this with $a=e^{2ix}$, $b=e^{-2ix}$ and you get

$$\begin{aligned} \int(\cos 2x)^4\,dx &= \frac{1}{16}\int\big\{e^{8ix}+4e^{4ix}+6+4e^{-4ix}+e^{-8ix}\big\}\,dx\\ &= \frac{1}{16}\Big\{\frac{1}{8i}e^{8ix}+\frac{4}{4i}e^{4ix}+6x+\frac{4}{-4i}e^{-4ix}+\frac{1}{-8i}e^{-8ix}\Big\}+C. \end{aligned}$$

We could leave this as the answer since we're done with the integral. However, we are asked to simplify our answer, and since we know ahead of time that the answer is a real function we should rewrite this as a real function. There are several ways of doing this, one of which is to carefully match complex exponential terms with their complex conjugates (e.g. e^{8ix} with e^{-8ix}.) This gives us

$$\int \left(\cos 2x\right)^4 \, dx = \frac{1}{16}\Big\{\frac{e^{8ix}-e^{-8ix}}{8i} + \frac{e^{4ix}-e^{-4ix}}{i} + 6x\Big\} + C.$$

Finally, we use the formula $\sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i}$ to remove the complex exponentials. We end up with the answer

$$\int \left(\cos 2x\right)^4 \, dx = \frac{1}{16}\big\{\tfrac{1}{4}\sin 8x + 2\sin 4x + 6x\big\} + C = \tfrac{1}{64}\sin 8x + \tfrac{1}{8}\sin 4x + \tfrac{3}{8}x + C.$$

(b) Use $\sin\theta = (e^{i\theta}-e^{-i\theta})/(2i)$:

$$\begin{aligned}
\int e^{-2x}\left(\sin ax\right)^2 \, dx &= \int e^{-2x}\Big(\frac{e^{iax}-e^{-iax}}{2i}\Big)^2 \, dx \\
&= \frac{1}{(2i)^2}\int e^{-2x}\left(e^{2iax} - 2 + e^{-2iax}\right) dx \\
&= -\tfrac{1}{4}\int \left(e^{(-2+2ia)x} - 2 + e^{(-2-2ia)x}\right) dx \\
&= -\tfrac{1}{4}\Big\{\underbrace{\frac{e^{(-2+2ia)x}}{-2+2ia}}_{A} - 2x + \underbrace{\frac{e^{(-2-2ia)x}}{-2-2ia}}_{B}\Big\} + C. \qquad (\dagger)
\end{aligned}$$

We are done with integrating. The answer must be a real function (being the integral of a real function), so we have to be able to write our answer in a real form. To get this real form we must expand the complex exponentials above, and do the division by $-2+2ia$ and $-2-2ia$. This is still a fair amount of work, but we can cut the amount of work in half by noting that the terms A and B are complex conjugates of each other, i.e. they are the same, except for the sign in front of i: you get B from A by changing all i's to $-i$'s. So once we have simplified A we immediately know B.

We compute A as follows

$$\begin{aligned}
A &= \frac{-2-2ia}{(-2-2ia)(-2+2ia)}\left(e^{-2x+2iax}\right) \\
&= \frac{(-2-2ia)e^{-2x}(\cos 2ax + i\sin 2ax)}{(-2)^2+(-2a)^2} \\
&= \frac{e^{-2x}}{4+4a^2}(-2\cos 2ax + 2a\sin 2ax) + i\frac{e^{-2x}}{4+4a^2}(-2a\cos 2ax - 2\sin 2ax).
\end{aligned}$$

Hence

$$B = \frac{e^{-2x}}{4+4a^2}(-2\cos 2ax + 2a\sin 2ax) - i\frac{e^{-2x}}{4+4a^2}(-2a\cos 2ax - 2\sin 2ax).$$

and

$$A + B = \frac{2e^{-2x}}{4+4a^2}(-2\cos 2ax + 2a\sin 2ax) = \frac{e^{-2x}}{1+a^2}(-\cos 2ax + a\sin 2ax).$$

Substitute this in (†) and you get the real form of the integral

$$\int e^{-2x}\bigl(\sin ax\bigr)^2\,dx = -\tfrac14\frac{e^{-2x}}{1+a^2}(-\cos 2ax + a\sin 2ax) + \frac{x}{2} + C.$$

(827) ***(a)*** This one can be done with the double angle formula, but if you had forgotten that, complex exponentials work just as well:

$$\begin{aligned}\int \cos^2 x\,\mathrm{d}x &= \int \Bigl(\frac{e^{ix}+e^{-ix}}{2}\Bigr)^2\,dx\\ &= \tfrac14\int \bigl\{e^{2ix}+2+e^{-2ix}\bigr\}\,dx\\ &= \tfrac14\bigl\{\tfrac{1}{2i}e^{2ix}+2x+\tfrac{1}{-2i}e^{-2ix}\bigr\}+C\\ &= \tfrac14\Bigl\{\frac{e^{2ix}-e^{-2ix}}{2i}+2x\Bigr\}+C\\ &= \tfrac14\bigl\{\sin 2x+2x\bigr\}+C\\ &= \tfrac14\sin 2x+\frac{x}{2}+C.\end{aligned}$$

(c), (d) using complex exponentials works, but for these integrals substituting $u = \sin x$ works better, if you use $\cos^2 x = 1 - \sin^2 x$.

(e) Use $(a-b)(a+b) = a^2 - b^2$ to compute

$$\cos^2 x\sin^2 x = \frac{(e^{ix}+e^{-ix})^2}{2^2}\frac{(e^{ix}-e^{-ix})^2}{(2i)^2} = \tfrac{1}{-16}(e^{2ix}+e^{-2ix})^2 = \tfrac{1}{-16}(e^{4ix}+2+e^{-4ix})$$

First variation: The integral is

$$\int \cos^2 x\sin^2 x\,dx = \tfrac{1}{-16}\bigl(\tfrac{1}{4i}e^{4ix}+2x+\tfrac{1}{-4i}e^{-4ix}\bigr)+C = \tfrac{1}{-32}\sin 4x - \tfrac18 x + C.$$

Second variation: Get rid of the complex exponentials before integrating:

$$\tfrac{1}{-16}(e^{4ix}+2+e^{-4ix}) = \tfrac{1}{-16}(2\cos 4x+2) = -\tfrac18(\cos 4x+1),$$

If you integrate this you get the same answer as above.

(j) and (l): Substituting complex exponentials will get you the answer, but for these two integrals you're much better off substituting $u = \cos x$ (and keep in mind that $\sin^2 x = 1-\cos^2 x$.)

(k) See (e) above.

(843) $y(t) = 2\dfrac{Ae^t+1}{Ae^t-1}$

(844) $y = Ce^{-x^3/3}$, $C = 5e^{1/3}$

(845) $y = Ce^{-x-x^3}$, $C = e^2$

(846) Implicit form of the solution $\tan y = -\frac{x^2}{2} + C$, so $C = \tan\pi/3 = \sqrt3$.

Solution $y(x) = \arctan\bigl(\sqrt3 - x^2/3\bigr)$

(847) Implicit form of the solution: $y + \frac{1}{2}y^2 + x + \frac{1}{2}x^2 = A + \frac{1}{2}A^2$. If you solve for y you get

$$y = -1 \pm \sqrt{A^2 + 2A + 1 - x^2 - 2x}$$

Whether you need the "+" or "−" depends on A.

(848) Integration gives $\dfrac{1}{2}\ln\left|\dfrac{y-1}{y+1}\right| = x + C$. Solve for y to get $\frac{y-1}{y+1} = \pm e^{2x+2C} = \left(\pm e^{2C}\right)e^{2x}$.

Let $B = \pm e^{2C}$ be the new constant and you get $\dfrac{y-1}{y+1} = Be^{2x}$ whence $y = \dfrac{1+Be^{2x}}{1-Be^{2x}}$.

The initial value $y(0) = A$ tells you that $B = \frac{A-1}{A+1}$, and therefore the solution with initial value $y(0) = A$ is $y = \dfrac{A+1+(A-1)e^{2x}}{A+1-(A-1)e^{2x}}$.

(849) $y(x) = \tan\bigl(\arctan(A) - x\bigr)$.

(850) $y = \sqrt{2(x - \frac{x^3}{3}) + 1}$

(851) $y = Ce^{-2x} - \frac{1}{3}e^x$

(852) $y = xe^{\sin x} + Ae^{\sin x}$

(853) Implicit form of the solution $\frac{1}{3}y^3 + \frac{1}{4}x^4 = C$; $C = \frac{1}{3}A^3$. Solution is $y = \sqrt[3]{A^3 - \frac{3}{4}x^4}$.

(857) General solution: $y(t) = Ae^{3t}\cos t + Be^{3t}\sin t$. Solution with given initial values has $A = 7$, $B = -10$.

(858) $y = 3e^x - e^{4x}$

(859) $y = Ae^x \sin(3x + B)$

(860) $y = Ae^t + Be^{-t} + C\cos t + D\sin t$

(861) The characteristic roots are $r = \pm\frac{1}{2}\sqrt{2} \pm \frac{1}{2}\sqrt{2}$, so the general solution is

$$y = Ae^{\frac{1}{2}\sqrt{2}t}\cos\tfrac{1}{2}\sqrt{2}t + Be^{\frac{1}{2}\sqrt{2}t}\sin\tfrac{1}{2}\sqrt{2}t + Ce^{-\frac{1}{2}\sqrt{2}t}\cos\tfrac{1}{2}\sqrt{2}t + De^{-\frac{1}{2}\sqrt{2}t}\sin\tfrac{1}{2}\sqrt{2}t.$$

(862) The characteristic equation is $r^4 - r^2 = 0$ whose roots are $r = \pm 1$ and $r = 0$ (double). Hence the general solution is $y = A + Bt + Ce^t + De^{-t}$.

(863) The characteristic equation is $r^4 + r^2 = 0$ whose roots are $r = \pm i$ and $r = 0$ (double). Hence the general solution is $y = A + Bt + C\cos t + D\sin t$.

(864) The characteristic equation is $r^3 + 1 = 0$, so we must solve

$$r^3 = -1 = e^{(\pi + 2k\pi)i}.$$

The characteristic roots are

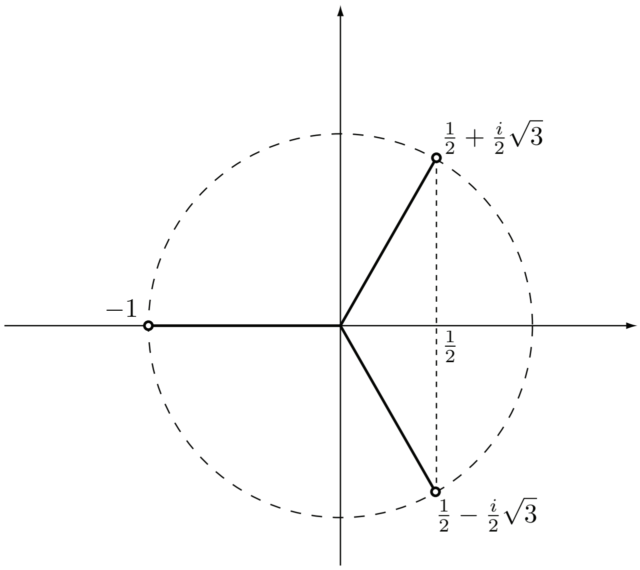
$$r = e^{\left(\frac{\pi}{3} + \frac{2}{3}k\pi\right)i}$$

where k is an integer. The roots for $k = 0, 1, 2$ are different, and all other choices of k lead to one of these roots. They are

$$k = 0: \qquad r = e^{\pi i/3} = \cos\frac{\pi}{3} + i\sin\frac{\pi}{3} = \frac{1}{2} + \frac{i}{2}\sqrt{3}$$

$$k = 1: \qquad r = e^{\pi i} = \cos\pi + i\sin\pi = -1$$

$$k = 2: \qquad r = e^{5\pi i/3} = \cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3} = \frac{1}{2} - \frac{i}{2}\sqrt{3}$$



The real form of the general solution of the differential equation is therefore

$$y = Ae^{-t} + Be^{\frac{1}{2}t}\cos\frac{\sqrt{3}}{2}t + Ce^{\frac{1}{2}t}\sin\frac{\sqrt{3}}{2}t$$

(865) $y = Ae^{t} + Be^{-\frac{1}{2}t}\cos\frac{\sqrt{3}}{2}t + Ce^{-\frac{1}{2}t}\sin\frac{\sqrt{3}}{2}t$

(866) $y(t) = c_1e^{\sqrt{3}t} + c_2e^{-\sqrt{3}t} + A\cos t + B\sin t.$

(867) Characteristic polynomial: $r^4 + 4r^2 + 3 = \left(r^2+3\right)\left(r^2+1\right)$.

Characteristic roots: $-i\sqrt{3}, -i, i, i\sqrt{3}$.

General solution: $y(t) = A_1\cos\sqrt{3}t + B_1\sin\sqrt{3}t + A_2\cos t + B_2\sin t$.

(868) Characteristic polynomial: $r^4 + 2r^2 + 2 = \left(r^2+1\right)^2 + 1$.

Characteristic roots: $r_{1,2}^2 = -1 + i$, $r_{3,4}^2 = -1 - i$.

Since $-1 + i = \sqrt{2}e^{\pi i/4 + 2k\pi}$ (k an integer) the square roots of $-1+i$ are $\pm 2^{1/4}e^{\pi i/8} = 2^{1/4}\cos\frac{\pi}{8} + i2^{1/4}\sin\frac{\pi}{8}$. The angle $\pi/8$ is not one of the familiar angles so we don't simplify $\cos\pi/8$, $\sin\pi/8$.

Similarly, $-1 - i = \sqrt{2}e^{-\pi i/4 + 2k\pi i}$ so the square roots of $-1 - i$ are $\pm 2^{1/4}e^{-\pi i/8} = \pm 2^{1/4}\left(\cos\frac{\pi}{8} - i\sin\frac{\pi}{8}\right)$.

If you abbreviate $a = 2^{1/4}\cos\frac{\pi}{8}$ and $b = 2^{1/4}\sin\frac{\pi}{8}$, then the four characteristic roots which we have found are

$$\begin{aligned} r_1 &= 2^{1/4}\cos\frac{\pi}{8} + i2^{1/4}\sin\frac{\pi}{8} = a + bi \\ r_2 &= 2^{1/4}\cos\frac{\pi}{8} - i2^{1/4}\sin\frac{\pi}{8} = a - bi \\ r_3 &= -2^{1/4}\cos\frac{\pi}{8} + i2^{1/4}\sin\frac{\pi}{8} = -a + bi \\ r_4 &= -2^{1/4}\cos\frac{\pi}{8} - i2^{1/4}\sin\frac{\pi}{8} = -a - bi \end{aligned}$$

The general solution is

$$y(t) = A_1 e^{at}\cos bt + B_1 e^{at}\sin bt + A_2 e^{-at}\cos bt + B_2 e^{-at}\sin bt$$

(871) Characteristic equation is $r^3 - 125 = 0$, i.e. $r^3 = 125 = 125e^{2k\pi i}$. The roots are $r = 5e^{2k\pi i/3}$, i.e.

$$5, \quad 5(-\tfrac{1}{2} + \tfrac{i}{2}\sqrt{3}) = -\tfrac{5}{2} + \tfrac{5}{2}i\sqrt{3}, \text{ and } 5(-\tfrac{1}{2} - \tfrac{i}{2}\sqrt{3}) = -\tfrac{5}{2} - \tfrac{5}{2}i\sqrt{3}.$$

The general solution is

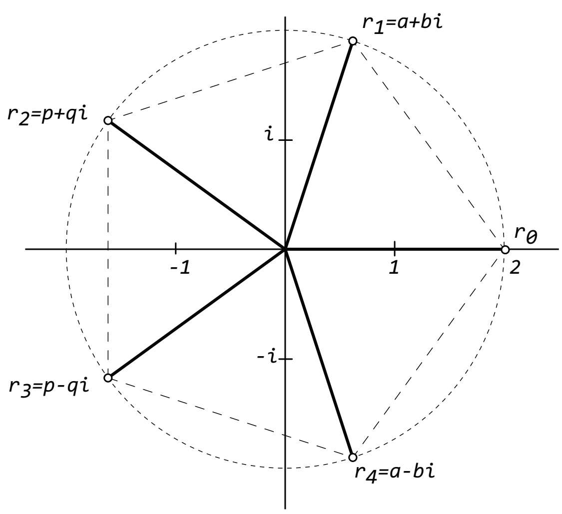
$$f(x) = c_1 e^{5x} + c_2 e^{-\frac{5}{2}x}\cos\tfrac{5}{2}\sqrt{3}x + c_3 e^{-\frac{5}{2}x}\sin\tfrac{5}{2}\sqrt{3}x.$$

(872) Try $u(x) = e^{rx}$ to get the characteristic equation $r^5 = 32$ which has solutions

$$r = 2,\ 2e^{\frac{2}{5}\pi i},\ 2e^{\frac{4}{5}\pi i},\ 2e^{\frac{6}{5}\pi i},\ 2e^{\frac{8}{5}\pi i},$$

i.e.

$$\begin{aligned} r_0 &= 2 \\ r_1 &= 2\cos\tfrac{2}{5}\pi + 2i\sin\tfrac{2}{5}\pi \\ r_2 &= 2\cos\tfrac{4}{5}\pi + 2i\sin\tfrac{4}{5}\pi \\ r_3 &= 2\cos\tfrac{6}{5}\pi + 2i\sin\tfrac{6}{5}\pi \\ r_4 &= 2\cos\tfrac{8}{5}\pi + 2i\sin\tfrac{8}{5}\pi. \end{aligned}$$



Remember that the roots come in complex conjugate pairs. By making a drawing of the roots you see that r_1 and r_4 are complex conjugates of each other, and also that r_2 and r_3 are complex conjugates of each other. So the roots are

$$2, \qquad 2\cos\tfrac{2}{5}\pi \pm 2i\sin\tfrac{2}{5}\pi, \text{ and } \quad 2\cos\tfrac{4}{5}\pi \pm 2i\sin\tfrac{4}{5}\pi.$$

The general solution of the differential equation is

$$u(x) = c_1e^{2x} + c_2e^{ax}\cos bx + c_3e^{ax}\sin bx + c_3e^{px}\cos qx + c_3e^{px}\sin qx.$$

Here we have abbreviated

$$a = 2\cos\tfrac{2}{5}\pi, b = 2\sin\tfrac{2}{5}\pi, p = 2\cos\tfrac{4}{5}\pi, q = 2\sin\tfrac{4}{5}\pi.$$

(874) Characteristic polynomial is $r^3 - 5r^2 + 6r - 2 = (r-1)(r^2 - 4r + 2)$, so the characteristic roots are $r_1 = 1, r_{2,3} = 2 \pm \sqrt{2}$. General solution:

$$y(t) = c_1e^t + c_2e^{(2-\sqrt{2})t} + c_3e^{(2+\sqrt{2})t}.$$

(876) Characteristic polynomial is $r^3 - 5r^2 + 4 = (r-1)(r^2 - 4r - 4)$. Characteristic roots are $r_1 = 1, r_{2,3} = 2 \pm 2\sqrt{2}$. General solution

$$z(x) = c_1e^x + c_2e^{(2+2\sqrt{2})x} + c_3e^{(2-2\sqrt{2})x}.$$

(877) General: $y(t) = A\cos 3t + B\sin 3t$. With initial conditions: $y(t) = \sin 3t$

(878) General: $y(t) = A\cos 3t + B\sin 3t$. With initial conditions: $y(t) = -3\cos 3t$

(879) General: $y(t) = Ae^{2t} + Be^{3t}$. With initial conditions: $y(t) = e^{3t} - e^{2t}$

(880) General: $y(t) = Ae^{-2t} + Be^{-3t}$. With initial conditions: $y(t) = 3e^{-2t} - 2e^{-3t}$

(881) General: $y(t) = Ae^{-2t} + Be^{-3t}$. With initial conditions: $y(t) = e^{-2t} - e^{-3t}$

(882) General: $y(t) = Ae^t + Be^{5t}$. With initial conditions: $y(t) = \frac{5}{4}e^t - \frac{1}{4}e^{5t}$

(883) General: $y(t) = Ae^t + Be^{5t}$. With initial conditions: $y(t) = (e^{5t} - e^t)/4$

(884) General: $y(t) = Ae^{-t} + Be^{-5t}$. With initial conditions: $y(t) = \frac{5}{4}e^{-t} - \frac{1}{4}e^{-5t}$

(885) General: $y(t) = Ae^{-t} + Be^{-5t}$. With initial conditions: $y(t) = \frac{1}{4}\big(e^{-t} - e^{-5t}\big)$

(886) General: $y(t) = e^{2t}\big(A\cos t + B\sin t\big)$. With initial conditions: $y(t) = e^{2t}\big(\cos t - 2\sin t\big)$

(887) General: $y(t) = e^{2t}\big(A\cos t + B\sin t\big)$. With initial conditions: $y(t) = e^{2t}\sin t$

(888) General: $y(t) = e^{-2t}\big(A\cos t + B\sin t\big)$. With initial conditions: $y(t) = e^{-2t}\big(\cos t + 2\sin t\big)$

(889) General: $y(t) = e^{-2t}\big(A\cos t + B\sin t\big)$. With initial conditions: $y(t) = e^{-2t}\sin t$

(890) General: $y(t) = Ae^{2t} + Be^{3t}$. With initial conditions: $y(t) = 3e^{2t} - 2e^{3t}$

(891) Characteristic polynomial: $r^3 + r^2 - r + 15 = (r+3)(r^2 - 2r + 5)$. Characteristic roots: $r_1 = -3$, $r_{2,3} = 1 \pm 2i$. General solution (real form) is

$$f(t) = c_1 e^{-3t} + Ae^t \cos 2t + Be^t \sin 2t.$$

The initial conditions require

$$f(0) = c_1 + A = 0, \quad f'(0) = -3c_1 + A + 2B = 1, \quad f''(0) = 9c_1 - 3A + 4B = 0.$$

Solve these equations to get $c_1 = -1/10$, $A = 1/10$, $B = 3/10$, and thus

$$f(t) = -\tfrac{1}{10} e^{-3t} + \tfrac{1}{10} e^t \cos 2t + \tfrac{3}{10} e^t \sin 2t.$$

(893) $y_P = \frac{1}{4}e^x + \frac{1}{2}x + \frac{1}{4}$

(894) $y = -2 + Ae^t + Be^{-t}$

(895) $y = Ae^t + Be^{-t} + te^t$

(896) $y = A\cos t + B \sin t + \frac{1}{6} t \sin t$

(897) $y = A\cos 3t + B\sin 3t + \frac{1}{8}\cos t$

(898) $y = A\cos t + B \sin t + \frac{1}{2} t \sin t$

(899) $y = A\cos t + B\sin t - \frac{1}{8}\cos 3t$

(901) **(i)** Homogeneous equation: try $z(t) = e^{rt}$, get characteristic equation $r^2 + 4r + 5 = 0$, with roots $r_{1,2} = -2 \pm i$. The general solution of the homogenous equation is therefore $z_h(t) = c_1 e^{-2t} \cos t + c_2 e^{-2t} \sin t$.

To find a particular solution try $z_p(t) = Ae^{it}$. You get $(i^2 + 4i + 5)Ae^{it} = e^{it}$, i.e. $(4+4i)A = 1$, so $A = \frac{1}{4+4i} = \frac{1}{4}\frac{1}{1+i} = \frac{1}{4}\frac{1-i}{2} = \frac{1}{8} - \frac{i}{8}$. So the general solution to the inhomogeneous problem is

$$z(t) = \frac{1-i}{8} e^{it} + c_1 e^{-2t} \cos t + c_2 e^{-2t} \sin t.$$

(903) Let $X(t)$ be the rabbit population size at time t. The rate at which this population grows is dX/dt rabbits per year.

$\frac{5}{100}X$ from growth at 5% per year

$\frac{2}{100}X$ from death at 2% per year

-1000 car accidents

$+700$ immigration from Sun Prairie

Together we get

$$\frac{dX}{dt} = \frac{3}{100}X - 300.$$

This equation is both separable and first order linear, so you can choose from two methods to find the general solution, which is

$$X(t) = 10,000 + Ce^{0.03t}.$$

If $X(1991) = 12000$ then

$$10,000 + Ce^{0.03\times 1991} = 12,000 \implies C = 2,000e^{-0.03\times 1991}\text{(don't simplify yet!)}$$

Hence

$$X(1994) = 10,000+2,000e^{-0.03\times 1991}e^{0.03\times 1994} = 10,000+2,000e^{0.03\times(1994-1991)} = 10,000+2,000e^{0.09} \approx 1$$

(904.ii) (i) Separate variables or find an integrating factor ($\frac{dT}{dt} - kT = -kA$). Both methods work here. You get $T(t) = A + Ce^{kt}$, where C is an arbitrary constant. Since $k < 0$ one has $\lim_{t\to\infty} e^{kt} = 0$, and hence $\lim_{t\to\infty} T(t) = \lim_{t\to\infty} A + Ce^{kt} = A + C \cdot 0 = A$.

(ii) Given $T(0) = 180$, $A = 75$, and $T(5) = 150$. This gives the following equations:

$$A + C = 180, \quad A + Ce^{5k} = 105 \implies C = 105, \quad 5k = \ln\frac{75}{105} = \ln\frac{5}{7} = -\ln\frac{7}{5}.$$

When is $T = 90$? Solve $T(t) = 90$ for t using the values for A, C, k found above (k is a bit ugly so we substitute it at the end of the problem):

$$T(t) = A + Ce^{kt} = 75 + 105e^{kt} = 90 \implies e^{kt} = \frac{15}{105} = \frac{1}{7}.$$

Hence

$$t = \frac{\ln 1/7}{k} = -\frac{\ln 7}{k} = \frac{\ln 7}{\ln 7/5}.$$

The limit as $t \to \infty$ of the temperature is $A = 75$ degrees.

(905) (a) Let $y(t)$ be the amount of "retaw" (in gallons) in the tank at time t. Then

$$\frac{dy}{dt} = \underbrace{\frac{5}{100}y}_{\text{growth}} - \underbrace{3}_{\text{removal}}.$$

(b) $y(t) = 60 + Ce^{t/20} = 60 + (y_0 - 60)e^{t/20}$.

(c) If $y_0 = 100$ then $y(t) = 60 + 40e^{t/20}$ so that $\lim_{t\to\infty} y(t) = +\infty$.

(d) $y_0 = 60$.

(906) Finding the equation is the hard part. Let $A(t)$ be the ***volume*** of acid in the vat at time t. Then $A(0) = 25\%$ of $1000 = 250$gallons.

$A'(t) =$ the volume of acid that gets pumped in minus the volume that gets extracted per minute. Per minute 40% of 20 gallons, i.e. 8 gallons of acid get added. The vat is well mixed, and $A(t)$ out of the 1000gallons are acid, so if 20 gallons get extracted, then $\frac{A}{1000} \times 20$ of those are acid. Hence

$$\frac{dA}{dt} = 8 - \frac{A}{1000} \times 20 = 8 - \frac{A}{50}.$$

The solution is $A(t) = 400 + Ce^{-t/50} = 400 + (A(0) - 400)e^{-t/50} = 400 - 150e^{-t/50}$.

The ***concentration*** at time t is

$$\text{concentration} = \frac{A(t)}{\text{total volume}} = \frac{400 - 150e^{-t/50}}{1000} = 0.4 - 0.15r^{-t/50}.$$

If you wait for very long the concentration becomes

$$\text{concentration} = \lim_{t\to\infty} \frac{A(t)}{1000} = 0.4.$$

(907) P is the volume of polluted water in the lake at time t. At any time the fraction of the lake water which is polluted is P/V, so if 24 cubic feet are drained then $\frac{P}{V} \times 24$ of those are polluted. Here $V = 10^9$; for simplicity we'll just write V until the end of the problem. We get

$$\frac{dP}{dt} = \text{"in minus out"} = 3 - \frac{P}{V} \times 24$$

whose solution is $P(t) = \frac{1}{8}V + Ke^{-\frac{24}{V}t}$. Here K is an arbitrary constant (which we can't call C because in this problem C is the concentration).

The concentration at time t is

$$C(t) = \frac{P(t)}{V} = \frac{1}{8} + \frac{K}{V}e^{-\frac{24}{V}t} = \frac{1}{8} + (C_0 - \frac{1}{8})e^{-\frac{24}{V}t}.$$

No matter what C_0 is you always have

$$\lim_{t\to\infty} C(t) = 0$$

because $\lim_{t\to\infty} e^{-\frac{24}{V}t} = 0$.

If $C_0 = \frac{1}{8}$ then the concentration of polluted water remains constant: $C(t) = \frac{1}{8}$.

(916) (a) $z(5) = 10.24$

(b) $b = -2\ln\frac{4}{5}$ and $k = \frac{b^2+\pi^2}{4}$

(917) No. Differentiating gives $z' = e^{-t}(\cos t - \sin t) = 0$ which is zero at $\pi/4$.

(918) Differentiating the equation $z'' + bz' + kz \equiv 0$ shows that z' is also a solution. Hence the zeros of z' are separated by the same intervals as the zeros of z. Hence its next peak will be at $t = 5$. Its height will be same as in problem 916 and the constants b, k will be the same. The actual solution (determined by the constants A and B) will be different.

(923) Hint: $(x-\alpha)^n = (x-\alpha)\cdot(x-\alpha)^{n-1}$

(928) $z = (-t + C_1)\cos t \quad + \quad (\ln|\sin t| + C_2)\sin t$

(929) Hint: you can say yourself some work if you remember that f and g were chosen to satisfy $f'z_1 + g'z_2 \equiv 0$.

(931) Hint: Multiply the first equation by d and the second by $-b$ then add them and solve for x.

(933)

$$f' = \frac{\det\begin{pmatrix} 0 & z_2 \\ b & z_2' \end{pmatrix}}{\det\begin{pmatrix} z_1 & z_2 \\ z_1' & z_2' \end{pmatrix}} \qquad g' = \frac{\det\begin{pmatrix} z_1 & 0 \\ z_1' & b \end{pmatrix}}{\det\begin{pmatrix} z_1 & z_2 \\ z_1' & z_2' \end{pmatrix}}$$

tells us by problem 931 that

$$f'z_1 + g'z_2 \equiv 0$$
$$f'z_1' + g'z_2' \equiv b$$

By problem 932

$$\mathcal{L}(fz_1 + gz_2) = f'z_1' + g'z_2'$$

and so it follows that

$$\mathcal{L}(fz_1 + gz_2) \equiv b$$

(936) (1) 3 (2) $\begin{pmatrix} 2 \\ -4 \\ 4 \end{pmatrix}$ (3) 36 (4) $\begin{pmatrix} 3 \\ -3 \\ 3 \end{pmatrix}$ (5) $\begin{pmatrix} 1 \\ -5 \\ 5 \end{pmatrix}$

(939) (a) Since $\begin{pmatrix} 1 \\ 2 \end{pmatrix} + \begin{pmatrix} x \\ x \end{pmatrix} = \begin{pmatrix} 1+x \\ 2+x \end{pmatrix}$ the number x would have to satisfy both $1+x=2$ and $2+x=1$. That's impossible, so there is no such x.

(b) No drawing, but $\vec{\boldsymbol{p}} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \begin{pmatrix} x \\ x \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} + x\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is the parametric representation of a straight line through the points $(1,2)$ (when $x=0$) and $(2,3)$ (when $x=1$).

(c) x and y must satisfy $\begin{pmatrix} x+y \\ 2x+y \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$. Solve $x+y=2$, $2x+y=1$ to get $x=-1$, $y=3$.

(940) Every vector is a position vector. To see of which point it is the position vector translate it so its initial point is the origin.

Here $\overrightarrow{AB} = \begin{pmatrix} -3 \\ 3 \end{pmatrix}$, so $\overrightarrow{AB}$ is the position vector of the point $(-3,3)$.

(941) One always labels the vertices of a parallelogram counterclockwise (see §14.4.3).

$ABCD$ is a parallelogram if $\overrightarrow{AB} + \overrightarrow{AD} = \overrightarrow{AC}$. $\overrightarrow{AB} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\overrightarrow{AC} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$, $\overrightarrow{AD} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$. So $\overrightarrow{AB} + \overrightarrow{AD} \neq \overrightarrow{AC}$, and $ABCD$ is not a parallelogram.

(942) (a) As in the previous problem, we want $\overrightarrow{AB} + \overrightarrow{AD} = \overrightarrow{AC}$. If D is the point (d_1, d_2, d_3) then $\overrightarrow{AB} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$, $\overrightarrow{AD} = \begin{pmatrix} d_1 \\ d_2-2 \\ d_3-1 \end{pmatrix}$, $\overrightarrow{AC} = \begin{pmatrix} 4 \\ -1 \\ 3 \end{pmatrix}$, so that $\overrightarrow{AB} + \overrightarrow{AD} = \overrightarrow{AC}$ will hold if $d_1 = 4$, $d_2 = 0$ and $d_3 = 3$.

(b) Now we want $\overrightarrow{AB} + \overrightarrow{AC} = \overrightarrow{AD}$, so $d_1 = 4$, $d_2 = 2$, $d_3 = 5$.

(947) (a) $\vec{\boldsymbol{x}} = \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} + t\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3-t \\ t \\ 1+t \end{pmatrix}$.

(b) Intersection with xy plane when $z=0$, i.e. when $t=-1$, at $(4,-1,0)$. Intersection with xz plane when $y=0$, when $t=0$, at $(3,0,1)$ (i.e. at A). Intersection with yz plane when $x=0$, when $t=3$, at $(0,3,4)$.

(948) (a) $\vec{L}[t] = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} + t\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$

(b) $\begin{pmatrix} 4 \\ 1 \\ 5 \end{pmatrix}$

(949) (a) $\vec{\boldsymbol{p}} = (\vec{\boldsymbol{b}} + \vec{\boldsymbol{c}})/2$, $\vec{\boldsymbol{q}} = (\vec{\boldsymbol{a}} + \vec{\boldsymbol{c}})/2$, $\vec{\boldsymbol{r}} = (\vec{\boldsymbol{a}} + \vec{\boldsymbol{b}})/2$.

(b) $\vec{\boldsymbol{m}} = \vec{\boldsymbol{a}} + \frac{2}{3}(\vec{\boldsymbol{p}} - \vec{\boldsymbol{a}})$ (See Figure 14.5, with AX twice as long as XB). Simplify to get $\vec{\boldsymbol{m}} = \frac{1}{3}\vec{\boldsymbol{a}} + \frac{1}{3}\vec{\boldsymbol{b}} + \frac{1}{3}\vec{\boldsymbol{c}}$.

(c) Hint : find the point N on the line segment BQ which is twice as far from B as it is from Q. If you compute this carefully you will find that $M = N$.

(951) To decompose $\vec{\boldsymbol{b}}$ set $\vec{\boldsymbol{b}} = \vec{\boldsymbol{b}}_\perp + \vec{\boldsymbol{b}}_{/\!/}$, with $\vec{\boldsymbol{b}}_{/\!/} = t\vec{\boldsymbol{a}}$ for some number t. Take the dot product with $\vec{\boldsymbol{a}}$ on both sides and you get $\vec{\boldsymbol{a}}\cdot\vec{\boldsymbol{b}} = t\|\vec{\boldsymbol{a}}\|^2$, whence $3 = 14t$ and $t = \frac{3}{14}$. Therefore

$$\vec{\boldsymbol{b}}_{/\!/} = \frac{3}{14}\vec{\boldsymbol{a}}, \qquad \vec{\boldsymbol{b}}_\perp = \vec{\boldsymbol{b}} - \frac{3}{14}\vec{\boldsymbol{a}}.$$

To find $\vec{\boldsymbol{b}}_{/\!/}$ and $\vec{\boldsymbol{b}}- \perp$ you now substitute the given values for $\vec{\boldsymbol{a}}$ and $\vec{\boldsymbol{b}}$.

The same procedure leads to $\vec{\boldsymbol{a}}_\perp$ and $\vec{\boldsymbol{a}}_{/\!/}$: $\vec{\boldsymbol{a}}_{/\!/} = \frac{3}{2}\vec{\boldsymbol{b}}$, $\vec{\boldsymbol{a}}_\perp = \vec{\boldsymbol{a}} - \frac{3}{2}\vec{\boldsymbol{b}}$.

(952) This problem is of the same type as the previous one, namely we have to decompose one vector as the sum of a vector perpendicular and a vector parallel to the hill's surface. The only difference is that we are not given the normal to the hill so we have to find it ourselves. The equation of the hill is $12x_1 + 5x_2 = 130$ so the vector $\vec{\boldsymbol{n}} = \left(\begin{smallmatrix}12\\5\end{smallmatrix}\right)$ is a normal.

The problem now asks us to write $\vec{\boldsymbol{f}}_{\text{grav}} = \vec{\boldsymbol{f}}_\perp + \vec{\boldsymbol{f}}_{/\!/}$, where $\vec{\boldsymbol{f}}_\perp = t\vec{\boldsymbol{n}}$ is perpendicular to the surface of the hill, and $\vec{\boldsymbol{f}}_{/\!/}$ id parallel to the surface.

Take the dot product with $\vec{\boldsymbol{n}}$, and you find $t\|\vec{\boldsymbol{n}}\|^2 = \vec{\boldsymbol{n}}\cdot\vec{\boldsymbol{f}}_{\text{grav}} \implies 169t = -5mg \implies t = -\frac{5}{169}mg$. Therefore

$$\vec{\boldsymbol{f}}_\perp = -\frac{5}{169}mg\begin{pmatrix}12\\5\end{pmatrix} = \begin{pmatrix}-\frac{60}{169}mg\\-\frac{25}{169}mg\end{pmatrix}, \qquad \vec{\boldsymbol{f}}_{/\!/} = \vec{\boldsymbol{f}}_{\text{grav}} - \vec{\boldsymbol{f}}_\perp = \begin{pmatrix}-\frac{60}{169}mg\\\frac{144}{169}mg\end{pmatrix},$$

(954) (i) $\|\vec{\boldsymbol{a}}-\vec{\boldsymbol{b}}\|^2 = \|\vec{\boldsymbol{a}}\|^2 - 2\vec{\boldsymbol{a}}\cdot\vec{\boldsymbol{b}} + \|\vec{\boldsymbol{b}}\|^2$; (ii) $\|2\vec{\boldsymbol{a}}-\vec{\boldsymbol{b}}\|^2 = 4\|\vec{\boldsymbol{a}}\|^2 - 4\vec{\boldsymbol{a}}\cdot\vec{\boldsymbol{b}} + \|\vec{\boldsymbol{b}}\|^2$; (iii) $\|\vec{\boldsymbol{a}}+\vec{\boldsymbol{b}}\| = \sqrt{54}$, $\|\vec{\boldsymbol{a}}-\vec{\boldsymbol{b}}\| = \sqrt{62}$ and $\|2\vec{\boldsymbol{a}}-\vec{\boldsymbol{b}}\| = \sqrt{130}$.

(956) Compute $\overrightarrow{AB} = -\overrightarrow{BA} = \begin{pmatrix}1\\1\end{pmatrix}$, $\overrightarrow{BC} = -\overrightarrow{CB} = \begin{pmatrix}-2\\2\end{pmatrix}$, $\overrightarrow{AC} = -\overrightarrow{CA} = \begin{pmatrix}-1\\3\end{pmatrix}$. Hence $\|\overrightarrow{AB}\| = \sqrt{2}$, $\|\overrightarrow{BC}\| = \sqrt{8} = 2\sqrt{2}$, $\|\overrightarrow{AC}\| = \sqrt{10}$.

And also $\overrightarrow{AB}\cdot\overrightarrow{AC} = 2 \implies \cos\angle A = \frac{\overrightarrow{AB}\cdot\overrightarrow{AC}}{\|\overrightarrow{AB}\|\,\|\overrightarrow{AC}\|} = \frac{2}{\sqrt{20}} = \frac{1}{\sqrt{5}}$.

A similar calculation gives $\cos\angle B = 0$ so we have a right triangle; and $\cos\angle C = \frac{2}{\sqrt{5}}$.

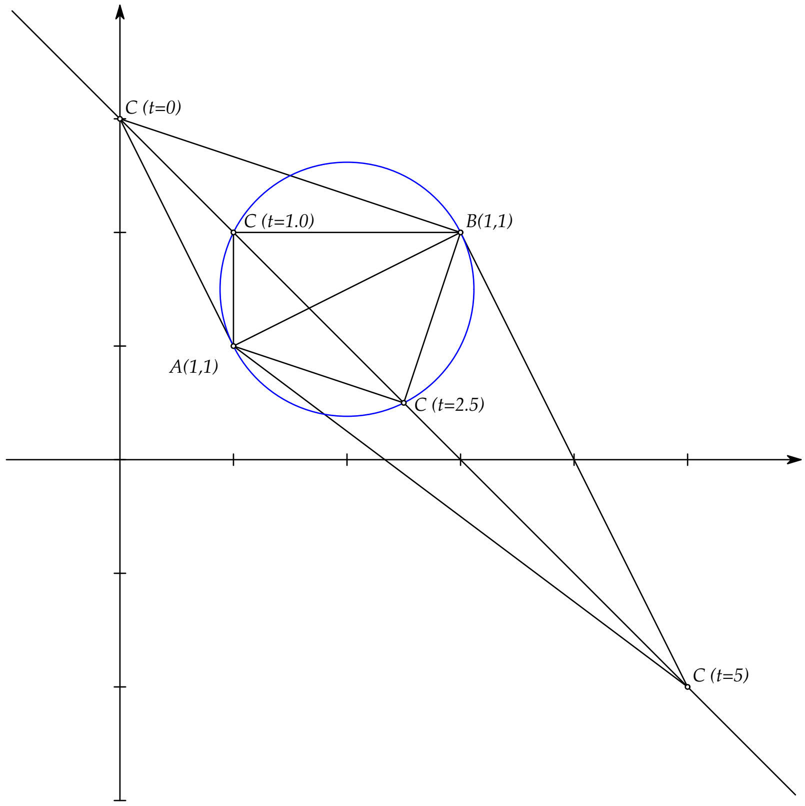
(957) $\overrightarrow{AB} = \left(\begin{smallmatrix}2\\1\end{smallmatrix}\right)$, $\overrightarrow{AC} = \left(\begin{smallmatrix}t-1\\2-t\end{smallmatrix}\right)$, $\overrightarrow{BC} = \left(\begin{smallmatrix}t-3\\1-t\end{smallmatrix}\right)$.

If the right angle is at A then $\overrightarrow{AB}\cdot\overrightarrow{AC} = 0$, so that we must solve $2(t-1) + (2-t) = 0$. Solution: $t = 0$, and $C = (0, 3)$.

If the right angle is at B then $\overrightarrow{AB}\cdot\overrightarrow{BC} = 0$, so that we must solve $2(t-3) + (1-t) = 0$. Solution: $t = 5$, and $C = (5, -2)$.

If the right angle is at C then $\overrightarrow{AC}\cdot\overrightarrow{BC} = 0$, so that we must solve $(t-1)(t-3) + (2-t)(1-t) = 0$. Note that this case is different in that we get a quadratic equation, and in that there are two solutions, $t = 1$, $t = \frac{5}{2}$.

This is a complete solution of the problem, but it turns out that there is a nice picture of the solution, and that the four different points C we find are connected with the circle whose diameter is the line segment AB:



(958.i) ℓ has defining equation $-\frac{1}{2}x + y = 1$ which is of the form $\vec{\boldsymbol{n}}\cdot\vec{\boldsymbol{x}}$ =constant if you choose $\vec{\boldsymbol{n}} = \begin{pmatrix} -1/2 \\ 1 \end{pmatrix}$.

(958.ii) The distance to the point D with position vector $\vec{\boldsymbol{d}}$ from the line ℓ is $\frac{\vec{\boldsymbol{n}}\cdot(\vec{\boldsymbol{d}}-\vec{\boldsymbol{a}})}{\|\vec{\boldsymbol{n}}\|}$ where $\vec{\boldsymbol{a}}$ is the position vector of any point on the line. In our case $\vec{\boldsymbol{d}} = \vec{\boldsymbol{0}}$ and the point $A(0,1)$, $\vec{\boldsymbol{a}} = \overrightarrow{OA} = \binom{0}{1}$, is on the line. So the distance to the origin from the line is $\dfrac{-\vec{\boldsymbol{n}}\cdot\vec{\boldsymbol{a}}}{\|\vec{\boldsymbol{n}}\|} = \dfrac{1}{\sqrt{(1/2)^2 + 1^2}} = 2/\sqrt{5}$.

(958.iii) $3x + y = 2$, normal vector is $\vec{\boldsymbol{m}} = \binom{3}{1}$.

(958.iv) Angle between ℓ and m is the angle θ between their normals, whose cosine is $\cos\theta = \frac{\vec{\boldsymbol{n}}\cdot\vec{\boldsymbol{m}}}{\|\vec{\boldsymbol{n}}\|\,\|\vec{\boldsymbol{m}}\|} = \frac{-1/2}{\sqrt{5/4}\sqrt{10}} = -\frac{1}{\sqrt{50}} = -\frac{1}{10}\sqrt{2}$.

(964.i) $\vec{\boldsymbol{0}}$ (the cross product of any vector with itself is the zero vector).

(964.iii) $(\vec{\boldsymbol{a}} + \vec{\boldsymbol{b}}) \times (\vec{\boldsymbol{a}} - \vec{\boldsymbol{b}}) = \vec{\boldsymbol{a}} \times \vec{\boldsymbol{a}} + \vec{\boldsymbol{b}} \times \vec{\boldsymbol{a}} - \vec{\boldsymbol{a}} \times \vec{\boldsymbol{b}} - \vec{\boldsymbol{b}} \times \vec{\boldsymbol{b}} = -2\vec{\boldsymbol{a}} \times \vec{\boldsymbol{b}}$.

(965) Not true. For instance, the vector $\vec{\boldsymbol{c}}$ could be $\vec{\boldsymbol{c}} = \vec{\boldsymbol{a}} + \vec{\boldsymbol{b}}$, and $\vec{\boldsymbol{a}} \times \vec{\boldsymbol{b}}$ would be the same as $\vec{\boldsymbol{c}} \times \vec{\boldsymbol{b}}$.

(966.i) A possible normal vector is $\vec{\boldsymbol{n}} = \overrightarrow{AB} \times \overrightarrow{AC} = \begin{pmatrix} -4 \\ 4 \\ -4 \end{pmatrix}$. Any (non zero) multiple of this vector is also a valid normal. The nicest would be $\frac{1}{4}\vec{\boldsymbol{n}} = \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}$.

(966.ii) $\vec{\boldsymbol{n}}\cdot(\vec{\boldsymbol{x}} - \vec{\boldsymbol{a}}) = 0$, or $\vec{\boldsymbol{n}}\cdot\vec{\boldsymbol{x}} = \vec{\boldsymbol{n}}\cdot\vec{\boldsymbol{a}}$. Using $\vec{\boldsymbol{n}}$ and $\vec{\boldsymbol{a}}$ from the first part we get $-4x_1 + 4x_2 - 4x_3 = -8$. Here you could replace $\vec{\boldsymbol{a}}$ by either $\vec{\boldsymbol{b}}$ or $\vec{\boldsymbol{c}}$. (Make sure you understand why; if you don't think about it, then ask someone).

(966.iii) Distance from D to $\mathcal{P}$ is $\frac{\vec{n}\cdot(\vec{d}-\vec{a})}{\|\vec{n}\|} = 4/\sqrt{3} = \frac{4}{3}\sqrt{3}$. There are many valid choices of normal $\vec{n}$ in part (i) of this problem, but they all give the same answer here.

Distance from O to $\mathcal{P}$ is $\frac{\vec{n}\cdot(\vec{0}-\vec{a})}{\|\vec{n}\|} = \frac{2}{3}\sqrt{3}$.

(966.iv) Since $\vec{n}\cdot(\vec{0}-\vec{a})$ and $\vec{n}\cdot(\vec{d}-\vec{a})$ have the same sign the point D and the origin lie on the same side of the plane $\mathcal{P}$.

(966.v) The area of the triangle is $\frac{1}{2}\|\overrightarrow{AB}\times\overrightarrow{AC}\| = 2\sqrt{3}$.

(966.vi) Intersection with x axis is A, the intersection with y-axis occurs at $(0,-2,0)$ and the intersection with the z-axis is B.

(967.i) Since $\vec{n} = \overrightarrow{AB}\times\overrightarrow{AC} = \begin{pmatrix}-3\\1\\1\end{pmatrix}$ the plane through A, B, C has defining equation $-3x+y+z = 3$. The coordinates $(2,1,3)$ of D do not satisfy this equation, so D is not on the plane ABC.

(967.ii) If E is on the plane through A, B, C then the coordinates of E satisfy the defining equation of this plane, so that $-3\cdot 1+1\cdot 1+1\cdot\alpha = 3$. This implies $\alpha = 5$.

(968.i) If $ABCD$ is a parallelogram then the vertices of the parallelogram are labeled A, B, C, D as you go around the parallelogram in a counterclockwise fashion. See the figure in §43.2. Then $\overrightarrow{AB}+\overrightarrow{AD} = \overrightarrow{AC}$. Starting from this equation there are now two ways to solve this problem.

(first solution) If D is the point (d_1,d_2,d_3) then $\overrightarrow{AD} = \begin{pmatrix}d_1-1\\d_2+1\\d_3-1\end{pmatrix}$, while $\overrightarrow{AB} = \begin{pmatrix}1\\1\\0\end{pmatrix}$ and $\overrightarrow{AC} = \begin{pmatrix}0\\3\\-1\end{pmatrix}$. Hence $\overrightarrow{AB}+\overrightarrow{AD} = \overrightarrow{AC}$ implies $\begin{pmatrix}d_1\\d_2+2\\d_3-1\end{pmatrix} = \begin{pmatrix}0\\3\\-1\end{pmatrix}$, and thus $d_1 = 0$, $d_2 = 1$ and $d_3 = 0$.

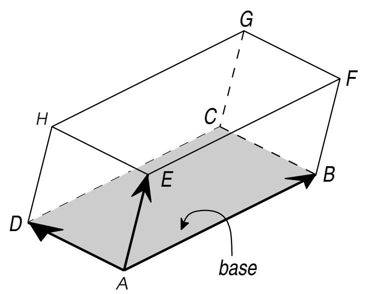
(second solution) Let $\vec{a},\vec{b},\vec{c},\vec{d}$ be the position vectors of A, B, C, D. Then $\overrightarrow{AB} = \vec{b}-\vec{a}$, etc. and $\overrightarrow{AB}+\overrightarrow{AD} = \overrightarrow{AC}$ is equivalent to $\vec{b}-\vec{a}+\vec{d}-\vec{a} = \vec{c}-\vec{a}$. Since we know $\vec{a},\vec{b},\vec{c}$ we can solve for $\vec{d}$ and we get $\vec{d} = \vec{c}-\vec{b}+\vec{a} = \begin{pmatrix}1\\-1\\1\end{pmatrix} - \begin{pmatrix}2\\0\\1\end{pmatrix} + \begin{pmatrix}1\\2\\0\end{pmatrix} = \begin{pmatrix}0\\1\\0\end{pmatrix}$.

(968.ii) The area of the parallelogram $ABCD$ is $\|\overrightarrow{AB}\times\overrightarrow{AD}\| = \left\|\begin{pmatrix}-1\\1\\3\end{pmatrix}\right\| = \sqrt{11}$.

(968.iii) In the previous part we computed $\overrightarrow{AB}\times\overrightarrow{AD} = \begin{pmatrix}-1\\1\\3\end{pmatrix}$, so this is a normal to the plane containing A, B, D. The defining equation for that plane is $-x+y+3z = 1$. Since $ABCD$ is a parallelogram any plane containing ABD automatically contains C.

(968.iv) $(-1,0,0)$, $(0,1,0)$, $(0,0,\frac{1}{3})$.

(969.i) Here is the picture of the parallelepiped (which you can also find on page 103):



Knowing the points A, B, D we get $\overrightarrow{AB} = \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}$, $\overrightarrow{AD} = \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$. Also, since $\frac{EFGH}{ABCD}$ is a parallelepiped, we know that all its faces are parallelogram, and thus $\overrightarrow{EF} = \overrightarrow{AB}$, etc. Hence: we find these coordinates for the points A, B, ...

$A(1, 0, 0)$, (given); $B(0, 2, 0)$, (given); $C(-2, 2, 1)$, since $\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{AD} = \begin{pmatrix} -3 \\ 2 \\ 1 \end{pmatrix}$;
$D(-1, 0, 1)$, (given); $E(0, 0, 2)$, (given)
$F(-1, 2, 2)$, since we know E and $\overrightarrow{EF} = \overrightarrow{AB} = \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}$
$G(-3, 2, 3)$, since we know F and $\overrightarrow{FG} = \overrightarrow{EH} = \overrightarrow{AD} = \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$
$H(-2, 0, 3)$, since we know E and $\overrightarrow{EH} = \overrightarrow{AD} = \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$.

(969.ii) The area of $ABCD$ is $\|\overrightarrow{AB} \times \overrightarrow{AD}\| = \sqrt{21}$.

(969.iii) The volume of $\mathfrak{P}$ is the product of its height and the area of its base, which we compute in the previous and next problems. So height$= \frac{\text{volume}}{\text{area base}} = \frac{6}{\sqrt{21}} = \frac{2}{7}\sqrt{21}$.

(969.iv) The volume of the parallelepiped is $\overrightarrow{AE} \cdot (\overrightarrow{AB} \times \overrightarrow{AD}) = 6$.

(971) The straight line $y = x + 1$, traversed from the top right to the bottom left as t increases from $-\infty$ to $+\infty$.

(972) The diagonal $y = x$ traversed from left to right, from upwards.

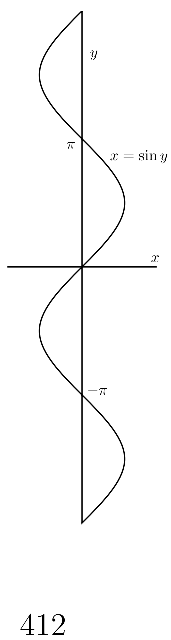
(973) The diagonal $y = x$ again, but since $x = e^t$ can only be positive we only get the part in the first quadrant. At $t = -\infty$ we start at the origin, as $t \to +\infty$ both x and y go to $+\infty$.

(974) The graph of $y = \ln x$, or $x = e^y$ (same thing), traversed in the upwards direction.

(975) The part of the graph of $y = 1/x$ which is in the first quadrant, traversed from left to right.

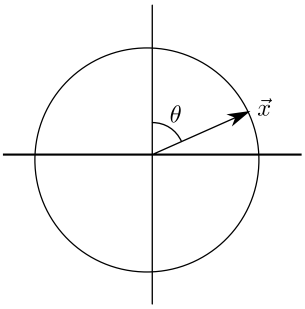
(976) The standard parabola $y = x^2$, from left to right.

(977) The graph $x = \sin y$. This is the usual Sine graph, but on its side.

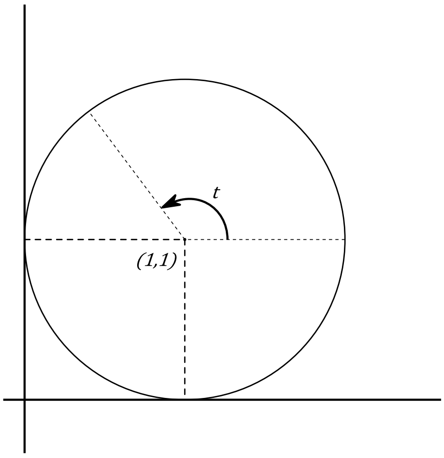


(978) We remember that $\cos 2\alpha = 1 - 2\sin^2\alpha$, so that $\vec{\boldsymbol{x}}(t)$ traces out a part of the parabola $y = 1 - x^2$. Looking at $x(t) = \sin t$ we see $\vec{\boldsymbol{x}}(t)$ goes back and forth on the part of the parabola $y = 1 - 2x^2$ between $x = -1$ and $x = +1$.

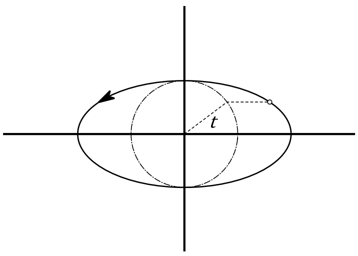
(979) The unit circle, traversed ***clockwise***, 25 times every 2π time units. Note that the angle $\theta = 25t$ is measured from the y-axis instead of from the x-axis.



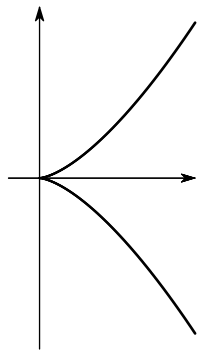
(980) Circle with radius 1 and center $(1, 1)$ (it touches the x and y axes). Traversed infinitely often in counterclockwise fashion.



(981) Without the 2 this would be the standard unit circle (dashed curve below). Multiplying the x component by 2 stretches this circle to an ellipse. So $\vec{\boldsymbol{x}}(t)$ traces out an ellipse, infinitely often, counterclockwise.



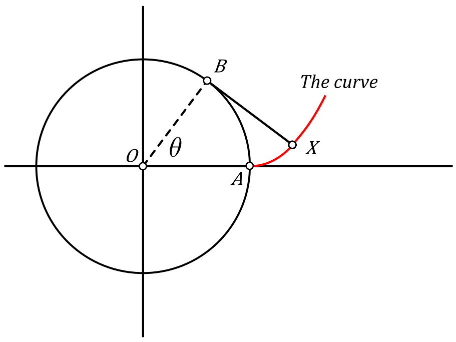
(982) For each $y = t^3$ there is exactly one t, namely, $t = y^{1/3}$. So the curve is a graph (with x as a function of y instead of the other way around). It is the graph of $x = y^{2/3} = \sqrt[3]{y^2}$.



The curve is called ***Neil's parabola***.

(983) If θ is the angle through which the wheel has turned, then $\vec{\boldsymbol{x}}(\theta) = \begin{pmatrix}\theta - a\sin\theta \\ 1 - a\cos\theta\end{pmatrix}$.

(986) Here's the picture:



The arc AB has length θ, and we are told the line segment BX has the same length. From this you get

$$\vec{\boldsymbol{x}}(\theta) = \begin{pmatrix}\cos\theta + \theta\sin\theta \\ \sin\theta - \theta\cos\theta\end{pmatrix}$$

This curve is called the ***evolute of the circle.***

(988.i) $\vec{\boldsymbol{x}}(0) = \vec{\boldsymbol{a}}$, $\vec{\boldsymbol{x}}(1) = \vec{\boldsymbol{c}}$ so the curve goes from A to C as t increases from $t = 0$ to $t = 1$. $\vec{\boldsymbol{x}}'(0) = 2(\vec{\boldsymbol{b}} - \vec{\boldsymbol{a}})$ so the tangent at $t = 0$ is parallel to the edge AB, and pointing from A to B. $\vec{\boldsymbol{x}}'(1) = 2(\vec{\boldsymbol{c}} - \vec{\boldsymbol{b}})$ so the tangent at $t = 1$ is parallel to the edge BC, and pointing from B to C. For an animation of the curve in this problem visit Wikipedia at

http://en.wikipedia.org/wiki/File:Bezier_2_big.gif

(988.ii) At $t = 1/2$. If you didn't get this, you can still get partial credit by checking that this answer is correct.

(990.i) Horizontal tangents: $t = 1/4$; Vertical tangents: $t = 0$; Directions: SouthEast $-\infty < t < 1/4$, NorthEast $1/4 < t < 0$, NorthWest $0 < t < \infty$.

(990.ii) This vector function is 2π periodic, so we only look at what happens for $0 \le t \le 2\pi$ (or you could take $-\pi \le t \le \pi$, or any other interval of length 2π).

Horizontal tangents: $t = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$; Vertical tangents: $t = \frac{\pi}{2}, \frac{3\pi}{2}$;

Directions: NE $0 < t < \frac{\pi}{4}$, SE $\frac{\pi}{4} < t < \frac{\pi}{2}$, SW $\frac{\pi}{2} < t < \frac{3\pi}{4}$, NW $\frac{3\pi}{4} < t < \frac{5\pi}{4}$, SW $\frac{5\pi}{4} < t < \frac{3\pi}{2}$, SE $\frac{3\pi}{2} < t < \frac{7\pi}{4}$, NE $\frac{7\pi}{4} < t < 2\pi$.

The curve traced out is a figure eight on its side, i.e. the symbol for infinity "∞".

(990.iii) Very similar to the previous problem. In fact both this vector function and the one from the previous problem trace out exactly the same curve. They just assign different values of the parameter t to points on the curve.

(990.iv) Horizontal points: $t = \pm\sqrt{a}$; Vertical points: $t = 0$; . Directions: SE $-\infty < t < -\sqrt{a}$, NE $-\sqrt{a} < t < 0$, NW $0 < t < \sqrt{a}$, SW $\sqrt{a} < t < \infty$.

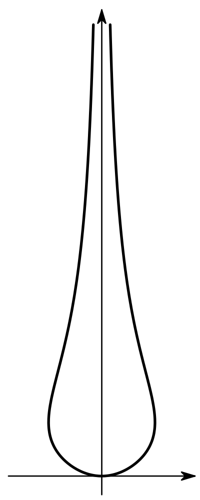
The curve looks like a "fish" (with some imagination.)

(990.v) No horizontal points; Vertical point: $t = 0$. Directions: NE $-\infty < t < 0$, NW $0 < t < \infty$.

(990.vi) This one has lots of horizontal and vertical tangents. If you replace the numbers 2 and 3 by other integers you get curves called "***Lissajous figures***". Get a graphing calculator/program and draw some. Or go to `http://en.wikipedia.org/wiki/Lissajous_curve` .

(990.vii) Horizontal point: $t = 0$; Vertical points: $t = \pm 1$; . Directions: SW $-\infty < t < -1$, SE $-1 < t < 0$, NE $0 < t < 1$, NW $1 < t < \infty$.

It sort of looks like this



(But this is really the graph of $\vec{\boldsymbol{x}}(t) = \begin{pmatrix} t/(1+t^4) \\ t^2 \end{pmatrix}$.)

(990.ix) This vector function traces out the right half of the parabola $y = 2(x-1)^2$ (i.e. the part with $x \ge 1$), going from right to left for $-\infty < t < 0$, and then back up again, from left to right for $0 < t < \infty$.

(998) Mercury's year is approximately 88 days.

Pluto is 3671 million miles from the sun.

(999) 539 miles.

(1000) 27 thousand miles.

(1001) 302942 to one.

(1002) 6778 miles.

An odd number of orbits does not work. The KMart space engineers knew that six orbits was too few but did not realize that 7 orbits is worse than six. The KMart7 satelite was tragically misnamed.

(1003) $A = \frac{1}{2}\int_0^T r^2 \frac{d\theta}{dt} dt = \frac{1}{2}\int_0^T \beta \, dt = \frac{1}{2} \, T \, \beta$

(1004) $a = \frac{D+d}{2}$ and $b = \sqrt{dD}$.

(1005) The furthest distance from sun is 3313 million miles. Its max and min speeds are 1086 and 17 million miles per year.